Regularized Acoustic Inversion for Towed-Array Shape Estimation

Nicole Collison and Stan Dosso
School of Earth and Ocean Sciences, Univ. of Victoria, P.O. Box 3055, Victoria, B.C., CANADA V8W 3Y2

BACKGROUND

This paper considers the inverse problem of determining the shape of a ship-towed hydrophone array using the relative travel times of direct and reflected arrivals from acoustic sources deployed by a pair of consort ships [1]. To date, this inversion has been solved as a least-squares problem (minimizing the squared data error), assuming straight-line acoustic propagation in the ocean and neglecting the inevitable errors in the source positions. This paper develops a new approach based on an iterated linearized inversion of the ray-tracing equations, which is solved using the method of regularization [2]. The 3-D positions of both sources and sensors are treated as unknowns, subject to *a priori* information. For the sources, the prior information consists of position estimates and uncertainties. For the sensors, the prior information is that the array shape is expected to be smooth; this is applied by minimizing the 3-D curvature of the array to obtain a minimum-structure solution. An example is given comparing least-squares and regularized inversion.

THEORY

The set of acoustic arrival times $\mathbf{t}$ measured in an acoustic positioning survey can be written in general vector form as

$$\mathbf{t} = \mathbf{T}(\mathbf{m}) + \mathbf{n}. \tag{1}$$

In (1), the forward mapping $\mathbf{T}$ represents the arrival times of the acoustic signals along ray paths between sources and receivers . The model $\mathbf{m}$ of unknown parameters consists of 3-D position variables x, y, z for each sensor, position variables x', y', z' for each source, and the source instant t_0 for each source. Finally, $\mathbf{n}$ represents the data errors (noise). The inverse problem of determining an estimate $\bar{\mathbf{m}}$ of $\mathbf{m}$ is functionally nonlinear; however, a local linearization can be obtained by expanding $\mathbf{T}(\bar{\mathbf{m}}) = \mathbf{T}(\mathbf{m_0} + \delta\mathbf{m})$ in a Taylor series to first order about an arbitrary starting model $\mathbf{m_0}$ to yield

$$\mathbf{t} = \mathbf{T}(\mathbf{m_0}) + \mathbf{J}(\mathbf{m_0})\,[\bar{\mathbf{m}} - \mathbf{m_0}], \tag{2}$$

where $\mathbf{J}$ represents the Jacobian matrix of partial derivatives $J_{ij} = \partial T_i / \partial m_j$ (analytic expressions for these are derived in [2]). Equation (2) can be written

$$\mathbf{J}\,\bar{\mathbf{m}} = \mathbf{t} - \mathbf{T}(\mathbf{m_0}) + \mathbf{J}\mathbf{m_0} \equiv \mathbf{d}, \tag{3}$$

where the explicit dependence on $\mathbf{m_0}$ has been suppressed. Note that $\mathbf{d}$ consists entirely of known or measured quantities, and may be considered modified data for the problem. Equation (3) represents a linear inverse problem for $\bar{\mathbf{m}}$. Since nonlinear terms have been neglected in (3), the linearized inversion may need to be repeated iteratively until the solution converges (i.e., update $\mathbf{m_0} \leftarrow \bar{\mathbf{m}}$ and repeat the inversion until $\bar{\mathbf{m}} = \mathbf{m_0}$).

Solving the linear inverse problem (3) at each iteration requires some attention. By treating both source and sensor positions as unknown, a straightforward application of least-squares yields an ill-posed solution. The method of regularization provides a unique, stable inversion by explicitly including *a priori* information regarding the solution. This is typically accomplished by minimizing an objective function Ψ which combines a term representing the data misfit and a regularizing term that imposes the *a priori* expectation that the model $\bar{\mathbf{m}}$ in some manner resembles a prior estimate $\hat{\mathbf{m}}$:

$$\Psi = |\mathbf{G}\,(\mathbf{J}\,\bar{\mathbf{m}} - \mathbf{d})|^2 + \mu\,|\mathbf{H}\,(\bar{\mathbf{m}} - \hat{\mathbf{m}})|^2. \tag{4}$$

In (4), $\mathbf{G}$ is a diagonal matrix with the reciprocals of the estimated data standard deviations on the main diagonal, $\mathbf{H}$ is an arbitrary weighting matrix for the regularization, and μ is a trade-off parameter controlling the relative importance of the two terms in the minimization. The regularized solution is obtained by minimizing Ψ with respect to $\bar{\mathbf{m}}$ to yield

$$\bar{\mathbf{m}} = \left[\mathbf{J}^T\mathbf{G}^T\mathbf{G}\mathbf{J} + \mu\mathbf{H}^T\mathbf{H}\right]^{-1}\left[\mathbf{J}^T\mathbf{G}^T\mathbf{G}\mathbf{d} + \mu\mathbf{H}^T\mathbf{H}\hat{\mathbf{m}}\right]. \tag{5}$$

The value for μ is generally chosen so that the χ^2 misfit (first term of eq. 4) achieves its expected value of N for N data, which applies the *a priori* information subject to ensuring that the data are fit to a statistically appropriate level.

The regularization matrix $\mathbf{H}$ in (4) and (5) provides flexibility in the application of *a priori* information in the inversion. For instance, if prior model parameter estimates $\hat{\mathbf{m}}$ are available, an appropriate weighting is given by

$$\mathbf{H} = \mathrm{diag}[1/\xi_1, \ldots, 1/\xi_M], \tag{6}$$

where ξ_j represents the uncertainty for jth parameter estimate $\hat{m}_j$. Alternatively, the *a priori* information can be applied to derivatives of the model parameters. For instance, if the *a priori* expectation is that the parameters are well approximated by a smooth function, then an appropriate choice is $\hat{\mathbf{m}} = \mathbf{0}$ and

$$\mathbf{H} = \begin{bmatrix} -1 & 2 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & -1 & 2 & -1 \end{bmatrix}. \tag{7}$$

For this choice of $\hat{\mathbf{m}}$ and $\mathbf{H}$, minimizing (4) effectively minimizes a discrete approximation to the second derivative of $\mathbf{m}$ (i.e., minimizes model curvature), subject to fitting the data.

The inverse problem considered here involves both types of *a priori* information described above. In particular, prior estimates are available for the source locations, and the prior expectation that the array shape is smooth can be applied by minimizing the curvature. A solution incorporating both regularizations can be written

$$\bar{\mathbf{m}} = \left[\mathbf{J}^T\mathbf{G}^T\mathbf{G}\,\mathbf{J} + \mu_1\,\mathbf{H}_1^T\mathbf{H}_1 + \mu_2\,\mathbf{H}_2^T\mathbf{H}_2\right]^{-1} \times \left[\mathbf{J}^T\,\mathbf{G}^T\mathbf{G}\,\mathbf{d} + \mu_1\,\mathbf{H}_1^T\mathbf{H}_1\,\hat{\mathbf{m}}_1 + \mu_2\,\mathbf{H}_2^T\mathbf{H}_2\,\hat{\mathbf{m}}_2\right]. \quad (8)$$

In (8), the first regularization term is taken to represent the *a priori* source-position estimates. Hence, $\hat{\mathbf{m}}_1$ consists of the prior estimates for these parameters, (with zeros for the remaining parameters), and $\mathbf{H}_1$ is of the form of (6). The second regularization term is taken to represent the *a priori* expectation of a smooth array shape. Hence, $\hat{\mathbf{m}}_2 = \mathbf{0}$, and $\mathbf{H}_2$ is of the form of (7) for the sensor position parameters.

EXAMPLE

This section illustrates the regularized approach to towed array shape estimation with a realistic synthetic example. The ocean is 1300 m deep, with a typical N. E. Pacific sound speed profile. The towed array consists of 32 sensors, each separated by 10 m, towed at a nominal depth of 300 m. The sources deployed by the two consort ships are nominally located at 200-m depth, 500-m range (from the array centre), and at angles of ±45° with respect to array broadside. The errors on the source positions are taken to be Gaussian-distributed random variables with a standard deviation of 10 m in x', y' and 5 m in z'. The data consists of the relative travel times of the direct and bottom-reflected acoustic arrivals, computed via 1D raytracing, with additive (Gaussian) errors. Cases are considered with data standard deviations of 1 ms and 0.5 ms. The (linearized) localization inversion was solved two ways: (i) using the regularized solution (8), treating source and sensor positions as unknown and solving for the smoothest array shape, as outlined in the previous section; and (ii) by applying standard least-squares inversion to minimize the misfit to the data, treating the source positions as known quantities. In each case, the linearization and inversion were applied to the 1-D raytracing equations.

The results of the inversions are shown in Fig. 1. Figure 1(a) and (b) show the array shape in the horizontal (x-y) and vertical (x-z) planes, respectively, for data errors of 1 ms. Figure 1(c) and (d) show the same results for data errors of 0.5 ms. The fairly large offsets of the inversion results are due to the errors in the source positions. Note, however, that by treating the source positions as (constrained) unknowns, the regularized solutions achieve significantly smaller offsets. In addition, the regularized inversion provides smooth models, with substantially less structure (random fluctuations) than the least-squares solution. The fluctuations in the least-squares solution result from the tendency of a minimum-misfit approach to over-fit the data, in effect fitting the noise as well as the data. The regularized inversion avoids this by trading off data misfit with physical *a priori* information.

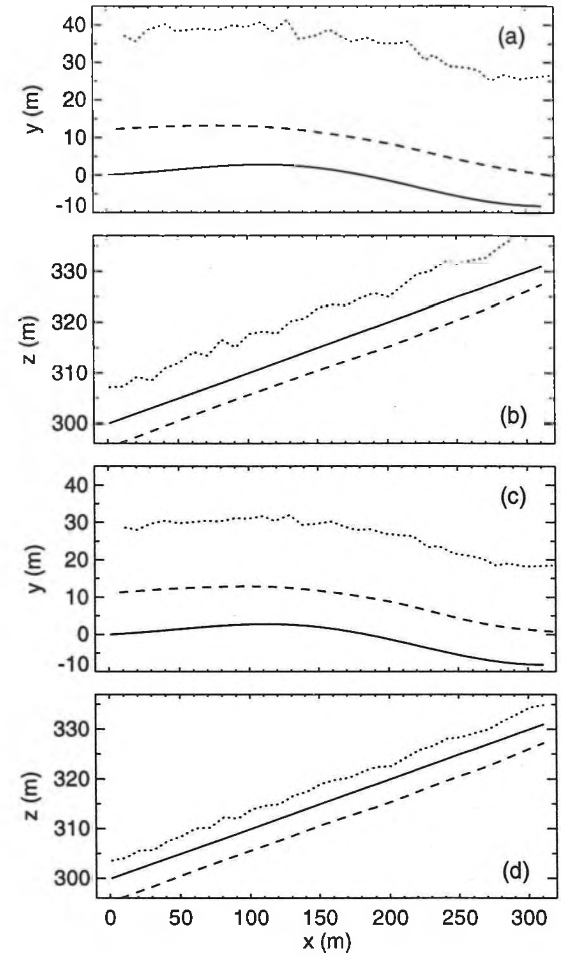

Fig. 1 Comparison of regularized and least-squares inversion for the synthetic test case described in text. Solid line indicates true array shape, dashed and dotted lines indicate the regularized and least-squares solutions, respectively. In (a) and (b) the data errors are 1 ms; in (c) and (d) they are 0.5 ms.

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REFERENCES

[1] E. C. Ballegooijen et al., 1989. Measurement of towed array position, shape and attitude, IEEE J. Ocean. Eng., **14**, 375–383.

[2] S. E. Dosso, M. R. Fallat, B. J. Sotirin and J. L. Newton, 1998. Array element localization for horizontal arrays via Occam's inversion. J. Acoust. Soc. Am., **104**, 846–859.