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REGULARIZED SEMIGROUPS, EXISTENCE FAMILIES AND THE ABSTRACT CAUCHY PROBLEM

Ralph deLaubenfels[1]
Scientia Research Institute, P.O. Box 988, Athens, Ohio 45701, USA

Sun Guozheng
Department of Mathematics, Anhui Normal University, Wuhu, 241000, People's Republic of China

Shengwang Wang
Department of Mathematics, Nanjing University, Nanjing, Jiangsu 210008, People's Republic of China

(Submitted by: Reza Aftabizadeh)

Abstract. For an arbitrary bounded linear operator C, on a Banach space, and a closable linear operator A, we introduce a C-regularized semigroup for A. We present equivalences between A having a C-regularized semigroup, the corresponding abstract Cauchy problem, well-posedness on a continuously embedded subspace, and (for exponentially bounded C-regularized semigroups) the Laplace transform.

0. Introduction. In dealing with the many physical problems that may be modeled as an *abstract Cauchy problem*

$$\frac{d}{dt}u(t,x) = A\left(u(t,x)\right) \quad (t \geq 0), \;\; u(0,x) = x, \tag{0.1}$$

where A is a linear operator on a Banach space X, and $t \mapsto u(t,x) \in C([0,\infty), X)$, well-posedness corresponds to A generating a strongly continuous semigroup. When (0.1) is not well-posed, at least in its original formulation, a useful concept for dealing with it is a *C-regularized semigroup* (Definition 2.1). When A generates a C-regularized semigroup, then (0.1) has a unique mild solution, for all initial data in the image of C, (0.1) has a unique strong solution for all x in $C(\mathcal{D}(A))$ and (0.1) is well-posed on a subspace continuously embedded between X and the image of C (see [9]).

However, in order that (0.1) have all these solutions, and the well-posedness on a subspace, it is not necessary that A itself generate a C-regularized semigroup, even if A is closed and C commutes with A. This is in contrast to the strongly continuous

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case, that is, when $C = I$; when A is closed and (0.1) has a unique mild solution, for all x in X, it follows automatically that A generates a strongly continuous semigroup.

Here is a simple example.

Counterexample 0.2. This is an example of a closed operator A and a bounded injective operator C such that $CA \subseteq AC$ and (0.1) has a unique bounded mild solution for all x in the image of C, but A does not generate a C-regularized semigroup.

Let $G \equiv \frac{d}{ds}$, on $X \equiv L^\infty(\mathbb{R})$, with maximal domain. Let B equal the restriction of G to $\mathcal{D}(G^2)$, the domain of G^2, that is,

$$\mathcal{D}(B) \equiv \mathcal{D}(G^2), \quad Bx = Gx, \ \forall x \in \mathcal{D}(B).$$

Let $A \equiv \overline{B}$, the closure of B. Let $C \equiv (1 - G)^{-2}$, and define a C-regularized semigroup $\{W(t)\}_{t \geq 0}$ by

$$(W(t)f)(s) \equiv (Cf)(t+s) \quad (t \geq 0, s \in \mathbb{R}).$$

It is not hard to show that G is the generator of $\{W(t)\}_{t\geq 0}$. The domain of A equals the graph closure of the domain of G^2, which may be shown to equal $(1 - G)^{-1}(BUC(\mathbb{R}))$, which does not equal $\mathcal{D}(G) = (1 - G)^{-1}(L^\infty(\mathbb{R}))$.

Note that the (proper) *extension* G, of A, generates $\{W(t)\}_{t\geq 0}$.

In this paper, we will write down exactly what conditions on A are equivalent to (0.1) having a unique mild solution, for all initial data in the image of C, when C commutes with A. We shall see that this is equivalent to (0.1) being well-posed on a subspace continuously embedded between X and the image of C, and automatically implies that (0.1) has a unique strong solution for all x in $C(\mathcal{D}(A))$ (Theorem 3.3).

We introduce a *regularized semigroup for* A (Definition 2.3). We will also say that A *has a regularized semigroup*. When C is injective and A is closed, having a C-regularized semigroup is equivalent to (0.1) having a unique mild solution, for all initial data in the image of C. In the language of this paper, the operator A of Counterexample 0.2 has a regularized semigroup, that it does not generate.

Thus, from the point of view of the abstract Cauchy problem (0.1), there is no difference between A having a regularized semigroup and A generating a regularized semigroup. In practice, it is much easier to verify that A has a regularized semigroup, than it is to show that A itself is the generator.

We also introduce a C-regularized semigroup when C may not be injective (Definition 2.1). This gives us the same solutions, but the solutions may not be unique. When C is injective, and A has a C-regularized semigroup, then the solutions of (0.1) are unique.

When A generates a C-regularized semigroup $\{W(t)\}_{t\geq 0}$, then $\{W(t)\}_{t\geq 0}$ is a regularized semigroup for A. However, as in Counterexample 0.2, it is sometimes sufficient to have an extension of A be the generator. This is desirable, because it is sometimes difficult to determine or describe the exact domain of the generator. The most natural choice of domain may produce an operator A, that has a regularized semigroup, but is not its generator.

In Section I we present some preliminary material on C-existence families for an operator A, and the consequences of their presence. In these definitions, C might not commute with A.

In Section II we introduce a C-regularized semigroup for A, and the generator of a C-regularized semigroup; C is bounded and commutes with A, but may not be injective. We give some basic properties of the generator, $\tilde{A}$, its relationship with A, and the relationship between the C-regularized semigroup for A and (0.1).

Theorem 3.1 gives a sufficient condition for an operator to have a regularized semigroup, that is often easy to verify, in practice. Theorem 3.3 contains numerous equivalences for A having a C-regularized semigroup, when A is closed and C is injective. In particular, it is equivalent to (0.1) having a unique mild solution, for all x in the image of C. When the image of C is dense, it is equivalent to $C^{-1}AC$ generating the C-regularized semigroup.

Theorem 3.7 and Theorem 3.12 characterize, respectively, exponentially bounded mild existence families for A and exponentially bounded regularized semigroups for A, in terms of a Laplace transform.

We give some examples, including the backwards heat equation and arbitrary systems of constant coefficient partial differential initial-value problems, in Section IV.

All operators are linear, on a Banach space X. We will write $\mathcal{D}(A)$ for the domain of the operator A. We will write $[\mathcal{D}(A)]$ for the normed vector space with the graph norm $\|x\|_{[\mathcal{D}(A)]} \equiv \|x\| + \|Ax\|$. We will write $B(X)$ for the space of bounded linear operators from X into itself. Throughout this paper, $C \in B(X)$. We will denote by $[Im(C)]$ the Banach space with norm $\|x\|_{[Im(C)]} \equiv \inf\{\|y\| : Cy = x\}$. Basic material on C-regularized semigroups and their generators, when C is injective, and existence families, may be found in [9], along with more extensive references.

I. Preliminaries. Existence families were introduced in [9], and, for the exponentially bounded case, in [7].

Definition 1.1. A *strong solution* of (0.1) is $u(t,x)$ such that

$$t \mapsto u(t,x) \in C([0,\infty),[\mathcal{D}(A)]) \cap C^1([0,\infty),X),$$

satisfying (0.1).

A *mild solution* of (0.1) is $u(t,x)$ such that $t \mapsto u(t,x) \in C([0,\infty),X)$, and for all $t \geq 0$, $\int_0^t u(s,x)\,ds \in \mathcal{D}(A)$, with

$$A\Big(\int_0^t u(s,x)\,ds\Big) = u(t,x) - x \quad (t \geq 0),\ u(0,x) = x.$$

Definition 1.2. A *mild C-existence family* for A is a strongly continuous family of operators $\{W(t)\}_{t\geq 0} \subseteq B(X)$ such that for any $x \in X$, $t \geq 0$, $\int_0^t W(s)x\,ds \in \mathcal{D}(A)$, with

$$A\Big(\int_0^t W(s)x\,ds\Big) = W(t)x - Cx.$$

The mild C-existence family $\{W(t)\}_{t\geq 0}$ is a *strong C-existence family* for A if

$$\{W(t)|_{[\mathcal{D}(A)]}\}_{t\geq 0}$$

is contained in $B([\mathcal{D}(A)])$ and is strongly continuous, with

$$\int_0^t AW(s)x\,ds = W(t)x - Cx,$$

for all $x \in \mathcal{D}(A)$.

Definition 1.3. If A is a closed operator such that all mild solutions of (0.1) are unique, then the *solution space* for A, denoted by Z, is the set of all x for which (0.1) has a mild solution, topologized by the seminorms

$$\|x\|_{a,b} \equiv \sup_{a\leq t\leq b} \|u(t,x)\| \quad (a, b \in Q^+).$$

In [9, Chapter 4], we show that Z is a Frechet space, and that $A|_Z$ generates a strongly continuous, locally equicontinuous semigroup.

When A is closed and has no eigenvalues in (ω, ∞), for some real ω, then it may be shown that all exponentially bounded mild solutions of (0.1) are unique (see Lemma 3.11). The *Hille-Yosida space* for A, denoted by Z_0, (see [9]) is then the set of all x for which (0.1) has a bounded uniformly continuous mild solution, topologized by

$$\|x\|_{Z_0} \equiv \sup_{t\geq 0} \|u(t,x)\|.$$

In [9, Chapter 5], we show that Z_0 is a Banach space and $A|_{Z_0}$ generates a strongly continuous semigroup of contractions.

Here are some consequences of having a C-existence family.

Lemma 1.4 [9, Theorem 2.6]**.**

(1) *If there exists a mild C-existence family, $\{W(t)\}_{t\geq 0}$, for A, then* (0.1) *has a mild solution for any $x \in Im(C)$, $u(t, Cy) \equiv W(t)y$, and the sequence of solutions $u(t, Cx_n)$ converge to zero, uniformly on compact subsets of* $[0, \infty)$, *whenever $x_n \to 0$.*

(2) *If there exists a strong C-existence family, $\{W(t)\}_{t\geq 0}$, for A, then* (0.1) *has a strong solution for any $x \in C(\mathcal{D}(A))$, $u(t, Cy) \equiv W(t)y$, and both $u(t, Cx_n)$ and $Au(t, Cx_n)$ converge to zero, uniformly on compact subsets of* $[0, \infty)$, *whenever x_n and Ax_n both converge to zero.*

The following is very useful for dealing with exponentially bounded solutions of (0.1).

Lemma 1.5 [1, Theorem 1.1]. *Suppose $f : (0,\infty) \to X$ and $M \geq 0$. Then the following are equivalent.*

(a) *There exists $F : [0,\infty) \to X$ such that $\|F(s) - F(t)\| \leq M|t-s|$, for all $s, t \geq 0, F(0) = 0$ and*

$$f(s) = s\int_0^\infty e^{-st}F(t)\,dt \quad \forall s > 0.$$

(b) *f is infinitely differentiable, with*

$$\|f^{(k)}(s)\| \leq M\frac{k!}{s^{k+1}}, \quad \forall k \in \mathbb{N}, s > 0.$$

Lemma 1.6 [9, Lemma 2.10]. *Suppose A is closed and u is a $O(e^{\omega t})$ mild solution of* (0.1). *Then, for $Re(z) > \omega$, $\int_0^\infty e^{-zt}u(t,x)\,dt \in \mathcal{D}(A)$, with*

$$(z - A)\int_0^\infty e^{-zt}u(t,x)\,dt = x.$$

II. Definitions and basic properties. When C commutes with an existence family, a more algebraic definition is possible (informally, think of $W(t) = e^{tA}C = Ce^{tA}$). When C is injective, the following was introduced in [4], and, independently, for the exponentially bounded case, in [5]; see [9].

Definition 2.1. The strongly continuous family of operators $\{W(t)\}_{t\geq 0} \subseteq B(X)$ is a *C-regularized semigroup* if

(1) $W(0) = C$;
(2) $W(t)W(s) = CW(t+s)$, for all $s, t \geq 0$;

$\{W(t)\}_{t\geq 0}$ is *nondegenerate* if $W(t)x \equiv 0$, for all $t \geq 0$, only when $x = 0$.

It is clear that $\{W(t)\}_{t\geq 0}$ is nondegenerate if C is injective. The converse is also true.

Proposition 2.2. *The C-regularized semigroup $\{W(t)\}_{t\geq 0}$ is nondegenerate if and only if C is injective.*

Proof. The sufficiency of being injective is clear. Conversely, suppose $\{W(t)\}_{t\geq 0}$ is nondegenerate and suppose $Cx = 0$. Then by (2) of Definition 2.1,

$$W(t)\,(W(s)x) = 0, \quad \forall s, t \geq 0,$$

thus since $\{W(t)\}_{t\geq 0}$ is nondegenerate, $W(s)x = 0$, for all $s \geq 0$. Now nondegeneracy again implies that $x = 0$. Thus C is injective, as desired. □

Definition 2.3. Suppose A is closable. We will say that the C-regularized semigroup $\{W(t)\}_{t\geq 0}$ is a *C-regularized semigroup for A* if

(1) $W(t)A \subseteq AW(t)$, for all $t \geq 0$; and
(2) $\{W(t)\}_{t\geq 0}$ is a mild C-existence family for A.

We will also say that A *has a C-regularized semigroup* or *has the C-regularized semigroup* $\{W(t)\}_{t\geq 0}$

We shall see that a strongly continuous family of bounded operators $\{W(t)\}_{t\geq 0}$ satisfying (1) and (2) of Definition 2.3 is automatically a C-regularized semigroup (see Theorem 3.3).

Proposition 2.4. *If A is closed and $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup for A, then $\{W(t)\}_{t\geq 0}$ is a strong C-existence family for A, such that*

$$W(t)x - Cx = \int_0^t W(s)Ax\,ds, \quad \forall x \in \mathcal{D}(A).$$

Proof. Suppose $x \in \mathcal{D}(A)$. For any $t \geq 0$, $W(t)x \in \mathcal{D}(A)$, with $AW(t)x = W(t)Ax$, thus

$$\|W(t)x\|_{[\mathcal{D}(A)]} \equiv \|W(t)Ax\| + \|W(t)x\| \leq \|W(t)\|(\|Ax\| + \|x\|) \equiv \|W(t)\|\|x\|_{[\mathcal{D}(A)]},$$

so that $W(t)|_{[\mathcal{D}(A)]} \in B([\mathcal{D}(A)])$.

Strong continuity of $\{W(t)|_{[\mathcal{D}(A)]}\}_{t\geq 0}$ follows similarly.

By the definition of C-regularized semigroup for A, and the fact that A is closed, for $x \in \mathcal{D}(A)$,

$$W(t)x - Cx = A\Big(\int_0^t W(s)x\,ds\Big) = \int_0^t AW(s)x\,ds = \int_0^t W(s)Ax\,ds,$$

concluding the proof.

Definition 2.5. Suppose $\{W(t)\}_{t\geq 0}$ is a nondegenerate C-regularized semigroup. Let $\mathcal{D}(\tilde{A})$ be the set of all $x \in X$ such that there exists $y \in X$ such that

$$W(t)x - Cx = \int_0^t W(s)y\,ds, \quad \forall t \geq 0.$$

Then $\tilde{A}x \equiv y$.

The operator $\tilde{A}$ is the *generator* of $\{W(t)\}_{t\geq 0}$.

An analogous definition of the generator of an integrated semigroup appears in [1] and [15].

Proposition 2.6. *Suppose $\{W(t)\}_{t\geq 0}$ is a nondegenerate C-regularized semigroup generated by $\tilde{A}$. Then we have the following.*

(1) *$\tilde{A}$ is closed.*

(2) *$\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup for $\tilde{A}$.*

(3) *If $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup for A, and A is closable, then*

$$C(\mathcal{D}(\tilde{A})) \subseteq \mathcal{D}(\overline{A}), \quad \text{and} \quad \overline{A} \subseteq \tilde{A}.$$

(4) *If $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup for A, then $Im(C) \subseteq \overline{\mathcal{D}(A)}$.*

(5)

$$\tilde{A}x = C^{-1}\big(\lim_{t\to 0}\frac{1}{t}(W(t)x - Cx)\big),$$

with maximal domain.

Note that (5) is asserting that Definition 2.5 is consistent with the usual definition of generator (see [9, Chapter 3]).

Proof. (1). Suppose $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{D}(\tilde{A})$, $x_n \to x$ and $\tilde{A}x_n \to y$, as $n \to \infty$. Then, for all $n \in \mathbb{N}$,

$$W(t)x_n - Cx_n = \int_0^t W(s)\tilde{A}x_n\, ds, \quad \forall t \geq 0,$$

thus, since $\|W(s)\|$ is uniformly bounded, for $s \in [0, t]$, we may conclude, by taking limits, that

$$W(t)x - Cx = \int_0^t W(s)y\, ds, \quad \forall t \geq 0,$$

so that $x \in \mathcal{D}(\tilde{A})$ and $\tilde{A}x = y$, as desired.

(2). For any $x \in X$, $r \geq 0$, by the definition of a C-regularized semigroup (Definition 2.1)

$$\begin{aligned}(W(t) - C)\Big(\int_0^r W(s)x\, ds\Big) &= \Big(\int_t^{r+t} - \int_0^r\Big)(W(s)Cx\, ds) \\ &= \Big(\int_r^{r+t} - \int_0^t\Big)(W(s)Cx\, ds) = \int_0^t W(s)(W(r)x - Cx)\, ds,\end{aligned}$$

thus $\int_0^r W(s)x\, ds \in \mathcal{D}(\tilde{A})$, with

$$\tilde{A}\Big(\int_0^r W(s)x\, ds\Big) = W(r)x - Cx,$$

so that (2) of Definition 2.3 is satisfied.

If $x \in \mathcal{D}(\tilde{A})$, and $r \geq 0$, then by applying $W(r)$ to both sides of the generator, we have

$$W(t)(W(r)x) - C(W(r)x) = \int_0^t W(s)(W(r)\tilde{A}x)\, ds, \quad \forall t \geq 0,$$

so that $W(r)x \in \mathcal{D}(\tilde{A})$, with

$$\tilde{A}W(r)x = W(r)\tilde{A}x,$$

so that (1) of Definition 2.3 is satisfied.

(3) For $x \in \mathcal{D}(\tilde{A})$, since

$$n\Big(\int_0^{\frac{1}{n}} W(s)x\, ds\Big) \to Cx,$$

and

$$A\Big(n\Big(\int_0^{\frac{1}{n}} W(s)x\, ds\Big)\Big) = n(W(\frac{1}{n})x - Cx) \to C\tilde{A}x = \tilde{A}Cx,$$

as $n \to \infty$, and A is closable, it follows that $Cx \in \mathcal{D}(\overline{A})$, with $\overline{A}Cx = \tilde{A}Cx$. Thus $C(\mathcal{D}(\tilde{A})) \subseteq \mathcal{D}(\overline{A})$. For $x \in \mathcal{D}(A)$, Definition 2.3 and the fact that A is closable imply that

$$W(t)x - Cx = A\Big(\int_0^t W(s)x\,ds\Big) = \int_0^t W(s)\overline{A}x\,ds,$$

thus $x \in \mathcal{D}(\tilde{A})$ and $Ax = \tilde{A}x$, that is, $A \subseteq \tilde{A}$. Since $\tilde{A}$ is closed, $\overline{A} \subseteq \tilde{A}$.
(4). For any $x \in X, n \in \mathbb{N}$, define

$$x_n \equiv n\int_0^{\frac{1}{n}} W(s)x\,ds.$$

Then $x_n \in \mathcal{D}(A)$, for all $n \in \mathbb{N}$, and $x_n \to Cx$, as $n \to \infty$, thus $Cx \in \overline{\mathcal{D}(A)}$.
(5). Define

$$Bx \equiv C^{-1}\big(\lim_{t\to 0}\frac{1}{t}(W(t)x - Cx)\big),$$

with maximal domain.

Differentiating at $t = 0$, in the definition of the generator, implies that $\tilde{A} \subseteq B$. Conversely, suppose $x \in \mathcal{D}(B)$. Then $\frac{d}{dt}W(t)x = W(t)Bx$, for all $t \geq 0$ (see [9, Theorem 3.4]), thus

$$W(t)x - Cx = \int_0^t W(s)Bx\,ds,$$

for all $t \geq 0$, which implies that $B \subseteq \tilde{A}$. $\square$

Example 2.7. Suppose B generates a strongly continuous semigroup $\{e^{tB}\}_{t\geq 0}$, and $\mathcal{D}(A) \equiv \mathcal{D}(B^2)$, with $Ax = Bx$, for $x \in \mathcal{D}(A)$. Fix $\lambda \in \rho(B)$ and define

$$W(t) \equiv (\lambda - B)^{-1}e^{tB} \quad (t \geq 0).$$

Then it is not hard to show that $\{W(t)\}_{t\geq 0}$ is a $(\lambda - B)^{-1}$-regularized semigroup for A. But A does not generate $\{W(t)\}_{t\geq 0}$; the generator is B.

In this example, A is not closed. An example of a closed operator A, that has a C-regularized semigroup it does not generate, is in Counterexample 0.2.

Definition 2.8. When $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup and C is injective, the *infinitesimal generator*, G, of $\{W(t)\}_{t\geq 0}$, is defined by

$$Gx \equiv \lim_{t\to 0}\frac{1}{t}(C^{-1}W(t)x - x),$$

with $\mathcal{D}(G)$ defined to be the set of all $x \in Im(C)$ such that the limit exists.

For $Im(C)$ dense, and $\{W(t)\}_{t\geq 0}$ exponentially bounded, this was introduced in [14].

Note that $\tilde{A}|_{C(\mathcal{D}(\tilde{A}))}$, where $\tilde{A}$ is the generator of $\{W(t)\}_{t\geq 0}$, is obviously contained in G; in fact, $C(\mathcal{D}(\tilde{A}))$ is the set of $x \in \mathcal{D}(G)$ such that $Gx \in Im(C)$.

Proposition 2.9. *Suppose A is closed, C is injective, and $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup for A. Then*

(1) *$\{W(t)\}_{t\geq 0}$ is unique;*
(2) *$C^{-1}AC$ is the generator of $\{W(t)\}_{t\geq 0}$; and*
(3) *$\overline{G} \subseteq A$.*

Proof. (1). Suppose, for $i = 1, 2$, that $\{W_i(t)\}_{t\geq 0}$ is a C-regularized semigroup for A. For any $x \in X$, $s, t \geq 0$,

$$\frac{d}{ds}\Big[W_1(t-s)\int_0^s W_2(r)x\,dr\Big] = W_1(t-s)W_2(s)x - W_1(t-s)A\int_0^s W_2(r)x\,dr$$
$$= W_1(t-s)Cx = CW_1(t-s)x,$$

and thus, by integrating in s from 0 to t, we obtain

$$C\int_0^t W_2(r)x\,dr = C\int_0^t W_1(r)x\,dr;$$

since C is injective, we may differentiate and conclude that $W_1(t) = W_2(t)$, for all $t \geq 0$, as desired.
(2) follows from Proposition 2.6(3) and [9, Proposition 3.11].
(3). Suppose $x \in \mathcal{D}(G)$. Then $x = Cy$, for some $y \in X$, so that for any $t > 0$,

$$\frac{1}{t}(C^{-1}W(t)x - x) = \frac{1}{t}(W(t)y - Cy) = A\Big[\frac{1}{t}\int_0^t W(s)y\,ds\Big],$$

so that, by taking the limit as $t \to 0$ and using the fact that A is closed, we conclude that $x \in \mathcal{D}(A)$, with $Ax = Gx$. $\square$

Corollary 2.10. *If A is closed and there exists a strongly continuous semigroup for A, then A is the generator.*

Example 2.11. If $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup for A, then it is not hard to show that $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup for $\overline{A}$. Here we give an example of a closable operator B such that $\overline{B}$ has a C-regularized semigroup but B does not.

Let X, G and A be as in Counterexample 0.2 and let $C \equiv (1 - G)^{-1}$. Define a C-regularized semigroup $\{W(t)\}_{t\geq 0}$ by

$$(W(t)f)(s) \equiv (Cf)(t + s) \quad (t \geq 0, s \in \mathbb{R}).$$

Let B equal the restriction of G to $(1 - G)^{-1}(BC^1(\mathbb{R}))$, where $BC^1(\mathbb{R})$ is the set of all bounded f such that f has a bounded continuous derivative, that is,

$$\mathcal{D}(B) \equiv (1 - G)^{-1}BC^1(\mathbb{R}), \;\; Bf \equiv f', \quad \forall f \in \mathcal{D}(B).$$

Then the following argument shows that $A = \overline{B}$.

Suppose $x \in \mathcal{D}(A)$. Then $x = (1-G)^{-1}y$, for some $y \in BUC(\mathbb{R})$. Let $\{y_n\}_n$ be a sequence in $BC^1(\mathbb{R})$ converging to y in X. Then $x_n \equiv (1-G)^{-1}y_n$ converges to x in X and

$$Bx_n = B(1-G)^{-1}y_n = G(1-G)^{-1}y_n \to G(1-G)^{-1}y = A(1-G)^{-1}y = Ax,$$

as $n \to \infty$.

It is clear that $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup for A, but not for B, since the integral of a function in $L^\infty(\mathbb{R})$ may not be in $BC^1(\mathbb{R})$. This implies that B has no C-regularized semigroup, for if it did, this C-regularized semigroup would also be a C-regularized semigroup for A; by Proposition 2.9(1), this would then imply that this C-regularized semigroup would be $\{W(t)\}_{t\geq 0}$.

III. Main results. We will begin with a simple sufficient condition for having a regularized semigroup, that is often satisfied in practice (see Section IV).

Theorem 3.1. *Suppose $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup generated by an extension of A, A is closed and densely defined and $W(t)$ leaves $\mathcal{D}(A)$ invariant, for all $t \geq 0$. Then $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup for A.*

Proof. Let $\tilde{A}$ be the generator of $\{W(t)\}_{t\geq 0}$. By Proposition 2.6(2) and the fact that $W(t)$ leaves $\mathcal{D}(A)$ invariant, for $x \in \mathcal{D}(A)$, $t \geq 0$,

$$W(t)Ax = W(t)\tilde{A}x = \tilde{A}W(t)x = AW(t)x.$$

Thus all that remains is to show that $\{W(t)\}_{t\geq 0}$ is a mild C-existence family for A. For $x \in \mathcal{D}(A)$, since A is closed, $\int_0^t W(s)x\,ds \in \mathcal{D}(A)$, with

$$A\Big(\int_0^t W(s)x\,ds\Big) = \int_0^t W(s)Ax\,ds = \int_0^t W(s)\tilde{A}x\,ds = W(t)x - Cx.$$

Since $\mathcal{D}(A)$ is dense, and A is closed, the same is true for all $x \in X$. $\square$

We shall see that, when C is injective and A has a C-regularized semigroup, then the solutions of (0.1) are unique. We will show this by showing that the C-existence family for A is also what we will call a *C-uniqueness family*. When $\{W(t)\}_{t\geq 0}$ is exponentially bounded, an equivalent version of the following definition appeared in [7].

Definition 3.2. If C is an injective, bounded operator, then a *C-uniqueness family for A* is a strongly continuous family of operators $\{W(t)\}_{t\geq 0} \subseteq B(X)$ such that

$$W(t)x - Cx = \int_0^t W(s)Ax\,ds \quad \forall x \in \mathcal{D}(A).$$

Now we present numerous relationships between having a regularized semigroup, having an existence family, and having unique solutions of (0.1), when C is injective and A is closed. It is interesting that, when C commutes with A, a mild existence family for A automatically commutes with A and is also a C-uniqueness family. When $Im(C)$ is dense, and $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup generated by $\tilde{A}$, then $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup for A if and only if $C\left(\mathcal{D}(\tilde{A})\right) \subseteq \mathcal{D}(A))$, if and only if $W(t)$ leaves $\mathcal{D}(A)$ invariant, for all $t \geq 0$. In fact, $\tilde{A}$ then equals $C^{-1}AC$.

In the following, note that, by Proposition 2.4, "mild C-existence family" could be replaced by "strong C-existence family".

Theorem 3.3. *Suppose A is closed, C is injective and $CA \subseteq AC$. Then the following are equivalent.*

(a) (0.1) *has a unique mild solution for all $x \in Im(C)$.*
(b) *All solutions of* (0.1) *are unique and $[Im(C)] \hookrightarrow Z$.*
(c) *All solutions of* (0.1) *are unique and there exists a mild C-existence family for A.*
(d) *There exists a mild C-existence family for A such that $W(t)A \subseteq AW(t)$, for all $t \geq 0$.*
(e) *There exists a mild C-existence family $\{W(t)\}_{t\geq 0}$ that is also a C-uniqueness family for A.*
(f) *There exists a C-regularized semigroup for A.*

In addition, if $Im(C)$ is dense, then (a)–(f) *are equivalent to the following.*

(g) *$C^{-1}AC$ generates a C-regularized semigroup.*
(h) *An extension of $A, \tilde{A}$, generates a C-regularized semigroup and $C(\mathcal{D}(\tilde{A})) \subseteq \mathcal{D}(A)$.*
(i) *An extension of A generates a C-regularized semigroup that leaves $\mathcal{D}(A)$ invariant.*

Proof. The equivalence of (a) through (c) is in [9, Chapter 4].
(c) $\rightarrow$ (f). Let $\{W(t)\}_{t\geq 0}$ be the mild C-existence family for A. For $s, t \geq 0$, $x \in X$,

$$W(t)W(s)x = A\Big(\int_0^t W(r)W(s)x\,dr\Big) + CW(s)x,$$

and, since $CA \subseteq AC$,

$$\begin{aligned} CW(t+s)x &= A\Big(\int_0^{t+s} CW(r)x\,dr\Big) + C^2x \\ &= A\Big(\int_0^t CW(s+r)x\,dr\Big) + A\Big(\int_0^s CW(r)x\,dr\Big) + C^2x \\ &= A\Big(\int_0^t CW(s+r)x\,dr\Big) + CW(s)x. \end{aligned}$$

By the uniqueness of solutions of (0.1),

$$W(t)W(s)x = CW(t+s)x, \quad \forall x \in X,$$

that is, $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup.

All that remains is to show that $W(t)A \subseteq AW(t)$, for all $t \geq 0$. Fix $x \in \mathcal{D}(A)$. We will show that

$$W(t)x = \int_0^t W(s)Ax\,ds + Cx. \tag{3.4}$$

To show (3.4), define

$$\tilde{W}(t)x \equiv \int_0^t W(s)Ax\,ds + Cx;$$

since A is closed, $CA \subseteq AC$ and $\{W(t)\}_{t\geq 0}$ is a mild existence family, it follows that $\int_0^t \tilde{W}(s)x\,ds$, $\big(\int_0^s W(r)Ax\,dr\big)$ and Cx are in the domain of A, with

$$\begin{aligned} A\int_0^t \tilde{W}(s)x\,ds &= \int_0^t A\Big(\int_0^s W(r)Ax\,dr\Big)\,ds + tCAx \\ &= \int_0^t W(s)Ax\,ds = \tilde{W}(t)x - Cx. \end{aligned}$$

Again by the uniqueness of solutions of (0.1), it follows that $\tilde{W}(t)x = W(t)x$, proving (3.4).

Assertion (3.4) and the fact that $\{W(t)\}_{t\geq 0}$ is a mild C-existence family now imply that

$$A\int_0^t W(s)x\,ds = \int_0^t W(s)Ax\,ds. \tag{3.5}$$

Since A is closed, we may differentiate both sides of (3.5) to conclude that $W(t)x \in \mathcal{D}(A)$, with $AW(t)x = W(t)Ax$, as desired.

(f) $\rightarrow$ (d) is obvious.

(d) $\rightarrow$ (e). This proof is the same as the proof of Proposition 2.4.

(e) $\rightarrow$ (c). It is sufficient to show that, if $\frac{d}{dt}v(t) = A(v(t))$ and is continuous, for $t \geq 0$, with $v(0) = 0$, then $v(t) \equiv 0$.

Since $\{W(t)\}_{t\geq 0}$ is a C-uniqueness family for A, for $s, t \geq 0$,

$$\frac{d}{ds}W(t-s)v(s) = 0.$$

Thus $W(t-s)v(s)$ is a constant function of s; letting s equal t, then 0, gives us

$$Cv(t) = W(t)v(0) = 0,$$

so that, since C is injective, $v(t) \equiv 0$.

We have shown the equivalence of (a)-(f). Now suppose $Im(C)$ is dense.

The equivalence of (g) and (h) is [9, Proposition 3.11] (this holds whether or not $Im(C)$ is dense).

(f) $\rightarrow$ (i) is obvious, and (i) $\rightarrow$ (f) is Theorem 3.1, since, by Proposition 2.6(4), $\mathcal{D}(A)$ is dense.

(f) → (g) is Proposition 2.9 (again, the denseness of $Im(C)$ is not required here).
(g) → (f). Let $\{W(t)\}_{t\geq 0}$ be the C-regularized semigroup generated by $C^{-1}AC$. Then, for any $x \in X$, $t \geq 0$,

$$W(t)x - Cx = C^{-1}AC \int_0^t W(s)x\, ds,$$

thus

$$W(t)Cx - C^2x = A \int_0^t W(s)Cx\, ds.$$

Since $Im(C)$ is dense and A is closed, this implies (2) of Definition 2.3.

If $x \in \mathcal{D}(A)$, it then follows that

$$A \int_0^t W(s)x\, ds = W(t)x - Cx = \int_0^t W(s)C^{-1}ACx\, ds = \int_0^t W(s)Ax\, ds, \quad \forall t \geq 0;$$

since A is closed, we may differentiate, to conclude that $W(t)x \in \mathcal{D}(A)$, with $AW(t)x = \frac{d}{dt}W(t)x = W(t)Ax$, so that (1) of Definition 2.3 is satisfied. □

Definition 3.6. The complex number λ is in $\rho_C(A)$, the *C-resolvent of A*, if $(\lambda - A)$ is injective and $Im(C) \subseteq Im(\lambda - A)$.

A Laplace transform definition of an exponentially bounded existence family for A appears in [7]. However, part of the definition given there asserted that $\int_0^t W(s)x\, ds \in \mathcal{D}(A)$, for all $x \in X$. This hypothesis is actually unnecessary, that is, it follows automatically from the existence of the desired Laplace transform. Thus we are led to the following simpler characterization of an exponentially bounded mild existence family, in terms of a Laplace transform.

Theorem 3.7. *Suppose A is closed, $\omega \in \mathbb{R}$, $\{W(t)\}_{t\geq 0}$ is a strongly continuous $O(e^{\omega t})$ family of bounded operators and $(s - A)$ is injective for all $s > \omega$. Then $\{W(t)\}_{t\geq 0}$ is a mild C-existence family for A if and only if*

(1) $(\omega, \infty) \subseteq \rho_C(A)$, *and*

(2)

$$(s - A)^{-1}Cx = \int_0^\infty e^{-st}W(t)x\, dt, \quad \forall x \in X, s > \omega.$$

Proof. Suppose $\{W(t)\}_{t\geq 0}$ is a mild C-existence family for A. Then (1) and (2) follow immediately from Lemma 1.6.

Conversely, suppose (1) and (2) hold. Without loss of generality (by translating A if necessary), suppose $\omega < 0$. Define, for any $t \geq 0$,

$$\tilde{W}(t)x \equiv \int_0^t W(s)x\, ds \quad (x \in X).$$

By (2),

$$(s - A)^{-1}Cx = s \int_0^\infty e^{-st}\tilde{W}(t)x\, dt, \quad \forall x \in X, s > 0. \tag{3.8}$$

Since $\omega < 0$, (3.8) implies that

$$\big\|(\frac{d}{ds})^k\big(\frac{1}{s}(s-A)^{-1}Cx\big)\big\| \leq Mk!s^{-(k+1)}\|x\| \quad (s > 0, k \in \mathbb{N}), \tag{3.9}$$

while (2) implies that

$$\|(\frac{d}{ds})^k\left((s-A)^{-1}Cx\right)\| \leq Mk!s^{-(k+1)}\|x\| \quad (s > 0, k \in \mathbb{N}), \tag{3.10}$$

for some constant M. Assertions (3.9) and (3.10), along with the identity

$$A\big(\frac{1}{s}(s-A)^{-1}Cx\big) = (s-A)^{-1}Cx - \frac{1}{s}Cx$$

imply that

$$\big\|(\frac{d}{ds})^k\big(\frac{1}{s}(s-A)^{-1}Cx\big)\big\|_{[\mathcal{D}(A)]} \leq M_1k!s^{-(k+1)}\|x\| \quad (s > 0, k \in \mathbb{N}),$$

for some constant M_1. By Lemma 1.5, there exists $W_x : [0,\infty) \to [\mathcal{D}(A)]$ such that $W_x(0) = 0$ and

$$\frac{1}{s}(s-A)^{-1}Cx = s\int_0^\infty e^{-st}W_x(t)\,dt.$$

Comparing this with (3.8) tells us that $\int_0^t \tilde{W}(s)x\,ds = W_x(t) \in \mathcal{D}(A)$, for any $x \in X$. Assertion (3.8) now implies that

$$tCx = \tilde{W}(t)x - A\int_0^t \tilde{W}(s)x\,ds.$$

Since A is closed, we may differentiate this, to conclude that $\int_0^t W(s)x\,ds = \tilde{W}(t)x \in \mathcal{D}(A)$, with

$$W(t)Cx - Cx = A\int_0^t W(s)x\,ds,$$

so that $\{W(t)\}_{t\geq 0}$ is a mild C-existence family for A. $\square$

Since, when $\tilde{A}$ generates an exponentially bounded C-regularized semigroup, and C is injective, $(s-\tilde{A})$ is automatically injective, for s sufficiently large, we will obtain, as a corollary, a characterization of A having an exponentially bounded regularized semigroup, in terms of the C-resolvent being a Laplace transform (of the C-regularized semigroup). First, we need a lemma that guarantees uniqueness of exponentially bounded solutions of (0.1). The following is an immediate corollary of Lemma 1.6.

Lemma 3.11 ([Proposition 2.9, 7])**.** *Suppose A is closed and there exists $\omega \in \mathbb{R}$ such that A has no eigenvalues in (ω,∞). Then all exponentially bounded solutions and mild solutions of* (0.1) *are unique.*

Theorem 3.12. *Suppose A is closed, $\{W(t)\}_{t\geq 0}$ is a strongly continuous $O(e^{\omega t})$ family of bounded operators and C is injective. Then the following are equivalent.*

(a) *$\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup for A.*
(b) *$Im(C) \subseteq Im(s-A)$ for all $s > \omega$, $CA \subseteq AC$ and $C^{-1}AC$ generates the regularized semigroup $\{W(t)\}_{t\geq 0}$.*
(c) *$Im(C) \subseteq Im(s-A)$ for all $s > \omega$, $CA \subseteq AC$ and an extension of A generates the regularized semigroup $\{W(t)\}_{t\geq 0}$.*
(d) *$(\omega, \infty) \subseteq \rho_C(A)$, $CA \subseteq AC$ and*

$$(s-A)^{-1}Cx = \int_0^\infty e^{-st}W(t)x\,dt, \quad \forall x \in X, s > \omega.$$

Proof. (a) $\rightarrow$ (b). By Definition 2.3, $CA \subseteq AC$ and by Lemma 1.6,

$$Cx = (s-A)\int_0^\infty e^{-st}W(t)x\,dt, \quad \forall x \in X, s > \omega,$$

Thus $Im(C) \subseteq Im(s-A)$, for $s > \omega$. By Proposition 2.9, $C^{-1}AC$ generates $\{W(t)\}_{t\geq 0}$.
(b) $\rightarrow$ (c) is obvious.
(c) $\rightarrow$ (d). Let $\tilde{A}$ be the extension of A that generates $\{W(t)\}_{t\geq 0}$. By [9, Chapter 17], $(\omega, \infty) \subseteq \rho_C(\tilde{A})$, with

$$(s-\tilde{A})^{-1}Cx = \int_0^\infty e^{-st}W(t)x\,dt, \quad \forall x \in X, s > \omega.$$

Since $Im(C) \subseteq Im((s-A))$, for all $s > \omega$, the same is true for A, that is, $(\omega, \infty) \subseteq \rho_C(A)$ and

$$(s-A)^{-1}Cx = \int_0^\infty e^{-st}W(t)x\,dt, \quad \forall x \in X, s > \omega.$$

(d) $\rightarrow$ (a). By Theorem 3.7, $\{W(t)\}_{t\geq 0}$ is a mild C-existence family for A, and by Lemma 3.11, all exponentially bounded mild solutions of (0.1) are unique. Thus, by the same proof as Theorem 3.3(c) $\rightarrow$ (f), $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup for A. $\square$

Remark 3.13. Laplace transform characterizations of exponentially bounded regularized semigroups, similar to Theorem 3.12(d), appear in [12, Definition 2.4] and [6, Proposition 3.4]. There is a misnomer in [12, Definition 2.4]; they define the closed operator A, satisfying (d) of Theorem 3.12, as *the* generator of $\{W(t)\}_{t\geq 0}$. As is demonstrated by Counterexample 0.2, this is ambiguous; that is, there can be more than one closed operator that has $\{W(t)\}_{t\geq 0}$ as a regularized semigroup, hence satisfies (d) of Theorem 3.12.

Corollary 3.14. *Suppose $CA \subseteq AC$, and A is closed and has no eigenvalues in $(0, \infty)$. Then the following are equivalent.*

(a) *(0.1) has a bounded uniformly continuous mild solution, for all $x \in Im(C)$.*
(b) $[Im(C)] \hookrightarrow Z_0$.
(c) *There exists a bounded strongly uniformly continuous mild C-existence family for A.*
(d) *There exists a bounded strongly uniformly continuous C-regularized semigroup for A.*

In addition, if C is injective, then (a)-(d) are equivalent to

(e) *$C^{-1}AC$ generates a bounded strongly uniformly continuous C-regularized semigroup and $Im(C) \subseteq Im(s - A)$, for all $s > 0$.*
(f) *An extension of A generates a bounded strongly uniformly continuous C-regularized semigroup and $Im(C) \subseteq Im(s - A)$, for all $s > 0$.*

Proof. By Lemma 3.11, all exponentially bounded solutions of (0.1) are unique. Thus the equivalence of (a)-(c) is in [9, Chapter 5], and the equivalence of (d) and (c) follows as did the equivalence of (c) and (f) in Theorem 3.3. The equivalence of (d), (e), and (f) is Theorem 3.12.

IV. Examples. In all these examples, the fact that the C-regularized semigroup is a C-regularized semigroup for A follows from Theorem 3.1.

Many examples of applications of regularized semigroups to differential equations may be found in [9]. Here we will focus on examples where either C is not injective, or A is not the generator of the C-regularized semigroup, so that the generality of this paper is required.

Example 4.1. Let C be a spectral projection for A, corresponding to a compact subset of the complex plane, Ω; by this we mean that $CA \subseteq AC$, the restriction of A to $[Im(C)]$ is in $B([Im(C)])$, and the spectrum of the restriction of A to $[Im(C)]$ is contained in Ω.

For example, when A is a self-adjoint operator on a Hilbert space, such spectral projections are defined, for any compact set Ω.

Then for any complex-valued polynomial p,

$$W(t) \equiv \sum_{k=0}^{\infty} \frac{t^k}{k!} \, (p(A)C)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \, (p(A))^k \, C$$

is a C-regularized semigroup for $p(A)$.

Note that, in this example, C is not injective.

Example 4.2. A model for linear elasticity is

$$(\frac{d}{dt})^2 u(t,x) = -B(u(t,x)) \quad (t \geq 0), \quad u(0,x) = x, \tag{4.3}$$

where B is a self-adjoint operator on a Hilbert space H and $x \in H$ (see [3] and [13]). This can be shown to be well-posed if and only if B is positive (see [16]).

When there are structural instabilities (for example, during an earthquake), then we do not expect (4.3) to be well-posed.

If we make the usual reduction to a first order problem,

$$A \equiv \begin{bmatrix} 0 & 1 \\ -B & 0 \end{bmatrix}, \quad \mathcal{D}(A) \equiv \mathcal{D}(B) \times H,$$

then (4.3) becomes (0.1), on $H \times H$.

It is clear that, if C is a spectral projection for B corresponding to a compact set, then CI_2 is a spectral projection for A corresponding to a (different) compact set. Thus we may apply Example 4.1 to (4.3).

See [10] for another approach.

Example 4.4. Suppose O is an open set in the complex plane, such that the boundary, ∂O, is a finite union of piecewise smooth, orientable (possibly unbounded) mutually nonintersecting arcs and the complement of O contains a half-line, the spectrum of A is contained in O, g is holomorphic in O, and for all $t \geq 0$, $|e^{tz}g(z)|$ is $O(|z|^{-2})$, for $z \in O$, and

$$\int_{\partial O} |e^{tz}g(z)| \|(z-A)^{-1}\| \, d|z| < \infty.$$

Then

$$W(t) \equiv \frac{1}{2\pi i} \int_{\partial O} e^{tz} g(z)(z-A)^{-1} \, dz$$

may be shown to define a $W(0)$-regularized semigroup for A (see [9, chapter XXII] for the details when $g(z) = (\lambda - z)^{-m}$).

In general, it is not clear when $W(0)$ is injective.

Example 4.5. The backward heat equation on a bounded domain $\Omega \subseteq \mathbb{R}^n$, with Dirichlet boundary conditions, is

$$\begin{aligned} &\frac{\partial}{\partial t} u(s,t) + \triangle u(s,t) = 0 \quad (s \in \Omega, \ t \geq 0) \\ &u(s,t) = 0 \quad (s \in \partial\Omega, \ t \geq 0) \\ &u(s,0) = f(s) \quad (s \in \Omega), \end{aligned}$$

where $\triangle$ is the Laplacian. For simplicity, we will assume that $\partial\Omega$, the boundary of Ω, is smooth.

This is the abstract Cauchy problem (0.1), when $-A$ generates a strongly continuous holomorphic semigroup of angle $\frac{\pi}{2}$. We may apply Example 4.4, with $O \equiv \{re^{i\phi} : r > 0, \, |\phi| < \theta\}$, for $0 < \theta < \frac{\pi}{4}, g(z) \equiv e^{-z^2}$.

In general, this is giving us reversibility of parabolic problems, for initial data in a dense set. See [9] or [8] for other approaches.

Example 4.6. We will now consider arbitrary systems of constant coefficient partial differential initial-value problems. We need some standard multivariable terminology.

Terminology 4.7. We will write $s = (s_1, \ldots s_n)$, for vectors in $\mathbb{R}^n$, $\alpha = (\alpha_1, \ldots \alpha_n)$ for vectors in $(\mathbb{N} \cup \{0\})^n$. We will write $s^\alpha \equiv s_1^{\alpha_1} \cdots s_n^{\alpha_n}, |s|^2 \equiv \sum_{k=1}^n |s_k|^2$, $|\alpha| \equiv \sum_{k=1}^n \alpha_k$.

For $1 \leq k \leq n$, let D_k be $i\frac{\partial}{\partial s_k}$, on a Banach space of complex-valued functions on $\mathbb{R}^n$, $X \equiv L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$, $BUC(\mathbb{R}^n), C_0(\mathbb{R}^n)$, or any space where translation is strongly continuous and uniformly bounded. We will write D for $(D_1, \ldots, D_n), D^\alpha$ for $(D_1)^{\alpha_1} \cdots (D_n)^{\alpha_n}$.

The Laplacian, $\triangle$, is defined as the generator of the strongly continuous semigroup

$$(e^{t\triangle} f)(s) \equiv (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4t}} f(s-y)\, dy.$$

Note that

$$\triangle = |D|^2 \equiv \sum_{k=1}^n (\frac{\partial}{\partial s_k})^2.$$

More generally, if p is a polynomial

$$p(s) \equiv \sum_{|\alpha| \leq N} a_\alpha s^\alpha,$$

then the constant coefficient differential operator $p(D)$ is defined by

$$p(D) \equiv \sum_{|\alpha| \leq N} a_\alpha D^\alpha,$$

with $\mathcal{D}(p(D)) \equiv \{f \in X : p(D)f \in X\}$, where $p(D)$ is taken in the sense of distributions.

We show in [9, Theorem 12.12] that $\mathcal{D}(\triangle^\ell) \subseteq \mathcal{D}(p(D))$, whenever $2\ell > N + \frac{n}{2}$.

Let $\mathcal{M} \equiv (p_{i,j})_{i,j=1}^m$ be an $m \times m$ matrix of polynomials. Let

$$N \equiv \max_{i,j}\{\text{degree of } p_{i,j}\}.$$

Define the operator $\mathcal{M}(D)$, on X^m, by

$$\mathcal{M}(D) \equiv (p_{i,j}(D)), \quad \mathcal{D}(\mathcal{M}(D)) \equiv \mathcal{D}(\triangle^\ell)^m, \;\; \ell \equiv 1 + [\frac{1}{2}(N + \frac{n}{2})].$$

Note that any system of constant coefficient partial differential initial-value problems may be written as (0.1), with $A \equiv \mathcal{M}(D)$, for some matrix of polynomials $\mathcal{M}$, that is,

$$\frac{d}{dt}\vec{u}(t, \vec{f}) = \mathcal{M}(D)\vec{u}(t, \vec{f}) \quad (t \geq 0), \quad \vec{u}(0, \vec{f}) = \vec{f}, \tag{4.8}$$

where $\vec{u}(t, \vec{f}) \in X^m$, for all $t \geq 0$.

The matrix of polynomials $\mathcal{M}$ is said to be *Petrowsky correct* if there exists $\omega \in \mathbb{R}$ such that for all $s \in \mathbb{R}^n$,

$$\sigma(\mathcal{M}(s)) \subseteq \{z \in \mathbb{C} : Re(z) \leq \omega\}.$$

Definition 4.9. We have shown (see [9, Theorems 13.9 and 14.1]) that an extension of $\mathcal{M}(D)$ generates a C-regularized semigroup, for appropriate C, that leaves the domain of $\mathcal{M}(D)$ invariant (see also [12] for the Petrowsky correct case). Thus $\mathcal{M}(D)$ is closable; let

$$A \equiv \overline{\mathcal{M}(D)}.$$

Then Theorem 3.1 and [9, Theorems 13.9 and 14.1] give us the following.

Theorem 4.10.

(a) *There exists injective C, with dense range, such that A has a C-regularized semigroup.*

(b) *If $\mathcal{M}$ is Petrowsky correct, then there exists $r \geq 0$ such that A has an exponentially bounded $(1 + \triangle)^{-r}$-regularized semigroup.*

This produces strong solutions of (4.8), for all $\vec{f}$ in a dense set. When $\mathcal{M}$ is Petrowsky correct, this produces mild solutions of (4.8), for all $\vec{f}$ in $\mathcal{D}(\triangle^r)$.

We remark that, in this example, it may be difficult to write down the domain of the generator of the C-regularized semigroup.

Example 4.6 is clearly a wide class of examples, that includes higher order constant coefficient initial-value problems. We give here two simple specific special cases, and refer the reader to [11] for others.

Example 4.11. The wave equation on $\mathbb{R}^n$ is

$$(\frac{\partial}{\partial t})^2 u(t,s) = \triangle u(t,s), \quad u(0,s) = f_1(s), \quad \frac{\partial}{\partial t} u(0,s) = f_2(s) \quad (t \geq 0, s \in \mathbb{R}^n).$$

This is clearly (4.8), with

$$\mathcal{M}(s) \equiv \begin{bmatrix} 0 & 1 \\ -|s|^2 & 0 \end{bmatrix}.$$

A simple calculation shows that this is Petrowsky correct.

Example 4.12. The equation describing sound propagation in a viscous gas is

$$(\frac{\partial}{\partial t})^2 u(t,s) = 2\frac{\partial^2}{\partial t \partial s^2} u(t,s) + \frac{\partial^2}{\partial s^2} u(t,s), \quad u(0,s) = f_1(s), \quad \frac{\partial}{\partial t} u(0,s) = f_2(s)$$
$$(t \geq 0,\ s \in \mathbb{R}).$$

This is (4.8) with

$$\mathcal{M}(s) \equiv \begin{bmatrix} 0 & 1 \\ -s^2 & -2s^2 \end{bmatrix}.$$

Again, this may be shown to be Petrowsky correct.

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