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REGULARITY STABILIZATION FOR THE POWERS OF GRADED $\mathfrak{M}$-PRIMARY IDEALS

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Abstract. This note provides first a generalization of the stabilization result of Eisenbud and Ulrich for the regularity of powers of an $\mathfrak{m}$-primary ideal to the case of ideals that are not generated in a single degree (see Theorem 1.5). We then partially extend our previous results expressing this stabilization degree in terms of the regularity of a specific graded strand of the Rees ring: The natural extension of the statement holds at least if the stabilization index or the regularity is greater than the number of variables. In any case, a precise comparison is given (see Theorem 4.1).

For simplicity, we do not introduce a graded module as in the work of Eisenbud and Ulrich. It can be done along the same lines, but makes the statements less transparent (see Remark 1.6 where we derive the key point for such an extension).

1. Stability and cohomology of a graded strand of the Rees ring

Let $A := k[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field k, $\mathfrak{m} := (x_1, \dots, x_n)$ and $I = (f_1, \dots, f_m)$ be a graded $\mathfrak{m}$-primary ideal.

Set $d := \min\{\mu \mid \exists p,\ (I_{\leq \mu})I^p = I^{p+1}\}$ and $I' := (I_{\leq d})$. By [Ko] there exists $b \geq 0$ such that

$$\operatorname{reg}(I^t) = dt + b, \quad \forall t \gg 0.$$

Furthermore,

Proposition 1.1 ([Ch2, 4.1]). *Set $t_0 := \min\{t \geq 1 \mid \mathfrak{m}^d I^t \subseteq I'\}$. Then*
(i) $d = \min\{\mu \mid (I_{\leq \mu})$ is $\mathfrak{m}$-primary$\}$.
(ii) The function $f(t) := \operatorname{reg}(I^t) - td$ is weakly decreasing for $t \geq t_0$.
(iii) One has

$$t_0 \leq \max\left\{1, \left\lceil \frac{\operatorname{reg}(I') - d}{d+1} \right\rceil\right\}$$

and $\operatorname{reg}(I') \leq (d-1)n + 1$.

In particular $t_0 \leq \left\lceil \frac{(n-1)(d-1)}{d+1} \right\rceil$ (unless $n = 1$ or $d = 1$). In Example 2.3 of [EU] $n = 4$ and $d = 5$ and this bound is sharp.

Similar related estimates concerning (ii) and (iii) are given in [Be].

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We now want to derive results concerning the stabilization index:

$$\mathrm{Stab}(I) := \min\{t \mid \mathrm{reg}(I^s) = ds + b,\ \forall s \geq t\}.$$

Let $(g_1, \dots, g_s)$ be generators of $I' = (I_{\leq d})$ with

(i) $\deg g_i = d$ for $1 \leq i \leq m$ and $\deg g_i \leq d$ for all i,

(ii) setting $J := (g_1, \dots, g_m)$, $J + (I_{<d})$ is a reduction of I.

Condition (ii) is satisfied if k is infinite, m is at least $n - \ell((I_{<d}))$, where ℓ stands for the analytic spread, and the g_i's are general elements in I_d.

Define $B := k[T_1, \dots, T_m]$, $\mathfrak{n} := (T_1, \dots, T_m)$ and $S := A[T_1, \dots, T_m]$.

We set $\mathrm{bideg}(T_i) := (0, 1)$ and $\mathrm{bideg}(a) := (\deg(a), 0)$ for $a \in A$. The natural inclusions $\mathcal{R}_J \subset \mathcal{R}_{I'} \subset \mathcal{R}_I := \bigoplus_{t \geq 0} I(d)^t = \bigoplus_{t \geq 0} I^t(td)$ and the onto map $S \rightarrow \mathcal{R}_J$ makes $\mathcal{R}_I$ a bigraded S-module. This module need not be finite over S. However $H^i_{\mathfrak{m}}(\mathcal{R}_I)_{\mu,*}$ is a finite graded B-module for every μ by [Ch1, 2.1(ii)], as $J + (I_{<d})$ is a reduction of I.

As I is $\mathfrak{m}$-primary, $H^1_{\mathfrak{m}}(\mathcal{R}_I) = \bigoplus_{t \geq 1} (A/I^t)(td)$, $H^n_{\mathfrak{m}}(\mathcal{R}_I) = \bigoplus_{t \geq 0} H^n_{\mathfrak{m}}(A)(td)$, and $H^i_{\mathfrak{m}}(\mathcal{R}_I) = 0$ for $i \notin \{1, n\}$.

Denote by N the finitely generated bigraded S-module $H^1_{\mathfrak{m}}(\mathcal{R}_I) = \bigoplus_{t \geq 1} (A/I^t)(td)$.

Lemma 1.2. *Let $t \geq 1$ and $\mu \geq 0$. Then*
(i) $N_{\mu,t} = 0 \Rightarrow N_{\mu+1,t} = 0$.
(ii) If $t \geq t_0$, then $N_{\mu,t} = 0 \Rightarrow N_{\mu,t+1} = 0$.

Proof. Claim (i) follows from the fact that $N_{*,t} = (A/I^t)(td)$ is cyclic and generated in degree $-td \leq \mu$ and (ii) is Proposition 1.1 (ii). □

Part (ii) shows that $N_{\mu,t} \neq 0$ if $0 \leq \mu < b$ and $t \geq t_0$.

Let

$$c := \max_{t \geq 1}\{\mathrm{reg}(I^t) - td\} = \mathrm{reg}_A(\mathcal{R}_I).$$

By Proposition 1.1, c is reached for a value of t equal at most to t_0.

Claim (i) implies that $e_\mu := \mathrm{end}(N_{\mu,*})$ is a weakly decreasing function of μ for $\mu \geq b$, which is equal to $-\infty$ for $\mu \geq c$. Hence,

Proposition 1.3.

$$\mathrm{reg}(I^t) = dt + b, \quad \forall t > e_b,$$

and $\mathrm{reg}(I^t) > dt + b$ *for any* $t_0 \leq t \leq e_b$.

Notice that, as the examples in [EU, 2.4] and [Be, Section 4] show, it may be that $N_{\mu,t} = 0$ for some $\mu \geq 0$ and $1 \leq t < \min\{e_\mu, t_0\}$.

Lemma 1.4. *There are natural bigraded isomorphisms*

$$H^1_{\mathfrak{m}+\mathfrak{n}}(\mathcal{R}_I) \simeq H^0_{\mathfrak{n}}(N) \simeq H^0_{\mathfrak{m}}(H^1_{\mathfrak{n}}(\mathcal{R}_I)).$$

Proof. The two spectral sequences arising from the double complex $\mathcal{C}^\bullet_{\mathfrak{n}}\mathcal{C}^\bullet_{\mathfrak{m}}\mathcal{R}_I$ abutting $H^\bullet_{\mathfrak{n}+\mathfrak{m}}(\mathcal{R}_I)$ have as first terms $\mathcal{C}^\bullet_{\mathfrak{n}}H^\bullet_{\mathfrak{m}}\mathcal{R}_I$ on one side. $H^0_{\mathfrak{m}}(\mathcal{R}_I) = 0$, in particular, the component of cohomological degree 1 in the abutment is $H^0_{\mathfrak{n}}(H^1_{\mathfrak{m}}(\mathcal{R}_I))$. As J has positive grade, $H^0_{\mathfrak{n}}(\mathcal{R}_I) = 0$ and the other spectral sequence has only one term of cohomological degree 1, which is $H^0_{\mathfrak{m}}(H^1_{\mathfrak{n}}(\mathcal{R}_I))$. □

This lemma shows that for $\mu \geq b$, $N_{\mu,t} = H^0_{\mathfrak{n}}(N_{\mu,t}) \subseteq H^1_{\mathfrak{n}}((\mathcal{R}_I)_{\mu,*})_t$, which in turn shows that

$$N_{\mu,t} = 0, \quad \forall t \geq \operatorname{reg}_B((\mathcal{R}_I)_{\mu,*}).$$

This implies the following:

Theorem 1.5. *With notation as above,*

$$\operatorname{reg}(I^t) = dt + b, \quad \forall t \geq \operatorname{end}(H^1_{\mathfrak{n}}((\mathcal{R}_I)_{b,*}) + 1,$$

hence $\operatorname{reg}(I^t) = dt + b$, *for* $t \geq \operatorname{reg}_B((\mathcal{R}_I)_{b,*})$. *More precisely,*

$$\operatorname{Stab}(I) = \operatorname{end}(H^0_{\mathfrak{m}}(H^1_{\mathfrak{n}}(\mathcal{R}_I))_{b,*}) + 1.$$

Proof. As $e_\mu := \operatorname{end}(N_{\mu,*})$ is a weakly decreasing function of μ for $\mu \geq b$, the result follows from Proposition 1.3 and the fact that

$$e_b = \operatorname{end}(H^0_{\mathfrak{m}}(H^1_{\mathfrak{n}}(\mathcal{R}_I))_{b,*}) \leq \operatorname{end}(H^1_{\mathfrak{n}}((\mathcal{R}_I)_{b,*}) \leq \operatorname{reg}_B((\mathcal{R}_I)_{b,*}) - 1$$

where the first equality is given by Lemma 1.4. □

We will see in Proposition 3.1 that $\operatorname{end}(H^0_{\mathfrak{m}}(H^1_{\mathfrak{n}}(\mathcal{R}_I))_{b,*}) = \operatorname{end}(H^1_{\mathfrak{n}}(\mathcal{R}_I)_{b,*})$ if J is $\mathfrak{m}$-primary.

Remark 1.6. Lemma 1.4 extends to the case where a graded A-module M such that M/IM is $\mathfrak{m}$-primary is involved, as in the work of Eisenbud and Ulrich. In that case, the arguments in the proof of Lemma 1.4 show that

$$H^1_{\mathfrak{m}+\mathfrak{n}}(M\mathcal{R}_I) \simeq H^0_{\mathfrak{m}}(H^1_{\mathfrak{n}}(M\mathcal{R}_I))$$

and $H^1_{\mathfrak{m}+\mathfrak{n}}(M\mathcal{R}_I)_{*,t} \simeq H^0_{\mathfrak{n}}(H^1_{\mathfrak{m}}(M\mathcal{R}_I))_{*,t}$ if $H^0_{\mathfrak{m}}(M) = 0$ or $t \geq 0$. In that setting, $H^i_{\mathfrak{m}}(MI^t) = H^i_{\mathfrak{m}}(M)$ for $i \geq 2$ and one has an exact sequence

$$0 \to H^0_{\mathfrak{m}}(I^tM) \to H^0_{\mathfrak{m}}(M) \to M/I^tM \to H^1_{\mathfrak{m}}(I^tM) \to H^1_{\mathfrak{m}}(M) \to 0,$$

which together with the variant of Lemma 1.2 deduced from [Ch2] provides an extension of [EU, 1.1] to the unequal degree case.

2. Finiteness of the regularity of graded strands of the Rees ring

Notice first that $\operatorname{reg}_S(\mathcal{R}_I) < \infty$ if $\mathcal{R}_I$ is finite over S, which in turn holds if J is a reduction of I; hence in that case $\operatorname{reg}_S((\mathcal{R}_I)_{\mu,*}) \leq \operatorname{reg}_S(\mathcal{R}_I) < \infty$, for any μ.

Let us also point out that $\operatorname{reg}_B(\mathcal{R}_I) = +\infty$ unless $I_{<d} = 0$, and if not, then $(\mathcal{R}_I)_{*,t+1}$ cannot be generated over B by $(\mathcal{R}_I)_{*,t}$, for any $t \geq 0$, for an obvious degree reason.

Next, the case of a complete intersection ideal will give us information. In that case one can do an explicit computation as follows.

Remark 2.1. If I is a complete intersection of n forms of degree d, then one can compute $\dim(A/I^t)_{\mu+td}$ explicitly, using [GVT, 2.3],

$$\dim_k(A/I^t)_{\mu+td} = \sum_{i=0}^{t-1} \binom{i+n-1}{n-1} \dim_k(A/I)_{\mu+(t-i)d}.$$

Hence for $t \gg 0$ (more precisely if $\mu + (t+1)d > n(d-1) = \operatorname{end}(A/I)$),

$$\dim_k(A/I^t)_{\mu+td} = \sum_{j\geq 1} \binom{t-j+n-1}{n-1} \dim_k(A/I)_{\mu+jd}.$$

It follows that $\dim_k(A/I^t)_{\mu+td} = D_\mu\binom{t+n-1}{n-1} + l.o.t.$ with $D_\mu := \sum_{j\geq 1}\dim_k(A/I)_{\mu+jd}$.

More generally, there exists a positive integer C_μ such that $\dim_k(A/I^t)_{\mu+td} = C_\mu\binom{t+m-1}{m-1} + l.o.t.$ if I is a complete intersection of degrees $d_1,\dots,d_n$ with $d = d_1 = \cdots = d_m > d_{m+1} \geq \cdots \geq d_n$, unless $\mu > \mathrm{end}(A/I) - d$, in which case $\dim_k(A/I^t)_{\mu+td} = 0$ for $t \geq 1$.

The remark above illustrates the fact that the finitely generated S-module $N_{\mu,*} = H^1_{\mathfrak{m}}(\mathcal{R}_I)_{\mu,*}$ is of Krull dimension $d(\mu) \leq n - \ell(I_{<d}) \leq n - \mathrm{codim}(I_{<d})$, and that this could be sharp. It in particular shows that $d(\mu) < n$ when $I_{<d} \neq 0$. Thus the equality $d(\mu) = n$ implies that I is generated in degrees $\geq d$, which is equivalent to the fact that it admits a homogeneous reduction generated in degree d.

The following lemma provides a criteria for the finiteness of the regularity.

Lemma 2.2. *Let M be a graded B-module. Then the following are equivalent:*

(i) $\dim_k(M_\mu) < +\infty$ *for some* $\mu \geq \mathrm{reg}(M)$,

(ii) $\dim_k(M_\mu) < +\infty$ *for all* $\mu \geq \mathrm{reg}(M)$.

If these conditions hold, then there exists a polynomial P of degree at most $m-1$ such that $\dim_k(M_\mu) = P(\mu)$ *for all* $\mu > \mathrm{reg}(M)$.

Proof. Let $H(\mu) := \dim_k(M_\mu)$. (ii) implies (i) and to prove the converse, it suffices to show that $H(\mu+1) \geq H(\mu)$ for $\mu \geq \mathrm{end}(H^0_{\mathfrak{n}}(M))$. Setting $\ell := U_1T_1 + \cdots + U_mT_m \in B \otimes_k k(U)$, $H(\mu) = \dim_{k(U)} M \otimes_k k(U)$ and

$$\ker(M_\mu \otimes_k k(U) \xrightarrow{\times \ell} M_{\mu+1} \otimes_k k(U)) \subseteq H^0_{\mathfrak{n}}(M)_\mu,$$

by the Dedekind-Mertens Lemma (see e.g. [CJR, 1.7]), which establishes our claim.

Notice that it suffices to show that there exists a polynomial P of degree at most $m-1$ such that $\dim_K(M_\nu) = P(\nu)$ for all $\nu > \mu$ if $\mu \geq \mathrm{reg}(M)$, with $\mu \in \mathbf{Z}$. Let $\mu \geq \mathrm{reg}(M)$ and $N := M_{\geq \mu}$. One has $\mathrm{reg}(N) = \mu$, unless $N = 0$, in which case $P = 0$ satisfies our claim. It follows that N is generated in degree μ, hence is finitely generated. As $\mathrm{reg}(N) = \mu$, and M and N coincide in degree at least μ, the conclusion follows. □

The useful consequence of Lemma 2.2 for our study is that any graded piece of the Rees algebra has infinite regularity whenever the ideal J has less than n generators.

Corollary 2.3. *If J has less than n generators (i.e. $m < n$), then* $\mathrm{reg}_B(\mathcal{R}_I)_{\mu,*} = +\infty$ *for any* $\mu \in \mathbf{Z}$.

Notice that $m < n$ can only happen if $I_{<d} \neq 0$.

Proof. By [Ch1, 2.1(ii)], the Hilbert function of $N_{\mu,*}$ is eventually a polynomial of degree at most $m-1$. As the dimension of $A_{\mu+td}$ over k is a polynomial in t of order $n-1$, it follows that it is also the case for $(\mathcal{R}_I)_{\mu,t}$ when $m < n$. From Lemma 2.2, we deduce the result as both (i) or (ii) are clear for $M := (\mathcal{R}_I)_{\mu,*}$. □

On the other hand, choosing J generated by n general elements in I_d (if k is infinite, else reducing to this case by faithfully flat extension) the ideal $J + (I_{<d})$ is a reduction of I, $N_{\mu,t}$ is finitely generated for any μ and J is $\mathfrak{m}$-primary.

3. Cohomology and regularity of graded strands of the Rees ring

In the context of the above two sections, the result below compares the cohomology of graded strands of $N = H^1_{\mathfrak{m}}(\mathcal{R}_I)$ and of $\mathcal{R}_I$, in order to obtain an estimate of the stabilization index in terms of the regularity of a graded strand of the Rees ring.

Proposition 3.1. *If the ideal J is $\mathfrak{m}$-primary, then*

(i) $H^j_{\mathfrak{n}}(\mathcal{R}_I)_{*,t} = H^0_{\mathfrak{m}}(H^j_{\mathfrak{n}}(\mathcal{R}_I)_{*,t})$ for any j and $t \geq 0$.

(ii) For any $\mu > -n$ and any $j > 0$, there exists a natural graded map of B-modules $\psi^\mu_j : H^j_{\mathfrak{n}}(N_{\mu,*}) \to H^{j+1}_{\mathfrak{n}}(\mathcal{R}_I)_{\mu,*}$, which is an isomorphism for $j > n$, onto for $j = n$, and an isomorphism in non-negative degree for any j.

(iii) For $\mu > -n$, let $d(\mu)$ be the Krull dimension of the support of the finitely generated graded B-module $N_{\mu,*}$. Then

$$\operatorname{reg}_B(N_{\mu,*}) \leq \max\{\operatorname{reg}_B((\mathcal{R}_I)_{\mu,*}) - 1, d(\mu)\},$$

$$\operatorname{reg}_B((\mathcal{R}_I)_{\mu,*}) \leq \max\{\operatorname{reg}_B(N_{\mu,*}) + 1, \operatorname{cd}_{B_+}((\mathcal{R}_I)_{\mu,*})\}.$$

(iv) $\operatorname{cd}_{B_+}((\mathcal{R}_I)_{\mu,*}) \leq n$ for $\mu > -n$.

This in particular shows that $\operatorname{reg}_B(N_{\mu,*}) = \operatorname{reg}_B((\mathcal{R}_I)_{\mu,*}) - 1$ if either

(1) $\operatorname{reg}_B(N_{\mu,*}) \geq n$ and $I_{<d} \neq 0$; or

(2) $\max\{\operatorname{reg}_B(N_{\mu,*}), \operatorname{reg}_B((\mathcal{R}_I)_{\mu,*})\} > n$.

Proof. First notice that after inverting any of the variables X_i, one has graded (but in general not bigraded) identifications

$$(\mathcal{R}_J)_{X_i} = (\mathcal{R}_J)_{X_i} = (\mathcal{R}_I)_{X_i} = \bigoplus_{t \geq 0} A_{X_i} T^t = A_{X_i}[T],$$

as $JA_{X_i} = JA_{X_i} = IA_{X_i} = A_{X_i}$. In the above identification, the positive part $\mathfrak{n}$ of S is mapped to the ideal generated by TJA_{X_i} which is the ideal $(T) = \bigoplus_{t>0} A_{X_i} T^t$.

As in the proof of Lemma 1.4, we will compare the two spectral sequences arising from the double complex $\mathcal{C}^\bullet_{\mathfrak{n}} \mathcal{C}^\bullet_{\mathfrak{m}} \mathcal{R}_I$.

From the identification above, it follows that the spectral sequence with first terms ${}'E^{i,j}_1 = \mathcal{C}^i_{\mathfrak{m}} H^j_{\mathfrak{n}} \mathcal{R}_I = H^j_{\mathfrak{n}} \mathcal{C}^i_{\mathfrak{m}} \mathcal{R}_I$ satisfies ${}'E^{i,j}_1 = 0$ unless $i = 0$, or $j = 1$. Also notice that $({}'E^{i,1})_{*,t} = 0$ for $t \geq 0$ and $i > 0$ by virtue of the identity $H^1_{(T)}(A_{X_i}[T]) = T^{-1} A_{X_i}[T^{-1}]$, proving claim (i).

Furthermore, its second terms ${}'E^{i,j}_2 = H^i_{\mathfrak{m}}(H^j_{\mathfrak{n}} \mathcal{R}_I)$ vanish for $i > n$, hence

$${}'E^{i,j}_2 = \begin{cases} H^j_{\mathfrak{n}}(\mathcal{R}_I) & \text{if } i = 0 \text{ and } j \neq 1, \\ H^i_{\mathfrak{m}}(H^1_{\mathfrak{n}}(\mathcal{R}_I)) & \text{if } 0 \leq i \leq n \text{ and } j = 1, \\ 0 & \text{else.} \end{cases}$$

It shows that $({}'E^{i,j}_2)_{\mu,t} = ({}'E^{i,j}_\infty)_{\mu,t} = 0$ for $i \neq 0$ if either $t \geq 0$ or $j > n$, which implies that $(H^j_{\mathfrak{n}}(\mathcal{R}_I))_{\mu,t} \simeq ({}'E^{0,j}_2)_{\mu,t} \simeq ({}'E^{0,j}_\infty)_{\mu,t}$ if either $t \geq 0$ or $j > n$.

On the other hand, $H^1_{\mathfrak{m}}(\mathcal{R}_I) = \bigoplus_{t \geq 1} (R/I^t)(td)$, $H^n_{\mathfrak{m}}(\mathcal{R}_I) = \bigoplus_{t \geq 0} H^n_{\mathfrak{m}}(R)(td)$, and $H^i_{\mathfrak{m}}(\mathcal{R}_I) = 0$ for $i \notin \{1, n\}$.

The second spectral sequence has second terms graded components:

$$({}''E^{i,j}_2)_{\mu,*} = \begin{cases} H^j_{\mathfrak{n}}(H^1_{\mathfrak{m}}(\mathcal{R}_I)_{\mu,*}) & \text{if } i = 1, \\ H^j_{\mathfrak{n}}(H^n_{\mathfrak{m}}(\mathcal{R}_I)_{\mu,*}) & \text{if } i = n, \\ 0 & \text{else,} \end{cases}$$

showing that $({}''E_2^{i,j})_{\mu,*} = 0$ for $i \neq 1$ and $\mu > -n$. This in turn shows that, if $\mu > -n$, there is a long exact sequence

$$\cdots \to H^{j-1}_{\mathfrak{m}}(H^1_{\mathfrak{n}}(\mathcal{R}_I)_{\mu,*}) \to H^{j-1}_{\mathfrak{n}}(N_{\mu,*}) \to H^j_{\mathfrak{n}}(\mathcal{R}_I) \to H^j_{\mathfrak{m}}(H^1_{\mathfrak{n}}(\mathcal{R}_I)_{\mu,*}) \to \cdots,$$

where the map $H^j_{\mathfrak{n}}(\mathcal{R}_I) \to H^j_{\mathfrak{m}}(H^1_{\mathfrak{n}}(\mathcal{R}_I)_{\mu,*})$ is ${}'d_j^{0,j}$ for $2 \leq j \leq n$ and 0 else.

This proves claim (ii), as we noticed above for (i) that $H^j_{\mathfrak{m}}(H^1_{\mathfrak{n}}(\mathcal{R}_I))_{\mu,t} = 0$ for $j \neq 0$ and $t \geq 0$.

Hence $H^{j-1}_{\mathfrak{n}}(N)_{\mu,t-j} = H^j_{\mathfrak{n}}(\mathcal{R}_I)_{\mu,t-j}$ if $j \geq 1$, $t \geq j$ and $\mu > -n$.

Property (iii) follows as $H^j_{\mathfrak{n}}(N)_{\mu,*} = H^j_{\mathfrak{n}}(N_{\mu,*}) = 0$ for $j > \mathrm{cd}_{B_+}(N_{\mu,*}) = d(\mu)$, and similarly $H^j_{\mathfrak{n}}(\mathcal{R}_I)_{\mu,*} = 0$ for $j > \mathrm{cd}_{B_+}((\mathcal{R}_I)_{\mu,*})$.

Last, notice that for $\mu > -n$, the above sequence shows that $H^{n+1}_{\mathfrak{n}}(\mathcal{R}_I)_{\mu,*} = 0$, proving (iv). ☐

Recall that, for any bigraded module M, $H^j_{\mathfrak{n}}(M)_{\mu,t-j} = 0$, $\forall j \geq 1$, implies $H^j_{\mathfrak{n}}(M)_{\mu,s-j} = 0$, $\forall j \geq 1$ and $s \geq t$. Together with property (ii) in the above proposition, it shows that if $H^j_{\mathfrak{n}}(N)_{\mu,t-j} = 0$ for $0 \leq j \leq d(\mu)$ and $t \geq \mathrm{cd}_{B_+}((\mathcal{R}_I)_{\mu,*})$, then $t > \mathrm{reg}(N_{\mu,*})$ if $\mu > -n$.

4. A simple stability result

Out of the results of sections 1 and 3, we can deduce the following result that provides a quite sharp estimate of the stabilization index in terms of the regularity of graded strands of the Rees ring for non-equigenerated ideals.

Theorem 4.1. *Let A be a standard graded polynomial ring in n variables over an infinite field, $\mathfrak{m} := A_+$ and I be an $\mathfrak{m}$-primary graded ideal.*

Set $d := \min\{\mu \mid (I_{\leq\mu})$ is $\mathfrak{m}$-primary$\}$ and $t_0 := \min\{t \geq 1 \mid \mathfrak{m}^d I^t \subseteq (I_{\leq d})\}$.

Let $J = (g_1, \ldots, g_n) \subseteq I$ be generated by n general elements in I_d.

Consider the standard bigraded ring $S := A[T_1, \ldots, T_n]$, $B := k[T_1, \ldots, T_n]$, $\mathfrak{n} := (T_1, \ldots, T_n)$ and denote by $\mathcal{R}_J = S/(\{g_iT_j - g_jT_i \mid 1 \leq i < j \leq n\})$ the Rees ring of the complete intersection ideal $J(d)$.

The inclusion $\mathcal{R}_J \subset \mathcal{R}_I := \bigoplus_t I(d)^t$ makes $\mathcal{R}_I$ a bigraded module over S.

Let b be defined by $\mathrm{reg}(I^t) = dt + b$ for $t \gg 0$. Then

$$\mathrm{Stab}(I) = \mathrm{end}(H^1_{\mathfrak{n}}(\mathcal{R}_I)_{b,*}) + 1$$

and

$$\mathrm{Stab}(I) \leq \mathrm{reg}_B((\mathcal{R}_I)_{b,*}) \leq \max\{\mathrm{Stab}(I), n\}.$$

In particular, $\mathrm{Stab}(I) = \mathrm{reg}_B((\mathcal{R}_I)_{b,*})$ *if* $\mathrm{Stab}(I) \geq n$ *or* $\mathrm{reg}_B((\mathcal{R}_I)_{b,*}) > n$.

Notice that if I has no generator of degree less than d, then J is a minimal reduction of I. Also recall that $t_0 \leq n-1$ by Proposition 1.1.

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