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REGULARITY THEOREMS FOR FRACTIONAL POWERS
OF A LINEAR ELLIPTIC OPERATOR ;

BY

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1. **Introduction.** — Let L be a linear elliptic operator with C^∞ coefficients in an open subset Ω of \mathbf{R}^n ($n \geq 2$). We suppose that L admits a (strictly) positive self-adjoint realisation \tilde{L} in $L^2(\Omega)$. Let $\{E_\lambda\}$ be the spectral resolution of \tilde{L} so that

$$\tilde{L} = \int \lambda dE_\lambda.$$

We consider the family of operators \tilde{L}^s , depending on a complex parameter s , defined by

$$\tilde{L}^s = \int \lambda^s dE_\lambda.$$

The operators \tilde{L}^s may be viewed as “fractional powers” of L . For $s = -1, -2, \dots$, we obtain the Green’s operator and its iterates.

We study in this paper the regularity properties of the operators \tilde{L}^s . For integral values of s , it is known that the operators \tilde{L}^s define kernels which are “very regular” in the sense of SCHWARTZ ([17], chap. V, § 6) and that if further the coefficients of L are analytic the kernels of \tilde{L}^s are analytically very regular. For positive integral values of s the results are trivial, for negative integral values of s these follow from well-known regularity theorems for elliptic operators [11]. The question arises whether these results are true for all values of s . We prove in this paper that this is in fact the case (Theorems 2 and 3). The case of elliptic operators with constant coefficients on a torus and on \mathbf{R}^n has already been dealt with respectively by S. BOCHNER [3] and L. SCHWARTZ ([16], chap. VII, § 10, ex. 7).

That the operators \tilde{L}^s possess kernels follows from regularity theorems for elliptic operators. In order to prove that the kernels are very regular,

we represent the kernels, for $Re(-s)$ sufficiently large, in terms of the Green's function $G(t, x, y)$ of the associated parabolic operator. By using some results of G. BERGENDAL [1] and S. D. EIDELMAN [6] and showing that $G(t, x, y)$ and its derivatives fall off exponentially as $t \rightarrow \infty$, we then prove that the kernel \tilde{L}^s is very regular.

The proof of analytic regularity, when the coefficients are analytic, is more difficult. It involves in the first instance estimates for the norms $\|A^k u\|_{L^2}$, where u is a function that is to be proved to be analytic and A a linear elliptic operator with analytic coefficients. Next we need to prove a general theorem (Theorem 1) to the effect that if A is a linear elliptic operator of order m with analytic coefficients in an open set Ω' of \mathbf{R}^n , and u is a function satisfying the inequalities

$$\|A^k u\|_{L^2(\Omega')} \leq (km)! c^{k+1}$$

for every integer $k \geq 0$, with a positive constant c independent of k , then u is analytic in Ω' .

This theorem is a natural one in as much as the conditions

$$\|A^k u\| \leq (km)! c^{k+1}$$

on every compact set are necessary for u to be analytic. We notice also that this theorem contains the well-known result: if A is linear elliptic operator and has analytic coefficients, and if $Au = f$ with f analytic, then u is analytic.

A weaker version of Theorem 1 has been proved by E. NELSON ([14], th. 7); he proves the analyticity of u under the stronger assumption

$$\|A^k u\| \leq k! c^{k+1}.$$

Theorem 1 is proved by suitably estimating the L^2 -norms of derivatives of order km of u in terms of L^2 -norms of $u, Au, \dots, A^k u$. The proof of this theorem uses some ideas of a paper of C. B. MORREY and L. NIRENBERG [13].

The use of the parabolic equation in the proofs of Theorems 2 and 3 was suggested by a paper of S. MINAKSHISUNDARAM [12].

For spaces of distributions we use the usual notation [17].

The results of this paper have been announced in [10].

2. Statement of the theorems. — Let Ω be an open subset of \mathbf{R}^n . Let $\mathcal{D}(\Omega)$ be the space of complex-valued C^∞ functions with compact support in Ω . $L^2(\Omega)$ is the Hilbert space of complex-valued square summable functions on Ω , with scalar product (φ, ψ) defined by

$$(\varphi, \psi) = \int_{\Omega} \varphi \cdot \bar{\psi} \, dx$$

for $\varphi, \psi \in L^2(\Omega)$; $\|\varphi\|_{L^2(\Omega)}$ means $(\varphi, \varphi)^{1/2}$.

Let A be a linear differential operator of order m ,

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

with sufficiently differentiable complex-valued coefficients $a_\alpha(x)$ defined in Ω , where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, α_i being integer ≥ 0 and we put :

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n, \\ D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

We say now A is an elliptic operator in Ω , if the homogeneous form of order m

$$\sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha \neq 0$$

for every $x \in \Omega$ and for every non vanishing real vector $\xi = (\xi_1, \xi_2, \dots, \xi_n)$.

THEOREM 1. — *Let Ω be an open subset of \mathbf{R}^n . Let A be a linear elliptic operator of order m with analytic coefficients in Ω . Let A^k be the k^{th} iterate of A . Suppose that a function u (of class C^∞) satisfies the inequality*

$$\|A^k u\|_{L^2(\Omega)} \leq (km)! c^{k+1}$$

for every integer $k \geq 0$ with a positive constant c independent of k . Then the function u is analytic in Ω .

REMARK. — The above theorem is also valid for elliptic systems; the demonstration is the same as for the scalar case.

As for the following theorems, we consider a linear elliptic operator L defined on Ω such that

$$(L\varphi, \psi) = (\varphi, L\psi)$$

for every $\varphi, \psi \in \mathcal{O}(\Omega)$.

Suppose further that L when defined on $\mathcal{O}(\Omega) (\subset L^2)$, where it is symmetric, has a strictly positive self-adjoint extension \tilde{L} .

Remark that these conditions entail that the form

$$L(x, \xi) = \sum_{|\alpha|=m} b_\alpha(x) \xi^\alpha$$

is real and definite for every $x \in \Omega$ and ξ real vector, when $L = \sum_{|\alpha| \leq m} b_\alpha(x) D^\alpha$

has sufficiently smooth coefficients.

Let $\{E_\lambda\}$ be the spectral resolution of \tilde{L} . By the hypothesis on \tilde{L} , we have $\lambda > c_0 > 0$ on the spectrum.

We can now define a family of operators \tilde{L}^s depending on the complex parameter s , by

$$\tilde{L}^s = \int \lambda^s dE_\lambda.$$

As we shall see in section 5, \tilde{L}^s thus defined is a continuous linear map of $\mathcal{O}(\Omega)$ into the space of distributions $\mathcal{O}'(\Omega)$ for every s , so that \tilde{L}^s defines a kernel $L^s(x, y)$ ([17], [19]); the theorems to be proved concern the regularity of the kernel $L^s(x, y)$.

THEOREM 2. — *Let L be a linear elliptic differential operator with C^∞ coefficients in an open set Ω of \mathbf{R}^n . We suppose further that L admits a strictly positive self-adjoint realisation*

$$\tilde{L} = \int \lambda dE_\lambda.$$

in $L^2(\Omega)$. Let s be a complex number. Then the operator

$$\tilde{L}^s = \int \lambda^s dE_\lambda.$$

defines a kernel which is very regular.

THEOREM 3. — *Let L be a linear elliptic differential operator with analytic coefficients in an open set Ω of \mathbf{R}^n , admitting a strictly positive self-adjoint realisation \tilde{L} in $L^2(\Omega)$. Then, for every complex number s , the kernel of the operator*

$$\tilde{L}^s = \int \lambda^s dE_\lambda$$

is analytically very regular.

For the definition of very regular kernels and analytically very regular kernels see ([17], chap. V, § 6).

As a consequence of the above theorems, $\tilde{L}^s(T)$ can be defined for T , a distribution with compact support and when L has the C^∞ (analytic) coefficients, $\tilde{L}^s(T)$ is an infinitely differentiable (resp. analytic) function in an open set of Ω where T is an infinitely differentiable (resp. analytic) function.

3. Preliminary lemmas. — We consider in this section some lemmas which are required in the proof of Theorem 1.

Let Ω' be any open subset of Ω . Let u be of class C^∞ on the closure $\bar{\Omega}'$ of Ω' . Let k be an integer ≥ 0 . We define the k -norm of $u \in C^\infty(\bar{\Omega}')$ by

$$\|u\|_{k, \Omega'} = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|D^\alpha u\|_{L^2(\Omega')},$$

where we put $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$ for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$.

LEMMA 3.1. — Let $k, k',$ be given integers ≥ 0 . Then we have

$$\|u\|_{k+k', \Omega'} = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|D^\alpha u\|_{k', \Omega'}$$

PROOF. — We have

$$\frac{(k+k')!}{\gamma!} = \sum_{\substack{|\alpha|=k \\ |\beta|=k' \\ \alpha+\beta=\gamma}} \frac{k!}{\alpha!} \frac{k'!}{\beta!}$$

The lemma follows immediately from this equality.

The next lemma is a refined version of Friedrichs' inequality [7]. The proof is a modification of Friedrichs' proof as in [13].

We denote by Ω_r the ball $|x| < r$ of radius r in \mathbf{R}^n .

LEMMA 3.2. — Let A be a linear elliptic operator of order m with C^∞ coefficients in Ω . Let r, δ be positive numbers such that $\delta < r$ and $\Omega_{r+\delta} \subset \Omega$. Then there exists a constant $c > 0$ independent of δ such that for every $u \in C^\infty(\Omega)$ we have

$$\|u\|_{m, \Omega_r} \leq c \{ \|Au\|_{0, \Omega_{r+\delta}} + \delta^{-m} \|u\|_{0, \Omega_{r+\delta}} \}.$$

PROOF. — Let $\zeta \in \mathcal{O}(\Omega)$ have its support in $\Omega_{r+\delta}$ and be such that $\zeta \equiv 1$ on Ω_r and satisfies

$$(3.1) \quad \sup_{\Omega_{r+\delta}} |D^\alpha \zeta(x)| \leq c_\alpha \delta^{-|\alpha|} \quad (\delta < r)$$

with $c_\alpha > 0$ depending only on α .

For any $u \in C^\infty(\Omega)$, we shall consider $\zeta^m u$, which is of class C^∞ having its support in $\Omega_{r+\delta}$. Since A is an elliptic operator with C^∞ coefficients, we have the well-known inequality [10]

$$(3.2) \quad \|\zeta^m u\|_{m, \Omega_{r+\delta}} \leq c \{ \|A(\zeta^m u)\|_{0, \Omega_{r+\delta}} + \|\zeta^m u\|_{0, \Omega_{r+\delta}} \}$$

with a constant $c > 0$ depending only on A and $\Omega_{r+\delta}$.

By using the estimate (3.1), we obtain

$$\|A(\zeta^m u)\|_{0, \Omega_{r+\delta}} \leq c' \left\{ \|\zeta^m A u\|_{0, \Omega_{r+\delta}} + \sum_{k=0}^{m-1} \delta^{-m+k} \|\zeta^k u\|_{k, \Omega_{r+\delta}} \right\},$$

$$\sum_{|\alpha|=m} \|\zeta^m D^\alpha u\|_{0, \Omega_{r+\delta}} \leq c'' \left\{ \|\zeta^m u\|_{m, \Omega_{r+\delta}} + \sum_{k=0}^{m-1} \delta^{-m+k} \|\zeta^k u\|_{k, \Omega_{r+\delta}} \right\}.$$

It follows then from (3.2),

$$(3.3) \quad \sum_{k=0}^m \delta^{-m+k} \|\zeta^k u\|_{k, \Omega_{r+\delta}} \leq c \left\{ \|\zeta^m A u\|_{0, \Omega_{r+\delta}} + \sum_{k=0}^{m-1} \delta^{-m+k} \|\zeta^k u\|_{k, \Omega_{r+\delta}} \right\}$$

with $c > 0$ independent of k .

To complete the proof of the lemma, we need the following fact: for every $\varepsilon, \delta > 0$, there exists a constant c independent of ε, δ and u such that

$$(3.4) \quad \sum_{|\alpha|=k} \|\zeta^k D^\alpha u\|_{0, \Omega} \leq \varepsilon \sum_{|\alpha|=k+1} \|\zeta^{k+1} D^\alpha u\|_{0, \Omega} + c(\varepsilon^{-1} + \delta^{-1}) \sum_{|\alpha|=k-1} \|\zeta^{k-1} D^\alpha u\|_{0, \Omega}$$

where $k \geq 1$.

In fact we have the equality

$$-(\zeta^k D^\alpha u, \zeta^k D^\alpha u) = (\zeta^{k-1} D^{\alpha'} u, \zeta^{k+1} D_1 D^\alpha u) + 2k((D_1 \zeta) \zeta^{k-1} D^\alpha u, \zeta^k D^\alpha u),$$

where $\alpha' = (\alpha_1 - 1, \alpha_2, \dots, \alpha_n)$ (we suppose $\alpha_1 \neq 0$) and $D_1 = \partial/\partial x_1$.

Now we can obtain the inequality (3.4) by Schwarz's inequality and by taking into account the estimate (3.1) for ζ .

In (3.4) we take $k = m - 1$ and choose ε as $\varepsilon = \delta/2c$. Bringing the inequality thus obtained in the right side of (3.3), we have

$$(3.5) \quad \sum_{k=0}^m \delta^{-m+k} \|\zeta^k u\|_{k, \Omega_{r+\delta}} \leq c \left\{ \|\zeta^m A u\|_{0, \Omega_{r+\delta}} + \sum_{k=0}^{m-2} \delta^{-m+k} \|\zeta^k u\|_{k, \Omega_{r+\delta}} \right\}$$

with $c > 0$ independent of k . Thus in the right side of (3.3), the terms corresponding to $k = m - 1$ can be absorbed in the left side. Repeating this procedure by using (3.4) with appropriate ε , we arrive finally at the desired inequality stated in the lemma.

LEMMA 3.3. — *Let q be positive integer such that $q < m$. Let $r < r_0$, r_0 being fixed. Then there exists a constant $c_m > 0$ depending only on m and r_0 such that for every $\varepsilon > 0$ and $u \in C^\infty(\Omega)$ one has*

$$\|u\|_q, \Omega_r \leq \varepsilon \|u\|_m, \Omega_r + c_m \varepsilon^{-q/(m-q)} \|u\|_0, \Omega_r.$$

A proof of this lemma can be given by using Fourier transforms after extending the functions suitably to \mathbf{R}^n . Another proof can be found in [15] (Appendix).

REMARK. — Let p be any integer ≥ 0 . By applying the above inequality to $D^\alpha u$ and by summing up the inequality thus obtained with respect to α such that $|\alpha| = mp$, we obtain from Lemma 3. 1,

$$\|u\|_{\rho m+q, \Omega_r} \leq \varepsilon \|u\|_{(\rho+1)m, \Omega_r} + c_m \varepsilon^{-q/(m-q)} \|u\|_{\rho m, \Omega_r}$$

with the same constant c_m as in the above lemma.

4. **Proof of theorem 1.** — In this section we shall prove Theorem 1. The proof is preceded by several lemmas which permit one to estimate suitably $\|u\|_{km}$ in terms of zero-norms of $u, Au, \dots, A^k u$.

We suppose throughout this section that A has analytic coefficients. In this section, $c(c_1, c_2, \dots, \text{etc.})$ will denote a positive constant, always independent of k , which may vary from place to place.

The first lemma gives an estimate for the commutator of the operator D^α and the operator of multiplication by an analytic function.

LEMMA 4.1. — Let a be an analytic function in $\bar{\Omega}'$. We define the commutator $[a, D^\alpha]$ by $[a, D^\alpha]u = a \cdot D^\alpha u - D^\alpha(a u)$, then we have for every integer $k > 0$.

$$(4.1) \quad \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|[a, D^\alpha]u\|_{0, \Omega'} \leq k! c^k \sum_{p=0}^{k-1} (p!)^{-1} c^{-p} \|u\|_{p, \Omega'}$$

with $c > 0$ independent of k .

PROOF. — Since a is analytic in $\bar{\Omega}'$, we have

$$(4.2) \quad \sup_{\bar{\Omega}'} |D^\alpha a| \leq \alpha! c^{|\alpha|+1}.$$

The Leibniz formula gives

$$D^\alpha(a u) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} (D^\beta a) (D^{\alpha - \beta} u)$$

where $\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$ and $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$ for each $i (i = 1, 2, \dots, n)$.

From (4.2) and the definition of $[a, D^\alpha]$, it follows immediately

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} \|[a, D^\alpha]u\|_{0, \Omega'} \leq \sum_{|\alpha|=k} \sum_{\substack{\gamma \leq \alpha \\ \gamma \neq \alpha}} \frac{k!}{\gamma!} c^{k-|\gamma|} \|D^\gamma u\|_{0, \Omega'}.$$

Now the number of α 's such that $\alpha > \gamma$ for fixed γ is at most of order $n^{k-|\gamma|}$, so that the right side is majorised by

$$\sum_{p=0}^k \frac{k!}{p!} (nc)^{k-p} \sum_{|\gamma|=p} \frac{p!}{\gamma!} \|D^\gamma u\|_{0, \Omega'};$$

this proves the lemma.

LEMMA 4.2. — Let r, δ be as in the lemma 3.2. Let ε be any positive number. Then there exist constants c, c_1 (c depending only on A and c_1 depending on A and ε) such that one has for every k and $u \in C^\infty(\Omega)$,

$$(4.3) \quad \|u\|_{(k+1)m, \Omega_r} \leq c \left\{ \|Au\|_{km, \Omega_{r+\delta}} + \delta^{-m} \|u\|_{km, \Omega_{r+\delta}} + \varepsilon \|u\|_{(k+1)m, \Omega_{r+\delta}} \right. \\ \left. + ((k+1)m)! c_1^{k+1} \sum_{p=0}^k ((pm)!)^{-1} c_1^p \|u\|_{pm, \Omega_{r+\delta}} \right\}.$$

PROOF. — From Lemma 3.1 and the Friedrichs' inequality (Lemma 3.2), we have

$$(4.4) \quad \|u\|_{(k+1)m, \Omega_r} = \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \|D^\alpha u\|_{m, \Omega_r} \\ \leq c \left\{ \|Au\|_{km, \Omega_{r+\delta}} + \delta^{-m} \|u\|_{km, \Omega_{r+\delta}} \right. \\ \left. + \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \|[A, D^\alpha]u\|_{0, \Omega_{r+\delta}} \right\}.$$

Now, witting A explicitly as $A = \sum_{|\beta| \leq m} a_\beta D^\beta$ with analytic coefficients a_β

and applying the Lemma 4.1 for $[A, D^\alpha]u = \sum_{|\beta| \leq m} [a, D^\alpha]D^\beta u$, we obtain

$$(4.5) \quad \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \|[A, D^\alpha]u\|_{0, \Omega_{r+\delta}} \leq \sum_{p=0}^{km-1} \sum_{q=0}^m \frac{(km)!}{p!} c_1^{km-p} \|u\|_{p+q, \Omega_{r+\delta}}.$$

Since we may suppose $c_1 > 1$ in (4.5), it follows immediately that there exists a constant $c_2 > 0$ independent of k such that

$$(4.6) \quad \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \|[A, D^\alpha]u\|_{0, \Omega_{r+\delta}} \leq \sum_{s=0}^{(k+1)m-1} \frac{((k+1)m)!}{s!} c_2^{(k+1)m-s} \|u\|_{s, \Omega_{r+\delta}}.$$

We wish now to majorize the right side of (4.6), containing terms $\|u\|_s$, for $s = 0, 1, \dots, (k+1)m - 1$, by an expression which contains only $\|u\|_{pm}$, for $p = 0, 1, \dots, (k+1)$.

For this purpose, we write s as $s = pm + q$ with $0 \leq p \leq k$, and $0 \leq q < m$. Then the remark of Lemma 3.3 gives

$$(4.7) \quad \|u\|_{pm+q, \Omega_{r+\delta}} \leq \varepsilon' \|u\|_{(p+1)m, \Omega_{r+\delta}} + c_m \varepsilon'^{-q/(m-q)} \|u\|_{pm, \Omega_{r+\delta}}$$

with c_m independent of ε' and δ .

In (4.7), we choose ε' as

$$\varepsilon' = \varepsilon \frac{(pm + q)!}{((p + 1)m)!} c_2^{-(m-q)}$$

where $\varepsilon (0 < \varepsilon < 1)$ is given.

Then we have

$$\varepsilon'^{-q/(m-q)} \leq \left(\frac{m}{\varepsilon}\right)^m \frac{(pm + q)!}{(pm)!} c_2^q$$

so that we obtain for $s = pm + q$,

$$(4.8) \quad \frac{c^{-s}}{s!} \|u\|_{s, \Omega_{r+\delta}} \leq \varepsilon \frac{c_2^{-(p+1)m}}{((p + 1)m)!} \|u\|_{(p+1)m, \Omega_{r+\delta}} + c_m \left(\frac{m}{\varepsilon}\right)^m \frac{c_2^{-pm}}{(pm)!} \|u\|_{pm, \Omega_{r+\delta}}.$$

Bringing this in the expression (4.6), we have

$$(4.9) \quad \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \|[A, D^\alpha]u\|_{0, \Omega_{r+\delta}} \leq m\varepsilon \|u\|_{(k+1)m, \Omega_{r+\delta}} + c'(\varepsilon) \sum_{\rho=0}^k \frac{((k + 1)m)!}{(pm)!} c_2^{(k+1)m - \rho m} \|u\|_{\rho m, \Omega_{r+\delta}}$$

where we put $c'(\varepsilon) = 1 + m\varepsilon + \left(\frac{m}{\varepsilon}\right)^m c_m$. We take now in (4.9) the constant c_2 large enough to absorb the constant $c'(\varepsilon)$ which is independent of k . Then, from (4.4), the desired inequality follows.

DEFINITION (see [13]). — Let λ be a positive number. For each integer $k \geq 0$, we define

$$\sigma^k(u, \lambda, R) = ((km)!)^{-1} \lambda^{-k} (R - r)^{km} \sup_{R/2 \leq r < R} \|u\|_{km, \Omega_r}.$$

LEMMA 4.3. — Let $R < 1$. There exists a constant λ depending only on A and R such that for every k and $u \in C^\infty(\Omega)$ we have

$$(4.10) \quad \sigma^{k+1}(u, \lambda, R) \leq [(km + 1) \dots ((k + 1)m)] \sigma^k(Au, \lambda, R) + \sum_{\rho=0}^k \sigma^\rho(u, \lambda, R).$$

PROOF. — Multiplying by $[((k + 1)m)!]^{-1} \lambda^{-(k+1)} (R - r)^{(k+1)m}$ on both sides of the inequality of Lemma 4.2 and taking the supremum for $R/2 \leq r < R$, we obtain

$$(4.11) \quad \sigma^{k+1}(u, \lambda, R) \leq \sup_{R/2 \leq r < R} (I_1 + \varepsilon I_2 + I_3 + I_4),$$

where

$$(4.12) \quad \begin{cases} I_1 = c [((k+1)m)!]^{-1} \lambda^{-(k+1)} (R-r)^{(k+1)m} \|Au\|_{km, \Omega_{r+\delta}}, \\ I_2 = c [((k+1)m)!]^{-1} \lambda^{-(k+1)} (R-r)^{(k+1)m} \|u\|_{(k+1)m, \Omega_{r+\delta}}, \\ I_3 = c [((k+1)m)!]^{-1} \lambda^{-(k+1)} (R-r)^{(k+1)m} \delta^{-m} \|u\|_{km, \Omega_{r+\delta}}, \\ I_4 = c \lambda^{-(k+1)} (R-r)^{(k+1)m} \sum_{p=0}^k \frac{c_1^{k+1-p}}{(pm)!} \|u\|_{pm, \Omega_{r+\delta}}. \end{cases}$$

We choose in what follows $\delta = \frac{R-r}{k+1}$; then we have

$$\left(\frac{R-r}{R-r-\delta}\right)^{km} = \left(1 - \frac{1}{k+1}\right)^{-km} < c_2$$

with c_2 independent of k . It follows now from the definition of $\sigma^k(u, \lambda, R)$,

$$(4.13) \quad I_1 \leq [(km+1) \dots ((k+1)m)]^{-1} \left(\frac{cc_2}{\lambda}\right) \sigma^k(Au, \lambda, R).$$

Similarly

$$(4.14) \quad I_2 \leq (cc_2) \sigma^{k+1}(u, \lambda, R).$$

For I_3 , we have

$$(4.15) \quad I_3 \leq \frac{c}{\lambda} \left(\frac{R-r}{R-r-\delta}\right)^{km} \left(\frac{R-r}{\delta}\right)^m \frac{(km)!}{((k+1)m)!} \sigma^k(u, \lambda, R).$$

Since we have from the definition of δ .

$$\left(\frac{R-r}{\delta}\right)^m = (k+1)^m$$

it follows from (4.15)

$$(4.16) \quad I_3 \leq \left(\frac{cc_2}{\lambda}\right) \sigma^k(u, \lambda, R).$$

Finally we obtain for I_4 ,

$$(4.17) \quad I_4 \leq \left(\frac{cc_1c_2}{\lambda}\right) \sum_{p=0}^k \left(\frac{c_1}{\lambda}\right)^{k-p} \sigma^p(u, \lambda, R) \quad (\lambda \geq 1).$$

It follows now for every $k \geq 0$,

$$(4.18) \quad (1 - \varepsilon c) \sigma^{k+1}(u, \lambda, R) \leq [(km+1) \dots ((k+1)m)]^{-1} \left(\frac{c_1}{\lambda}\right) \sigma^k(Au, \lambda, R) \\ + \left(\frac{c_1}{\lambda}\right) \sum_{p=0}^k \left(\frac{c_1}{\lambda}\right)^{k-p} \sigma^p(u, \lambda, R)$$

for sufficiently large constants $c, c_1 > 0$, c being independent of ε , while c_1 depends on ε . After we have chosen $\varepsilon = 1/2c$ in (4.18) c_1 is a constant dependent only on A and R so that it is possible to find λ independent of k such that $\lambda > 2c_1$; thus we obtain the inequality (4.10).

LEMMA 4.4. — *Let λ be the same constant as in lemma 4.3; we have then*

$$(4.19) \quad \sigma^{k+1}(u, \lambda, R) \leq \sum_{p=0}^{k+1} 2^{k-p+1} \binom{k+1}{p} ((mp)!)^{-1} \sigma^0(A^p u, \lambda, R).$$

PROOF. — The proof is by induction on k . For $k = 0$, the Lemma is valid (see Lemma 4.3). Suppose that the lemma is valid upto $k - 1$. Applying the induction hypothesis to the function Au , we have

$$(4.20) \quad \sigma^k(Au, \lambda, R) \leq \sum_{p=0}^k 2^{k-p} \binom{k}{p} ((pm)!)^{-1} \sigma^0(A^{p+1}u, \lambda, R).$$

Also, we have for $q \leq k$,

$$(4.21) \quad \sigma^q(u, \lambda, R) \leq \sum_{p=0}^q 2^{q-p} \binom{q}{p} ((pm)!)^{-1} \sigma^0(A^p u, \lambda, R).$$

From Lemma 4.3, we get

$$(4.22) \quad \begin{aligned} \sigma^{k+1}(u, \lambda, R) &\leq [(km + 1) \dots ((k + 1)m)]^{-1} \\ &\quad \times \sum_{p=0}^k 2^{k-p} \binom{k}{p} ((pm)!)^{-1} \sigma^0(A^{p+1}u, \lambda, R) \\ &\quad + \sum_{q=0}^k \sum_{p=0}^q 2^{q-p} \binom{q}{p} ((pm)!)^{-1} \sigma^0(A^p u, \lambda, R). \end{aligned}$$

Now, let c_p be the coefficient of $\sigma^0(A^p u, \lambda, R)$. Then for $0 \leq p \leq k$

$$\begin{aligned} c_p &= [(km + 1) \dots ((k + 1)m)]^{-1} 2^{k-p+1} \binom{k}{p-1} [((p-1)m)!]^{-1} \\ &\quad + \sum_{q=p}^k 2^{q-p} \binom{q}{p} [(mp)!]^{-1}. \end{aligned}$$

Since

$$\sum_{q=p}^k 2^{q-p} \binom{q}{p} \leq 2^{k-p+1} \binom{k}{p}$$

we get

$$c_p \leq 2^{k-p+1} \binom{k+1}{p} ((pm)!)^{-1}.$$

On the other hand, for $p = k + 1$, we have evidently,

$$c_{k+1} = [((k + 1)m)!]^{-1}.$$

Hence, it follows

$$(4.23) \quad \sigma^{k+1}(u, \lambda, R) \leq \sum_{p=0}^{k+1} 2^{k-p+1} \binom{k+1}{p} [((pm)!)^{-1} \sigma^0(A^p u, \lambda, R)];$$

this is the inequality which we wanted to prove; thus the induction is completed.

PROOF OF THEOREM 1. — Let $u \in C^\infty(\Omega)$ such that

$$(4.24) \quad \|A^k u\|_{L^2(\Omega')} \leq (km)! c^{k+1}$$

for Ω' an open set of Ω and for all $k \geq 0$ with a constant c independent of k .

Since the analyticity is a local property, we may suppose that the origin of \mathbf{R}^n belongs to Ω' and it is sufficient to prove analyticity at the origin. Take $R < 1$ with $\Omega_R \subset \Omega'$, then

$$(4.25) \quad \sigma^0(A^k u, \lambda, R) = \|A^k u\|_{L^2(\Omega_R)} \leq km! c^{k+1}.$$

Now from Lemma 4.4, we have

$$(4.26) \quad \begin{aligned} \sigma^{k+1}(u, \lambda, R) &\leq \sum_{p=0}^{k+1} 2^{k-p+1} \binom{k+1}{p} [((pm)!)^{-1} \sigma^0(A^p u, \lambda, R)] \\ &\leq \sum_{p=0}^{k+1} 2^{k-p+1} c^{p+1} \binom{k+1}{p} = c(c+2)^{k+1}. \end{aligned}$$

From the definition of $\sigma^{k+1}(u, \lambda, R)$ we obtain

$$\|u\|_{((k+1)m, \Omega_{R/2}} \leq ((k+1)m)! \cdot c^{k+1}$$

with a certain constant c independent of k .

Then, Lemma 3.3 permits us to estimate $\|u\|_p$ for $p = 0, 1, \dots$ by $\|u\|_{((k+1)m}$ for $k = 0, 1, \dots$ and we have

$$(4.27) \quad \|u\|_{p, \Omega_{R/2}} \leq p! c^{p+1}$$

for all $p (= 0, 1, \dots)$, where c is a constant depending only on A and Ω_R . Now, by Sobolev's lemma [13], we see that u is analytic at the origin. Hence, the proof of Theorem 1 is completed.

5. **Regularity of the kernel of \tilde{L}^s .** — We denote by $D(\tilde{L}^s)$ the domain of \tilde{L}^s , that is the set of elements $f \in L^2(\Omega)$ such that $\int |\lambda^s|^2 d\|E_\lambda f\| < \infty$.

Then, under our hypothesis on \tilde{L} , it is easy to see that

$$(5.1) \quad D(\tilde{L}^s) \subseteq D(\tilde{L}^{s'}) \quad \text{if } Rls \geq Rls',$$

(5.2) for every complex number s ,

$$\tilde{L}^s f \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k) \quad \text{if } f \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k).$$

Let $f \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k)$. It follows from (5.1), (5.2) that $f \in D(\tilde{L}^{s+k})$

and $\tilde{L}^s f \in D(\tilde{L}^k)$ for every complex number s and integer $k \geq 0$. We have then

$$(5.3) \quad \tilde{L}^k \tilde{L}^s f = \tilde{L}^s \tilde{L}^k f = \tilde{L}^{s+k} f$$

(for these properties, see [16], § 228; [18], p. 222).

PROPOSITION 5.1. — *For any complex number s , \tilde{L}^s defines a kernel $L^s(x, y)$, that is, a distribution in the product space $\Omega \times \Omega$.*

PROOF. — We first consider the case $Rls < 0$. In this case, \tilde{L}^s is a continuous map of $L^2(\Omega)$ into itself. For, by hypothesis on $\tilde{L} = \int \lambda dE_\lambda$, we have a positive constant c_0 such that $\lambda > c_0$ on the spectrum, hence $\lambda^{Rls} \leq c_0^{Rls}$ for $\lambda \leq 1$ and $\lambda^{Rls} \leq 1$ for $\lambda > 1$, since $Rls < 0$.

Thus, λ^s is bounded on the spectrum of \tilde{L} . Hence \tilde{L}^s is a continuous linear map of $L^2(\Omega)$ into itself. *A fortiori*, \tilde{L}^s is a continuous linear map of $\mathcal{O}(\Omega)$ into $\mathcal{O}'(\Omega)$. By the kernel theorem of L. SCHWARTZ [19], \tilde{L}^s defines a kernel.

For general s , we take a positive integer m such that $Rl(s - m) < 0$. Then, as seen above, \tilde{L}^{s-m} is a continuous map of $\mathcal{O}(\Omega)$ into $\mathcal{O}'(\Omega)$ while \tilde{L}^m , m^{th} iterate of L with C^∞ coefficients, is evidently a continuous map of $\mathcal{O}(\Omega)$ into itself.

Now, the proposition follows from (5.3), by remarking that

$$\tilde{L}^s \varphi = \tilde{L}^{s-m} \tilde{L}^m \varphi \quad \text{for } \varphi \in \mathcal{O}(\Omega) \quad \text{since } \mathcal{O}(\Omega) \subseteq \bigcap_{k=0}^{\infty} D(\tilde{L}^k).$$

From now on, we denote by $L^s(x, y)$ the kernel of \tilde{L}^s .

PROPOSITION 5.2. — *For every complex number s , the kernel $L^s(x, y)$ is regular.*

PROOF. — We have to prove that \tilde{L}^s maps continuously $\mathcal{O}(\Omega)$ into $\mathcal{E}(\Omega)$ and can be extended to a continuous linear map of $\mathcal{E}'(\Omega)$ into $\mathcal{O}'(\Omega)$.

Suppose that for every s , \tilde{L}^s maps continuously $\mathcal{O}(\Omega)$ into $\mathcal{E}(\Omega)$. Let φ, ψ be in $\mathcal{O}(\Omega)$. We have then $(\tilde{L}^s \varphi, \psi) = (\varphi, \tilde{L}^{\bar{s}} \psi)$, \bar{s} denoting the conjugate complex of s , this implies that \tilde{L}^s can be identified on the dense subspace $\mathcal{O}(\Omega)$ of $\mathcal{E}'(\Omega)$ with the transpose of $\tilde{L}^{\bar{s}}$, while the transpose of $\tilde{L}^{\bar{s}}$ is a continuous map of $\mathcal{E}'(\Omega)$ into $\mathcal{O}'(\Omega)$ when $\tilde{L}^{\bar{s}}$ is a continuous map of $\mathcal{O}(\Omega)$ into $\mathcal{E}(\Omega)$. Hence, \tilde{L}^s can be extended to a continuous map of $\mathcal{E}'(\Omega)$ into $\mathcal{O}'(\Omega)$.

It remains now to prove that \tilde{L}^s maps continuously $\mathcal{O}(\Omega)$ into $\mathcal{E}(\Omega)$.

Remark first that the image of $\mathcal{O}(\Omega)$ by \tilde{L}^s is contained in $\mathcal{E}(\Omega)$. For, if $\varphi \in \mathcal{O}(\Omega)$, then $\varphi \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k)$, so that by (5.2) we have $\tilde{L}^s \varphi \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k)$. From the regularity theorem for a linear elliptic operator with C^∞ coefficients ([7], [15]), it follows that $\tilde{L}^s \varphi$ is of class C^∞ .

As for the continuity of the mapping \tilde{L}^s , it is sufficient [17] to verify that the image of every bounded set in $\mathcal{O}(\Omega)$ by \tilde{L}^s is also a bounded set in $\mathcal{E}(\Omega)$.

Let s be such that $Re\,s < 0$. Let B be a bounded set in $\mathcal{O}(\Omega)$. Then, by definition [17], the image $\tilde{L}^k(B)$ of B by \tilde{L}^k is bounded in $\mathcal{O}(\Omega)$, *a fortiori*, bounded in $L^2(\Omega)$. Now \tilde{L}^s is a continuous map of $L^2(\Omega)$ into itself, so that $\tilde{L}^s \tilde{L}^k(B)$ is bounded in $L^2(\Omega)$. On the other hand, $\tilde{L}^s(B)$ is a family of C^∞ functions belonging to the domain of \tilde{L}^k ; hence it follows from (5.3) that $\tilde{L}^k \tilde{L}^s(B)$ is bounded in $L^2(\Omega)$, from this, we see, according to Lemma 3.2 and Sobolev's lemma [13], that $\tilde{L}^s(B)$ is a family of C^∞ functions whose derivatives of orders $mk - \left[\frac{n}{2} \right] - 1$ are uniformly bounded on every compact of Ω . Since k is arbitrary, this proves that $\tilde{L}^s(B)$ is bounded in $\mathcal{E}(\Omega)$.

For general s , as in the proof of Proposition 5.1, choose m so large that $Re(s - m) < 0$ and remark that $\tilde{L}^s \varphi = \tilde{L}^{s-m} \tilde{L}^m \varphi$ for $\varphi \in \mathcal{O}(\Omega)$, then \tilde{L}^m and \tilde{L}^{s-m} map respectively $\mathcal{O}(\Omega)$ into $\mathcal{O}(\Omega)$ and $\mathcal{E}(\Omega)$ continuously. This completes the proof.

6. Estimates for the Green's function of the associated parabolic operator. — Consider the family of operators $G_t = \int e^{-\lambda t} dE_\lambda$ for $t > 0$.

G_t is a bounded and Hermitian operator in $L^2(\Omega)$. Associated with these operators we have a C^∞ function in $\mathbf{R} \times \Omega \times \Omega$,

$$G(t, x, y) = \int e^{-\lambda t} de(\lambda, x, y),$$

where $e(\lambda, x, y)$ denotes the spectral function of \tilde{L} [8].

We have then

$$\left(\frac{\partial}{\partial t} + L_x\right)G(t, x, y) = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial t} + L_y\right)\overline{G(t, x, y)} = 0$$

for $t > 0$.

The next lemma shows that the function $G(t, x, y)$ and its derivatives fall off exponentially as $t \rightarrow \infty$.

LEMMA 6.1. — *Let H be a compact in $\Omega \times \Omega$. Under our assumption that \tilde{L} is strictly positive operator ($\lambda > c_0 > 0$ on the spectrum), we have*

$$\left| \left(\frac{\partial}{\partial t}\right)^p D_x^\alpha D_y^\beta G(t, x, y) \right| \leq c e^{-c_0 t/2}$$

for $t > 1$ and uniformly for $(x, y) \in H$, where c depends on p, α, β and H .

PROOF. — Denote by \bar{L} the elliptic operator with conjugate complex coefficients of L .

Consider the operator :

$$L_x + \bar{L}_y = L\left(x, \frac{\partial}{\partial x}\right) + \bar{L}\left(y, \frac{\partial}{\partial y}\right)$$

which is evidently elliptic with C^∞ coefficients in the product space $\Omega \times \Omega$.

Now, by Lemma 3.2 and Sobolev's lemma [13] applied to $(L_x + \bar{L}_y)$, it is easy to see that the desired estimate is a simple consequence of the following : let U be a relatively compact open subset in Ω such that $H \subset U \times U$. Then for every positive integers k', k'' , we have

$$\left| (L_x + \bar{L}_y)^{k'} \left(\frac{\partial}{\partial t}\right)^{k''} G(t, x, y) \right| \leq c e^{-c_0 t/2}$$

for $t > 1$ and for $(x, y) \in U \times U$. Since

$$L_x G(t, x, y) = \bar{L}_y G(t, x, y) = -\frac{\partial}{\partial t} G(t, x, y) \quad \text{for } t > 0,$$

it is sufficient to estimate $\left(\frac{\partial}{\partial t}\right)^k G(t, x, y)$ for every positive integer k .

Let m be a sufficiently large positive integer such that \tilde{L}^{-m} has a kernel $K(x, y)$ of the Carleman type ([4], [5], [8]). For $x \in \Omega$, let $K_x \in L^2$ denote the function $K(x, \cdot)$.

Now

$$\begin{aligned} \left| \left(\frac{\partial}{\partial t} \right)^k G(t, x, y) \right| &= \left| \left(\frac{\partial}{\partial t} \right)^k \int e^{-\lambda t} de(\lambda, x, y) \right| \\ &= \left| \int e^{-\lambda t} (-\lambda)^k de(\lambda, x, y) \right| \\ &= \left| \int e^{-\lambda t} (-\lambda)^k \lambda^{2m} de(E_\lambda K_x, K_y) \right| \\ &\leq e^{-c_0 t/2} \int e^{-\lambda/2} \lambda^{2m+k} |d(E_\lambda K_x, K_y)| \end{aligned}$$

since $\lambda > c_0$ and $t \geq 1$. Now the variation of $(E_\lambda K_x, K_y)$ in \mathbf{R} is majorised by $\|K_x\|_{L^2} \|K_y\|_{L^2}$ ([16], § 126) and $\|K_x\|_{L^2} \|K_y\|_{L^2} \leq c(U)$ for $(x, y) \in U \times U$, where $c(U)$ is a constant depending only on U and \tilde{L} .

It follows that

$$\left| \left(\frac{\partial}{\partial t} \right)^k G(t, x, y) \right| \leq c e^{-c_0 t/2}$$

for $t > 1$ and $(x, y) \in H$ with a constant c depending on k, H and \tilde{L} . Thus Lemma 5.1 is proved.

We next consider the behaviour of $G(t, x, y)$ and its derivatives as $t \rightarrow 0$. The required information is given by the results of G. BERGENDAL [1] and S. D. EIDELMAN [6].

Let K be a relatively compact open subset of Ω . Consider now the parabolic operator $\left(\frac{\partial}{\partial t} + L \right)$ on $\mathbf{R} \times K$ associated with L . According to S. D. EIDELMAN, we have a fundamental solution $E(t, x, y)$ of $\left(\frac{\partial}{\partial t} + L_x \right)$. It is of class C^∞ in (t, x, y) when $t > 0$ and satisfies near $t = 0$ the following estimate.

LEMMA 6.2 (S. D. EIDELMAN). — For $0 < t < 1$ and $(x, y) \in K \times K$, we have

$$\left| \left(\frac{\partial}{\partial t} \right)^p D_x^\alpha D_y^\beta E(t, x, y) \right| \leq c t^{-(pm + |\alpha| + |\beta| + n)/m} e^{-c_1 |x-y|^{1+\mu} t^{-\mu}},$$

where $\mu = 1/(m - 1)$ and c_1 depends only on L, K , while c depends also on p, α, β .

As for the behaviour of $G(t, x, y)$ we have

LEMMA (6.3) (G. BERGENDAL). — Let H be a compact subset of $\Omega \times \Omega$ such that $H \subset K \times K$. Let $E(t, x, y)$ be the same as in lemma 6.2. Then there exist positive constants c, c_1 such that

$$\left| \left(\frac{\partial}{\partial t} \right)^p D_x^\alpha D_y^\beta [G(t, x, y) - E(t, x, y)] \right| \leq c e^{-c_1 t^{-\mu}}$$

for $0 < t < 1$ and for $(x, y) \in H$, where c_1 depends only on L and H , while c depends also on p, α, β .

For $p + |\alpha| + |\beta| = 0$, this is proved in [1]. The general case can be proved in a similar fashion (see [2], § 2.3).

7. A representation for the kernel $L^s(x, y)$ in terms of the Green's function $G(t, x, y)$.

PROPOSITION 7.1. — *Let s be a complex number such that $Re\ s < -n/m$. Then we have*

$$(7.1) \quad L^s(x, y) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} G(t, x, y) dt.$$

The integral on the right converges uniformly on every compact subset of $\Omega \times \Omega$ and represents a continuous function of (x, y) in $\Omega \times \Omega$, where we denote by $\Gamma(-s)$ the Gamma function.

PROOF. — From Lemma 6.1 we have for $t \geq 1$ and for $(x, y) \in H$,

$$(7.2) \quad |G(t, x, y)| \leq c e^{-c_1 t^{1/2}}$$

while for $0 < t < 1$ and for $(x, y) \in H$, it follows from Lemma 6.2 and Lemma 6.3,

$$(7.3) \quad |G(t, x, y)| \leq |E(t, x, y)| + |(E - G)(t, x, y)| \leq c t^{-n/m} + c e^{-c_1 t^{\alpha}}$$

with positive constants c, c_1 depending on H .

From these estimates, it is easy to see that the integral converges uniformly for $(x, y) \in H$ when $Re\ s < -n/m$ and represents a continuous function of (x, y) since $G(t, x, y)$ is of class C^∞ for $t > 0$.

We shall prove now the equality stated in proposition 7.1. For $\varphi, \psi \in \mathcal{O}(\Omega)$, consider

$$P = \frac{1}{\Gamma(-s)} \left\langle \int_0^\infty t^{-s-1} G(t, x, y) dt, \varphi(x) \bar{\psi}(y) \right\rangle,$$

where $\langle \ , \ \rangle$ denote the scalar product between $\mathcal{O}'(\Omega \times \Omega)$ and $\mathcal{O}(\Omega \times \Omega)$. By what has been seen,

$$\begin{aligned} P &= \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} dt \int_{\Omega \times \Omega} G(t, x, y) \varphi(x) \bar{\psi}(y) dx dy \\ &= \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} dt \int_{c_0}^\infty e^{-t} d(E_t \varphi, \psi), \end{aligned}$$

where the integration $\int e^{-\lambda t} d(E_\lambda \varphi, \psi)$ is taken in the sense of the Radon-Stieltjes integral with respect to the complex-valued function of bounded variation $(E_\lambda \varphi, \psi)$ in $-\infty < \lambda < \infty$.

Let

$$(E_\lambda \varphi, \psi) = [\rho_1(\lambda) - \rho_2(\lambda)] + i[\rho_3(\lambda) - \rho_4(\lambda)]$$

be the canonical resolution of $(E_\lambda \varphi, \psi)$ with the real valued monotone increasing functions of bounded variation $\rho_k(\lambda)$, $k = 1, 2, 3, 4$ ([20], p. 202).

Then we have

$$\int_{c_0}^\infty e^{-\lambda t} d(E_\lambda \varphi, \psi) = \sum_{k=1}^4 \varepsilon_k \int_{c_0}^\infty e^{-\lambda t} d\rho_k(\lambda)$$

where $\varepsilon_1 = -\varepsilon_2 = -i\varepsilon_3 = i\varepsilon_4 = 1$.

Consider now

$$\int_c^\infty t^{-s-1} dt \int_{c_0}^\infty e^{-\lambda t} d\rho_k(\lambda).$$

Since $t^{-s-1} e^{-\lambda t}$ is a continuous function of (t, λ) in the integration domain: $0 < t < \infty$, $c_0 < \lambda < \infty$ and the obvious estimate $|t^{-s-1} e^{-\lambda t}| \leq t^{-Rl s-1} e^{-c_0 t}$ implies that it is integrable there with respect to the product measure $dt d\rho_k(\lambda)$ when $Rl s < 0$.

By *Fubini's theorem*, we have,

$$\int_0^\infty t^{-s-1} dt \int_{c_0}^\infty e^{-\lambda t} d\rho_k(\lambda) = \int_{c_0}^\infty d\rho_k \int_0^\infty t^{-s-1} e^{-\lambda t} dt.$$

Noting that $\int_0^\infty t^{-s-1} e^{-\lambda t} dt = \Gamma(-s)\lambda^s$ and summing up the above integral with respect to k , we have

$$P = \sum_{k=1}^4 \varepsilon_k \int \lambda^s d\rho_k(\lambda)$$

which is equal to

$$\int \lambda^s d(E_\lambda \varphi, \psi) = (\tilde{L}^s \varphi, \psi).$$

This completes the proof.

8. Proof of theorem 2. — As in paragraph 5, we see that it is sufficient to prove Theorem 2 for $Rl s < -\frac{n}{m}$. Since we have already proved that $L^s(x, y)$ is regular, it is sufficient to prove that $L^s(x, y)$ is of class C^∞ outside the diagonal [17].

For $Re\ s < -\frac{n}{m}$, we have by Proposition 7.1,

$$(8.1) \quad L^s(x, y) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} G(t, x, y) dt.$$

If (x, y) belongs to a compact set H in the complement of the diagonal we see from Lemmas 6.1, 6.2 and 6.3 that

$$(8.2) \quad \left| \left(\frac{\partial}{\partial t} \right)^k D_x^\alpha D_y^\beta G(t, x, y) \right| \leq c e^{-c_1(t+t^{-1})} \quad (0 < t < \infty)$$

with positive constants c, c_1 , where c_1 is independent of k, α, β .

It now follows from (8.1) and (8.2) that $L^s(x, y)$ is of class C^∞ outside the diagonal, since we may differentiate under the integral sign any number of times.

9. Proof of theorem 3. — In this section $c, c_i (i = 1, 2, \dots)$ will denote positive constants independent of k . We suppose that L has analytic coefficients.

To prove Theorem 3, it is sufficient to prove the following two statements :

(i) $L^s(x, y)$ is an analytic function in the complement of the diagonal in $\Omega \times \Omega$.

(ii) For each $\varphi \in \mathcal{O}(\Omega)$, $\tilde{L}^s \varphi$ is an analytic function in every open set where φ is analytic.

PROOF OF (i). — $(L_x + \bar{L}_y)^k$ is a linear elliptic operator of order m with analytic coefficients in $\Omega \times \Omega$. Applying Theorem 1, we see that to prove (i) it is sufficient to prove the following : for each compact set H in the complement of the diagonal, there exists a constant c independent of k such that

$$(9.1) \quad \sup_{(x,y) \in H} |(L_x + \bar{L}_y)^k L^s(x, y)| \leq (mk)! c^{k+1}.$$

It is sufficient to consider the case $Re\ s < -\frac{n}{m}$.

As in paragraph 8, we start from the integral representation of $L^s(x, y)$:

$$(9.2) \quad L^s(x, y) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} G(t, x, y) dt.$$

If $(x, y) \in H$, we have the estimate (8.2) which permits us to differentiate under the integral sign, so that we have

$$(9.3) \quad (L_x + \bar{L}_y)^k L^s(x, y) = \frac{(-1)^k 2^k}{\Gamma(-s)} \int_0^\infty t^{-s-1} \left(\frac{\partial}{\partial t} \right)^k G(t, x, y) dt.$$

For we have

$$\left(\frac{\partial}{\partial t} + L_x\right)G(t, x, y) = \left(\frac{\partial}{\partial t} + \bar{L}_y\right)G(t, x, y) = 0$$

for $t > 0$. Let us first suppose that s is not a negative integer. By integration by parts in (9.3) [which is permitted by (8.2)] we obtain

$$(9.4) \quad (L_x + \bar{L}_y)^k L^s(x, y) = \frac{2^k}{\Gamma(-s)} (-s-1)(-s-2)\dots(-s-k) \int_0^\infty t^{-s-k-1} G(t, x, y) dt.$$

Now as a special case of (8.2) we have

$$|G(t, x, y)| \leq c e^{-c_1(t+t^{-2})}$$

uniformly for $(x, y) \in H$ with positive constants c, c_1 depending on H .

Remembering that $\mu = (m-1)^{-1}$, it follows from a simple calculation that

$$(9.5) \quad \sup_{(x,y) \in H} \left| \int_0^\infty t^{-s-k-1} G(t, x, y) dt \right| \leq ((m-1)k)! c^{k+1},$$

c being independent of k , which gives evidently, from (9.4),

$$\begin{aligned} \sup_{(x,y) \in H} |(L_x + \bar{L}_y)^k L^s(x, y)| \\ \leq \frac{2^k}{|\Gamma(-s)|} |(-s-1)(-s-2)\dots(-s-k)| ((m-1)k)! c^{k+1} \leq (mk)! c_1^k. \end{aligned}$$

If s is a negative integer, we see that the integral

$$\int_0^\infty t^{-s-1} \left(\frac{\partial}{\partial t}\right)^k G(t, x, y) dt \quad (x, y) \in H$$

vanishes for all large k and (9.1) is trivially valid. So (i) is proved.

PROOF OF (ii). — Let $\varphi \in \mathcal{O}(\Omega)$. We suppose φ is analytic in an open subset Ω_0 of Ω . We shall show that $\tilde{L}^s \varphi$ is analytic in Ω_0 .

Let Ω_1, Ω_2 be any relatively compact open subsets of Ω_0 such that

$$\bar{\Omega}_1 \subset \Omega_2 \subset \bar{\Omega}_2 \subset \Omega_0.$$

Let $\alpha \in \mathcal{O}(\Omega_0)$ and $\alpha \equiv 1$ on Ω_2 . One has then

$$\tilde{L}^s(\varphi) = \tilde{L}^s(\alpha\varphi) + \tilde{L}^s((1-\alpha)\varphi).$$

Now, $(1-\alpha)\varphi \in \mathcal{O}(\Omega)$ and its support does not intersect Ω_1 ; by what has been seen in (i), $L^s(x, y)$ is an analytic function of (x, y) outside the diagonal in $\Omega \times \Omega$, so that it follows immediately from the integral representation of $L^s(x, y)$ that $\tilde{L}^s((1-\alpha)\varphi)$ is analytic in Ω_1 .

It remains to show that $\tilde{L}^s(\alpha\varphi)$ is analytic in Ω_1 . It is sufficient to consider the case $Re\ s < -\frac{n}{m}$. Then we have for each integer $k \geq 0$

$$(9.6) \quad L^k \tilde{L}^s(\alpha\varphi)(x) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} dt \int_\Omega G(t, x, y) (\alpha L^k \varphi)_y dy \\ + \frac{1}{\Gamma(-s)} \sum_{p=0}^{k-1} L_x^p \int_0^\infty t^{-s-1} dt \\ \times \int_\Omega G(t, x, y) ([L, \alpha] L^{k-p-1} \varphi)_y dy,$$

where $[L, \alpha]$ is the commutator of L and α .

Consider the second term in the above expression, which we write as

$$(9.7) \quad \frac{1}{\Gamma(-s)} \sum_{p=0}^{k-1} F_p(x),$$

where

$$F_p(x) = L_x^p \int_0^\infty t^{-s-1} dt \int_\Omega G(t, x, y) ([L, \alpha] L^{k-p-1} \varphi)_y dy.$$

Now $[L, \alpha]$ is a differential operator of order $(m-1)$ whose coefficients have their supports in $(\Omega_0 - \Omega_2)$, so that if we consider x in Ω_1 we may perform the differentiation L_x^p under the integral sign as in paragraph 8 and we obtain,

$$(9.8) \quad F_p(x) = (-s-1)(-s-2)\dots(-s-p) \int_0^\infty t^{-s-p-1} dt \\ \times \int_\Omega G(t, x, y) ([L, \alpha] L^{k-p-1} \varphi)_y dy, \text{ for } s \text{ non-integral} \\ = 0 \text{ for all large } p \text{ if } s \text{ is a negative integer.}$$

Since the coefficients of $[L, \alpha]$ have their supports in $(\Omega_0 - \Omega_2)$ and φ is analytic in Ω_0 by hypothesis, we have

$$(9.9) \quad \sup_{x \in \Omega_0} |[L, \alpha] L^{k-p-1} \varphi| \leq ((k-p)m)! c^{k-p+1}$$

with c independent of k and p . Further we have (see § 8)

$$(9.10) \quad \sup_{(x,y) \in \Omega_1 \times (\Omega_0 - \Omega_2)} |G(t, x, y)| \leq c e^{-c_1(t+t^{-1})},$$

we obtain from (9.8), (9.9) and (9.10)

$$\sup_{x \in \Omega} |F_p(x)| \leq (pm)! ((k-p)m)! c^{k+1}$$

with a constant c independent of k, p .

Consequently, we have

$$(9.11) \quad \sup_{x \in \Omega_1} \left| \frac{1}{\Gamma(-s)} \sum_{p=0}^{k-1} F_p(x) \right| \leq (km)! c^{k+1}.$$

On the other hand, since φ is analytic in Ω_0 , we have

$$\sup_{x \in \Omega_1} |\alpha L^k \varphi| \leq (km)! c^{k+1}$$

and from the results of paragraph 6 [see (7.2), (7.3)], we have

$$\sup_{(x,y) \in \Omega_1 \times \Omega_0} |G(t, x, y)| \leq c t^{-n/m} e^{-c_1 t}$$

so that it follows for $Rs < -n/m$,

$$(9.12) \quad \sup_{x \in \Omega_1} \left| \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} dt \int_{\Omega_0} G(t, x, y) (\alpha L^k \varphi)_y dy \right| \leq (km)! c^{k+1}.$$

From (9.6), (9.11) and (9.12) we obtain finally

$$\sup_{x \in \Omega_1} |L^k \tilde{L}^s(\alpha\varphi)| \leq (km)! c^{k+1}.$$

with c independent of k ; now from Theorem 1 we see that $\tilde{L}^s(\alpha\varphi)$ is analytic in Ω_1 , which was an arbitrary open subset of Ω_0 . This proves (ii) and the proof of Theorem 3 is thus completed.

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