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REGULARIZING EFFECTS OF HOMOGENEOUS EVOLUTION EQUATIONS

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ABSTRACT

It is well-known that solving the initial-value problem for the heat equation forward in time takes a "rough" initial temperature into a temperature which is smooth at later times $t > 0$. One aspect of this is the validity of certain estimates on tu_t when u is a solution of the heat equation. In this paper we prove related estimates on nonlinear evolution equations which are governed by homogeneous nonlinearities. The results apply to classes of nonlinear diffusion equations and to conservation laws. The results are interesting from the point of view of identifying a new "regularization" mechanism and the estimates themselves cast new light on the nature of the solutions of some initial-value problems with rough initial data.

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SIGNIFICANCE AND EXPLANATION

It is well-known that solving the initial-value problem for the heat equation forward in time takes a "rough" initial temperature into a temperature which is smooth at later times $t > 0$. One aspect of this is the validity of certain estimates on tu_t when u is a solution of the heat equation. In this paper we prove related estimates on nonlinear evolution equations which are governed by homogeneous nonlinearities. The results apply to classes of nonlinear diffusion equations and to conservation laws. The results are interesting from the point of view of identifying a new "regularization" mechanism and the estimates themselves cast new light on the nature of the solutions of some initial-value problems with rough initial data.

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REGULARIZING EFFECTS OF HOMOGENEOUS EVOLUTION EQUATIONS

Michael G. Crandall and Philippe Bénilan

Introduction

Each of the three evolution equations

$$(1)_\alpha \qquad \frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(|u|^{\alpha-1} u) \qquad t > 0,\ x \in \mathbb{R} ,$$

$$(2)_\alpha \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(|u|^{\alpha-1} u) \qquad t > 0,\ x \in \mathbb{R} ,$$

and

$$(3)_\alpha \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{\alpha-1} \frac{\partial u}{\partial x}\right) \qquad t > 0,\ x \in \mathbb{R} ,$$

may, if $\alpha > 0$, be solved subject to the initial condition

$$\text{(IC)} \qquad u(x,0) = u_0(x), \qquad x \in \mathbb{R} ,$$

where $u_0 \in L^1(\mathbb{R})$ for a solution $u(x,t)$ in such a way that $t \to u(\cdot,t)$, which we call $u(t)$, is a continuous curve in $L^1(\mathbb{R})$ and if $S(t)$ is defined by $S(t)u_0 = u(t)$ then it is a nonexpansive self-map of $L^1(\mathbb{R})$ for each $t \geq 0$. (This is discussed further in Section 2, but the reader need know nothing of these matters beforehand.) A main result of this note, as applied to these three problems, establishes that

$$(4) \qquad \lim_{h \downarrow 0} \frac{\|u(t+h)-u(t)\|}{h} = \frac{2\|u_0\|}{|\alpha-1|t} ,$$

with $\|\ \|$ the norm of $L^1(\mathbb{R})$, provided $\alpha \neq 1$. Indeed, it is shown that (4) holds for any evolution equation $u'(t) = B(u(t))$ in which B is homogeneous of degree $\alpha > 0$, $\alpha \neq 1$, and for which there is an associated nonexpansive family $S(t)$. In addition, each of the problems $(1)_\alpha$, $(2)_\alpha$, $(3)_\alpha$ give rise to operators $S(t)$ which respect the natural order on $L^1(\mathbb{R})$. It will follow from another abstract result of this paper that therefore the pointwise estimate

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(5) $$(\alpha-1)\frac{\partial u}{\partial t} \geq -\frac{1}{t}u \qquad t > 0,\ x \in \mathbb{R} \quad ,$$

holds for nonnegative solutions of any once of the above problems ((5) being understood in the sense of distributions).

Estimates of the form (4), (5) are types of "regularizing effects" in that the quantities estimated for $t > 0$ need not be sensible at $t = 0$. We comment on the rather subtle implications of the estimates (4) and (5) in particular cases at some length in Section 2, and we pay there due respect to the difference between the assertion (4), which is an estimate of the "speed", and the stronger assertion that the "velocity" $u'(t)$ exists and admits the corresponding estimate.

This note is divided into two sections. Section 1 presents the abstract results concerning solutions of the equation $u' = B(u)$ and its perturbations under various assumptions (always including that B is homogeneous). These results are elementary estimates on the difference quotients $h^{-1}(u(t+h)-u(t))$. Section 2 discusses the interaction of the abstract results with particular problems, including those listed above, and it is partly expository.

Estimates in evolution problems of velocities $u'(t)$ by expressions involving $1/t$ are familiar in several contexts. Perhaps the closest in spirit to those given here occur when B is the (linear) infinitesimal generator of a strongly continuous semigroup, in which case an estimate of $h^{-1}\|u(t+h)-u(t)\|$ in the form $C\|u(0)\|/t$ is essentially equivalent to B being the generator of a holomorphic semigroup (see, e.g., [17], [29]). (This is the case for the linear problems $(1)_1 = (3)_1$ in a variety of spaces.) Another known case is the result of Brezis [7, chp. III] which applies if $B = \partial\phi$ is the subdifferential of a convex function on a Hilbert space. Brezis' estimates apply to variants of $(1)_\alpha$, $(3)_\alpha$ to give L^2 based results like (4) which do not use the homogeneity of the right-hand sides. See [3, pg. 200]. Other regularizing effects are to be found in [5], [16], [26].

Section 1

Let $\| \ \|$ be a semi norm on the vector space X and $B : D(B) \subseteq X \to X$ be an operator in X which is homogeneous of degree $\alpha > 0$. That is

$$\text{(H)} \qquad B(rx) = r^{\alpha}B(x) \quad \text{for} \quad r \geq 0 \quad \text{and} \quad x \in D(B) \quad ,$$

where it is understood in (H) that $rD(B) \subseteq D(B)$ for $r \geq 0$. We are interested in the evolution problem

$$\text{(E)} \qquad \begin{cases} \dfrac{du}{dt} = B(u) \\ u(0) = x \end{cases} ,$$

but rather than deal with (E) directly we shall work here only with its solutions. These solutions are assumed to be presented to us by some theory or construction in the form $u(t) = S(t)x$ where each $S(t)$, $t \geq 0$, is a mapping $S(t) : C \to X$ and C is subset of X. The property (H) of B is taken to be reflected in S by the identities

$$\text{(HS)} \qquad \lambda^{\frac{1}{\alpha-1}} S(\lambda t)x = S(t)(\lambda^{\frac{1}{\alpha-1}} x) \quad \text{for} \quad t,\lambda \geq 0 \quad \text{and} \quad x \in C \quad .$$

This is arrived at in the following way: If $u(t)$ is a classical solution of (E) and (H) holds and $\lambda > 0$, then $v(t) = \lambda^{\frac{1}{\alpha-1}} u(\lambda t)$ satisfies $\frac{dv}{dt}(t) = \lambda^{\frac{1}{\alpha-1}} \lambda u'(\lambda t) = \lambda^{\frac{\alpha}{\alpha-1}} B(u(\lambda t)) = B(\lambda^{\frac{1}{\alpha-1}} u(\lambda t)) = B(v(t))$ so that v is again a classical solution of (E) and further satisfies $v(0) = \lambda^{\frac{1}{\alpha-1}} u(0)$. The corresponding property of the notion of solution of (E) provided by S is what is requested by (HS). It is understood in (HS) that $rC \subseteq C$ for $r \geq 0$. The other major requirement we place upon S is the Lipschitz condition

$$\text{(L)} \qquad \|S(t)x - S(t)\hat{x}\| \leq L\|x-\hat{x}\| \quad \text{for} \quad t \geq 0, \ x,\hat{x} \in C \quad ,$$

where $\| \ \|$ is the norm in X.

Theorem 1. Let $C \subseteq X$ and $S(t) : C \to X$ for $t \geq 0$ and satisfy (HS) with $\alpha > 0$, $\alpha \neq 1$, (L) and $S(t)0 \equiv 0$. Then for $x \in C$ and $t,h > 0$,

$$
(6) \quad \begin{cases} \|S(t+h)x - S(t)x\| \le 2L\|x\| \; \left|1 - (1 + \frac{h}{t})^{\frac{1}{1-\alpha}}\right| \; , \\ \text{and, in particular,} \\ \limsup\limits_{h \downarrow 0} \frac{\|S(t+h)x - S(t)x\|}{h} \le (\frac{2L\|x\|}{|\alpha-1|})\frac{1}{t} \; . \end{cases}
$$

Proof. We use (HS) with

$$
(7) \qquad \lambda = (1 + \frac{h}{t})
$$

to compute

$$
(8) \qquad S(t+h)x - S(t)x = S(\lambda t)x - S(t)x = \lambda^{\frac{1}{1-\alpha}} S(t)(\lambda^{\frac{1}{\alpha-1}} x) - S(t)x
$$

$$
= \lambda^{\frac{1}{1-\alpha}}(S(t)(\lambda^{\frac{1}{\alpha-1}}x) - S(t)x) + (\lambda^{\frac{1}{1-\alpha}}-1)S(t)x \; ;
$$

Now (8), (L) and $S(t)0 \equiv 0$ imply

$$
\|S(t+h)x - S(t)x\| \le \lambda^{\frac{1}{1-\alpha}} L|\lambda^{\frac{1}{\alpha-1}}-1| \; \|x\| + |\lambda^{\frac{1}{1-\alpha}}-1|L\|x\|
$$

$$
= 2L\|x\|L \; |1 - \lambda^{\frac{1}{1-\alpha}}| \; .
$$

The estimates (6) follow from this and (7).

For the next result we assume that X is equipped with a relation $\ge$ under which it is an ordered vector space and that $S(t)$ respects this order.

Theorem 2. Let X be an ordered vector space with the order relation denoted by $\ge$. Let $S(t)$ satisfy

$$
(O) \qquad S(t)x \ge S(t)y \quad \text{if} \quad x,y \in C \quad \text{and} \quad x \ge y \; ,
$$

and satisfy (HS) with $\alpha > 0$, $\alpha \ne 1$. If $x \in C$, $x \ge 0$ and $t,h > 0$ then

$$
(9) \qquad (\alpha-1)(S(t+h)x - S(t)x) \ge (\alpha-1)((1 + \frac{h}{t})^{\frac{1}{1-\alpha}}-1)S(t)x \; .
$$

Proof. There are two cases. If $\alpha > 1$, we return to (8) and observe that $\lambda = (1 + \frac{h}{t}) > 1$ and $\lambda^{\frac{1}{\alpha-1}} > 1$. Thus $\lambda^{\frac{1}{\alpha-1}} x \ge x$ and so $\lambda^{\frac{-1}{\alpha-1}}(S(t)(\lambda^{\frac{1}{\alpha-1}} x) - S(t)x) \ge 0$ by (O). The inequality (9) is obtained by dropping this nonnegative term from the right hand side of (8) and multiplying by $\alpha-1 > 0$. The parallel reasoning if $0 < \alpha < 1$ shows that the term just dropped is now nonpositive, so we have the opposite inequality than above coming from (8), which becomes (9) again upon multiplication by $\alpha-1 < 0$.

In applications of Theorem 2 there is sometimes a nonnegative linear functional $\Lambda : X \to \mathbb{R}$ which is preserved by $S(t)$, i.e.

$$\Lambda S(t)x = \Lambda x \quad \text{for} \quad t \geq 0,\ x \in C,\ x \geq 0 \quad , \tag{10}$$

and X is a lattice. The notation $x^+ = \sup\{x,0\}$, $x^- = -\inf\{x,0\}$ will be used.

<u>Corollary 3</u>. In addition to the conditions of Theorem 2 assume that X is a vector lattice, Λ is a nonnegative linear functional on X and (10) holds. Let $x \in C$, $x \geq 0$ and $u(t) = S(t)x$. The following estimate is valid for $t,h > 0$ and $\nu \in \{+,-\}$:

$$\Lambda((u(t+h) - u(t))^\nu) \leq \left|1 - \left(1 + \frac{h}{t}\right)^{\frac{1}{1-\alpha}}\right| \Lambda x \quad . \tag{11}$$

<u>Proof</u>. From (10) we have

$$\Lambda((u(t+h) - u(t))^+) - \Lambda((u(t+h) - u(t))^-) = \Lambda u(t+h) - \Lambda u(t) = 0 \quad . \tag{12}$$

From (9) and $S(t)x \geq 0$ we also have

$$\begin{cases} (u(t+h) - u(t))^- \leq \left(1 - \left(1 + \frac{h}{t}\right)^{\frac{1}{1-\alpha}}\right) S(t)x & \text{if} \quad \alpha > 1 \\ (u(t+h) - u(t))^+ \leq \left(\left(1 + \frac{h}{t}\right)^{\frac{1}{1-\alpha}} - 1\right) S(t)x & \text{if} \quad 0 < \alpha < 1 \quad . \end{cases} \tag{13}$$

Applying Λ to the inequalities (13) and using (12) implies (11).

<u>Remarks</u>. If Λ is as above and if (10) holds, then (O) is essentially equivalent to the property

$$\Lambda((S(t)x - S(t)y)^+) \leq \Lambda(x-y)^+ \quad . \tag{14}$$

See [14]. Hence (11) represents a slight refinement of (6) with $\|x\| = \Lambda x^+$. Also the proof shows (11) is valid for $\alpha > 1$ if (10) is weakened to $\Lambda S(t)x \leq \Lambda x$.

We turn now to the "forced" problem

$$\text{(FE)} \qquad \begin{cases} \dfrac{du}{dt} = B(u) + f(t) \\ u(0) = x \end{cases}$$

where $f : [0,T] \to X$ for some $T > 0$. Again, it is most efficient to assume that solutions of (FE) are presented to us in the form $u(t) = S(t,x,f)$ and lay our conditions directly upon S. Computing the equation satisfied by $v(t) = \lambda^{\frac{1}{\alpha-1}} u(\lambda t)$ if u is a classical solution of (FE) and B is homogeneous of degree α leads to

$$\text{(15)} \qquad v'(t) = B(v(t)) + \lambda^{\frac{\alpha}{1-\alpha}} f(\lambda t) \quad .$$

In order to minimize bookkeeping problems we will assume simply that X is a Banach space and $C \subseteq X \times L^1_{loc}(0,\infty : X)$ is given together with

$$S : [0,\infty) \times C \to X$$

such that $u(t) = S(t,x,f)$ is the solution of (FE) of interest for $(x,f) \in C$. We let

$$\text{(16)} \qquad f_\lambda(t) = f(\lambda t)$$

and assume $(x,f) \in C \Rightarrow (x, \lambda^{\frac{\alpha}{1-\alpha}} f_\lambda) \in C$ for $\lambda \geq 0$. The equation (15) satisfied by $\lambda^{\frac{1}{1-\alpha}} u(\lambda t)$ is to be reflected in S by

$$\text{(FH}_S\text{)} \qquad \lambda^{\frac{1}{\alpha-1}} S(\lambda t,x,f) = S(t, \lambda^{\frac{1}{\alpha-1}} x, \lambda^{\frac{\alpha}{\alpha-1}} f_\lambda) \quad .$$

Motivated by known existence theories (see Section 2) the Lipschitz condition (L) is generalized to

$$\text{(FL)} \qquad \|S(t,x,f) - S(t,\hat{x},\hat{f})\| \leq L(\|x-\hat{x}\| + \int_0^t \|f(\tau) - \hat{f}(\tau)\| d\tau)$$

when the arguments lie in the domain of (S).

<u>Theorem 4</u>. Let S satisfy (FH$_S$), (FL) and $S(t,0,0) \equiv 0$. If $t,h > 0$, $(x,f) \in C$, $\alpha > 0$, $\alpha \neq 1$ and $u(t) = S(t,x,f)$, then

(17) $\|u(t+h) - u(t)\| \le L(|1 - (1 + \frac{h}{t})^{\frac{1}{1-\alpha}}|(2\|x\| + \int_0^t \|f(\tau)\|d\tau)$

$$+ |(1 + \frac{h}{t}) - (1 + \frac{h}{t})^{\frac{1}{1-\alpha}}|\int_0^t \|f(\tau + \frac{h}{t}\tau)\|d\tau + (1 + \frac{h}{t})^{\frac{1}{1-\alpha}}\int_0^t \|f(\tau + \frac{h}{t}\tau) - f(\tau)\|d\tau) .$$

In particular, if

(18)
$$V(t,f) = \limsup_{\xi \downarrow 0} \int_0^t \frac{\|f(\tau+\xi\tau) - f(\tau)\|}{\xi}\,d\xi$$

then

(19)
$$\limsup_{h \downarrow 0} \frac{\|u(t+h) - u(t)\|}{h} \le \frac{L}{t}\left(\frac{2\|x\| + (1+\alpha)\int_0^t \|f(\tau)\|d\tau}{|\alpha-1|} + V(t,f)\right)$$

and u is Lipschitz continuous on each compact subset of $(0,T]$ if $V(T,f) < \infty$.

Proof. Of course the argument is just as before. The relation (FH_S) yields, with $\lambda = (1 + \frac{h}{t})$,

(20)
$$\begin{aligned} u(t+h) - u(t) &= S(\lambda t,x,f) - S(t,x,f) \\ &= \lambda^{\frac{1}{1-\alpha}}[(S(t,\lambda^{\frac{1}{\alpha-1}}x,\lambda^{\frac{\alpha}{\alpha-1}}f_\lambda) - S(t,x,f_\lambda)) + (S(t,x,f_\lambda) - S(t,x,f))] \\ &\quad + (\lambda^{\frac{1}{1-\alpha}}-1)S(t,x,f) . \end{aligned}$$

Using (LH) in conjunction with (20) proves (17) and (19) follows by taking the indicated limit.

Remark 5. If f is absolutely continuous and differentiable almost everywhere on each compact subset of $(0,T]$, then

(21)
$$V(t,f) = \int_0^t \tau\|f'(\tau)\|d\tau .$$

In general $V(T,f) < \infty$ is equivalent to $t \to tf(t)$ being of (essentially) finite variation on $[0,T]$.

Remark 6. It is quite interesting that Theorem 2 has a forced analogue. If $S(t,x,f)$ is nondecreasing in x and f (where $f \geq g$ means $f(t) \geq g(t)$ a.e.) and $t \to (\alpha-1)(t^{\frac{\alpha}{\alpha-1}}f(t))$ is nondecreasing in t, then (20) implies (9) with $u(t) = S(t,x,f)$ in place of $S(t)x$.

The final abstract case we consider is (E) perturbed by a Lipschitz continuous function $p : D(p) \subseteq X \to X$. That is

$$\text{(PE)} \qquad \begin{cases} \dfrac{du}{dt} = B(u) + p(u) \\ u(0) = x \end{cases}$$

where p satisfies

$$\text{(22)} \qquad \|p(x) - p(y)\| \leq M\|x-y\| \quad \text{for } x,y \in D(p)$$

and some $M \geq 0$. We regard (PE) as a special case of (FE) in the sense that we assume solutions $S(t,x,f)$ of (FE) are known and understand a solution u of (PE) to be a function u with values in $D(p)$ such that $(x,p(u)) \in C$ and $u(t) = S(t,x,p(u))$. The results will be a modulus of continuity of any solution u of (PE).

Theorem 7. Let S satisfy the assumptions of Theorem 4 with $\alpha > 0$, $\alpha \neq 1$. Let $p : D(p) \subseteq X \to X$ satisfy (22), $u \in C([0,\infty) : D(p))$, $(x,p(u)) \in C$ and $u = S(t,x,p(u))$. Then for each $T > 0$

$$\sup_{\substack{0 \leq t \leq T \\ 0 \leq h \leq t}} \frac{t}{h}\|u(t+h) - u(t)\| \leq c(T,\alpha,\|x\|,L,M)$$

where the right hand side above depends only on the indicated quantities. In particular, u is Lipschitz continuous on compact subsets of $(0,T]$ for each $T > 0$.

Proof. The Lipschitz condition (22) implies

$$\text{(23)} \qquad \|p(u(t))\| \leq a + M\|u(t)\|$$

for some a. Using (23), (LF) and $S(t,0,0) \equiv 0$ one deduces that

$$\|u(t)\| = \|S(t,x,p(u)) - S(t,0,0)\|$$

$$\leq L(\|x\| + at + M\int_0^t \|u(\tau)\|d\tau)$$

from which flows the estimate

(24) $$\|u(t)\| \leq L(\|x\| + aT)e^{LMT} \quad \text{for} \quad 0 \leq t \leq T \ .$$

Next we use (17) with $f(t) = p(u(t))$ and the estimates (23), (24) to conclude that for $T > 0$ and $0 \leq t \leq t+h \leq T$ there is a constant $C = C(T,\alpha,\|x\|,L,M)$ for which

(25) $$\frac{t}{h}\|u(t+h) - u(t)\| \leq C[\frac{t}{h}|1 - (1 + \frac{h}{t})^{\frac{1}{1-\alpha}}| + \frac{t}{h}|(1 + \frac{h}{t}) - (1 + \frac{h}{t})^{\frac{1}{1-\alpha}}|$$

$$+ \frac{t}{h}(1 + \frac{h}{t})^{\frac{1}{1-\alpha}} \int_0^t M\|u(\tau + \frac{h}{t}\tau) - u(\tau)\|d\tau] \ .$$

Set $\xi = h/t$ above and

(26) $$g(t,\xi) = \frac{\|u(t(1+\xi)) - u(t)\|}{\xi} \ .$$

Then (25) implies

(27) $$g(t,\xi) \leq C[1 + \int_0^t g(\tau,\xi)d\tau]$$

for some new constant C and $0 \leq t \leq T/(1+\beta)$ $0 \leq \xi \leq \beta \leq 1$, where β is chosen in $(0,1]$. The estimate (27) gives a new estimate

(28) $$g(t,\xi) \leq \hat{C} \quad \text{for} \quad 0 \leq t \leq T/(1+\beta), \ 0 \leq \xi \leq \beta \leq 1$$

where $\hat{C}$ is yet another constant, whose precise structure we leave to the reader, but depends only on allowed quantities. T being arbitrary the proof is complete.

Section 2. Examples and Applications

We begin by reviewing one abstract theory for (E) which guarantees that (FHS) and (FL) hold whenever B satisfies (H) and one additional condition. The theory encompasses the three classes of examples $(1)_\alpha$, $(2)_\alpha$, $(3)_\alpha$ and generalizations of them as well as the equation

$$(26)_\alpha \qquad \frac{\partial u}{\partial t} = \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{\alpha-1} \frac{\partial u}{\partial x_i} \qquad t > 0,\ x \in \mathbb{R}^N$$

and a host of other possibilities.

Following this we discuss briefly the two classes of examples $(1)_\alpha$, $(2)_\alpha$ in their simplest setting to make various points and orient the reader. We make no attempt to write down the new results which obviously flow from the estimates of Section 2 even as applied to the examples mentioned here.

Given a Banach space X and $B : D(B) \subseteq X \to X$, $T > 0$, and $f \in L^1(0,T : X)$ we call $u \in C([0,T] : X)$ a <u>mild solution</u> of

$$(\mathrm{EF}) \qquad u' = B(u) + f$$

on $[0,T]$ provided for every $\varepsilon > 0$ we can find a partition $\{0 = t_0 < t_1 < \cdots < t_n\}$ of $[0,t_n]$ and finite sequences $\{x_i\}_{i=0}^n$, $\{f_i\}_{i=1}^n$ in X such that

$$(27) \qquad \begin{cases} \text{(i)} & \dfrac{x_{i+1} - x_i}{t_{i+1} - t_i} - B(x_{i+1}) = f_{i+1}, & i = 0,1,\ldots,n-1 \\ \text{(ii)} & t_{i+1} - t_i < \varepsilon & i = 1,\ldots,n-1 \\ \text{(iii)} & 0 \le T - t_n < \varepsilon & \\ \text{(iv)} & \displaystyle\sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \|f_i - f(s)\|\,ds < \varepsilon\ , & \end{cases}$$

and

$$(28) \qquad \|u_\varepsilon(t) - u(t)\| \le \varepsilon \quad \text{on} \quad [0,t_n)$$

where

(29) $$u_\varepsilon(t) = x_i \quad \text{for} \quad t_i \le t < t_{i+1},\ i = 1,\dots,n-1 \ .$$

A function u_ε, piecewise constant as in (29), is called an <u>ε-approximate solution</u> of (EF)' when the various conditions of (27) are satisfied. Roughly, (27) defines a simple implicit Euler approximation of (EF)' and we are <u>defining</u> solutions of (EF)' to be the uniform limits of solutions of these difference approximations. We have:

<u>Proposition 8</u>. Let B be homogeneous of degree $\alpha > 0$, $\alpha \neq 1$. Let $T > 0$, $\lambda > 0$ and $f \in L^1(0,T : X)$. If $u \in C([0,T] : X)$ is a mild solution of (EF), then $v(t) = \lambda^{\frac{1}{\alpha-1}} u(\lambda t)$ is a mild solution of (EF) on $[0,T/\lambda]$ with f replaced by $\lambda^{\alpha/\alpha-1} f(\lambda t)$.

The proof is left to the reader. If B is also dissipative (equivalently, $-B$ is accretive - see, e.g., [3], [11], [16]) one has:

<u>Proposition 9</u>. Let B be dissipative. Let $x \in$ closure$(D(B))$, $T > 0$ and $f \in L^1(0,T : X)$. If for each $\varepsilon > 0$ there is an ε-approximate solution u_ε of (EF)' satisfying $\|u_\varepsilon(0) - x\| < \varepsilon$ then (EF)' has a mild solution u on $[0,T]$. Moreover, if $f,\hat{f} \in L^1(0,T : X)$ and $u,\hat{u}$ are mild solutions of $u' = B(u) + f$, $\hat{u}' = B(\hat{u}) + \hat{f}$ respectively, then

$$\|u(t) - \hat{u}(t)\| \le \|u(0) - \hat{u}(0)\| + \int_0^t \|f(s) - \hat{f}(s)\| ds$$

for $0 \le t \le T$.

This is proved in [13], although the definition of "mild solution" is not given there. See also [18].

It follows from Propositions 8 and 9 that letting $u(t) = S(t,u(0),f)$ when u is a mild solution of (EF)' and B is dissipative and homogeneous of degree α defines an operator S with the desired properties (FHS) and (FL) with $L = 1$. (The special case $f = 0$ gives (HS) and (L) with $L = 1$.)

Associated with each of the problems $(1)_\alpha$, $(2)_\alpha$, $(3)_\alpha$, $\alpha > 0$, is a densely defined m-dissipative operator B_α in $L^1(\mathbb{R})$, that is B_α is dissipative and the range of $(I - \lambda B_\alpha)$ is $L^1(\mathbb{R})$ for $\lambda > 0$. This in conjunction with Propositions 8 and 9, guarantees the existence of mild solutions. Moreover, each operator S so obtained is order preserving with respect to the natural order on L^1. This provides one precise sense in which these problems fall under the scope of this paper. (One may, of course, treat these problems by any other suitable method which provides the information (FHS) and (FL), etc.) Some references are: (i) [6] which shows how to make precise the m-dissipative operator in $L^1(\mathbb{R}^N)$ associated with equations $u_t = \Delta\varphi(u)$ for more general nonlinearities than in $(1)_\alpha$ and in any number of dimensions N, (ii) [4] and [9] which contain results defining m-dissipative operators associated with initial-boundary value problems for $u_t = \Delta\varphi(u)$, (iii) [2], [23] which contain results defining m-dissipative operators in L^p spaces, $1 \le p \le \infty$, associated with variants of $(3)_\alpha$ (which must be modified for the pure initial value problem), (iv) [10], [4] which establish m-dissipative operators for generalizations of $(2)_\alpha$. The equation $(26)_\alpha$, $1 < \alpha \le 2$, corresponds to an m-dissipative operator in the space of uniformly continuous functions on $\mathbb{R}^N$, as is proved in [25].

Of course, there is a huge literature concerning other approaches and results for these problems. We continue in this section by choosing $(1)_\alpha$ and $(2)_\alpha$ for further discussion to illustrate the significance of the results of Section 1 in applications and something of the relationship with known results.

The problem $(1)_\alpha$ for $\alpha = 1$ is the initial-value for the linear heat equation which is solved by

$$(30) \qquad u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-|x-y|^2}{4t}} u_0(y)\,dy \quad .$$

If X is any one of the Banach spaces $L^p(\mathbb{R})$, $1 \le p < \infty$, of $BU(\mathbb{R})$ (the bounded uniformly continuous functions on $\mathbb{R}$) equipped with the usual norm and $u_0 \in X$, then $t \to u(\cdot,t)$ with u given by (30) is a continuous curve in X for $t > 0$. Moreover $u(t) = u(\cdot,t) \to u_0$ in X as $t \downarrow 0$. In each space X define an operator B by

$$(31) \qquad D(B) = \{v \in X : v'' \in X\} \text{ and } Bv = v'' \text{ for } v \in D(B) \quad ,$$

where differentiation is in the sense of distributions. Then it is very well-known that B is m-dissipative and the mild solution of $u' = Bu$, $u(0) = u_0$ is $S(t)u_0 = u(t)$. Direct examination of (30) shows that if $u_0 \in X$, than $u(t)$ is differentiable, $u(t) \in D(B)$ and

$u'(t) = Bu(t)$ for $t > 0$. Moreover

$$\left\|\frac{du}{dt}(t)\right\| = \|Bu(t)\| \le \frac{c}{t}\|u(0)\| \quad \text{for} \quad t > 0 \quad . \tag{32}$$

Thus one has very explicit regularizing here, most convincingly illustrated by the formula (30). The estimate (32) implies that $S(t)$ is an analytic semigroup in X (see, e.g., [17], [29]). We note that while (32) has much the same character as the estimate (6) of Theorem 1, Theorem 1 does not apply here for $\alpha = 1$.

If $\alpha > 0$ and $\alpha \neq 1$, we do not know a formula for the solution of (1). However, the operator B_α in $X = L^1(\mathbb{R})$ given by

$$\begin{cases} D(B_\alpha) = \{v \in L^1(\mathbb{R}) : |v|^{\alpha-1}v \in L^1_{loc}(\mathbb{R}) \text{ and } (|v|^{\alpha-1}v)'' \in L^1(\mathbb{R})\} \\ B_\alpha v = (|v|^{\alpha-1}v)'' \quad \text{for} \quad v \in D(B_\alpha) \end{cases} \tag{33}$$

is m-accretive in $L^1(\mathbb{R})$ ([6]). The mild solutions provided by this B_α are uniquely characterized as solutions of $(1)_\alpha$ in the sense of distributions (see [8]). Thus for $\alpha > 0$, $\alpha \neq 1$ Theorems 1 and 2 apply with these choices and we conclude that the solution u of $(1)_\alpha$ satisfies

$$\limsup_{h\downarrow 0} \int_{\mathbb{R}} \frac{|u(x,t+h) - u(x,t)|}{h}\,dx \le \frac{1}{t}\,\frac{2}{|\alpha-1|}\int_{\mathbb{R}} |u_0(x)|\,dx \quad , \tag{34}$$

and also

$$\frac{\partial u}{\partial t} \ge -\frac{1}{(\alpha-1)t}\, u \tag{35}$$

provided $u_0 \ge 0$ so $u \ge 0$. The analogue of (34) in $L^p(\mathbb{R})$ or $BU(\mathbb{R})$ does not hold in view of explicit examples. See, e.g. [24]. The relation (35) follows from Theorem 2 applied to this example by dividing (9) by h and letting $h \downarrow 0$. (The limit of $(u(t+h,x) - u(t,x))/h$ is taken in the sense of distributions.) The curve $t \to u(t)$ in $L^1(\mathbb{R})$ solving $(1)_\alpha$ thus has a "speed" bounded in the form c/t for $t > 0$, as was true in the linear case, but we cannot so easily assert here the existence of the velocity $\lim_{h\to 0} h^{-1}(u(t+h) - u(t)) = u'(t)$ in $L^1(\mathbb{R})$ or that $u(t) \in D(B_\alpha)$ even for almost all $t > 0$. If $u_0 = u(0) \ge 0$, these desirable properties hold true - see [1]. The current proof of this is a long story beginning with (35). (Ongoing work of various investigators indicates that the results of [1] extend to u_0 not necessarily of fixed sign and to more general nonlinearities.) However, it is known that if $S(t)$ is constructed from an (abstract) m-dissipative B as explained above, then $\lim_{h\downarrow 0} \|S(t+h)x - S(t)\|/h \le M < \infty$ exactly when there is a sequence $\{x_n\} \subset D(B)$ with $x_n \to S(t)x$ and $\limsup_{n\to\infty} \|Bx_n\| \le M$. (See, e.g., [12], [14]). Thus (34) itself and the explicit nature (33) of B_α imply that $(|u(t)|^{\alpha-1}u(t))''$ is a measure on $\mathbb{R}$ of variation at most $2/t|\alpha-1|$.

With respect to other literature about $(1)_\alpha$ and variants, we mention in particular that the L^1-non expansiveness is noted in [27], that [3], [22], [24] are of interest and the references listed therein provide access to the large literature, that the estimate (34) is not new if $u_0 \ge 0$ (see [1]), and that the result of our paper applied to $(1)_\alpha$ and generalizations of it with u_0 not of fixed sign and the equation either perturbed or forced seem to be new.

The distinction between finite speed and possessing a velocity is clearly illustrated by the class of problems $(2)_\alpha$. The linear problem $\alpha = 1$ is explicitly solved by $u(x,t) = u_0(x+t)$. If $u_0 \in X$ and X is one of the spaces $L^p(\mathbb{R})$, $1 \le p < \infty$ or $BU(\mathbb{R})$, then $u(t) = u_0(\cdot+t)$ is a continuous curve in X. The velocity $u'(t)$ exists at some t if and only if it exists for $t = 0$ if and only if $u_0 \in D(B) = \{v \in X : v' \in X\}$. The speed $\lim_{h\downarrow 0} h^{-1}\|u(t+h) - u(t)\|$ is independent of t and is finite if and only if

$$
(36) \qquad \begin{cases} \text{(i)} \ \ u_0 \in D(B) \ \text{ when } X = L^p(\mathbb{R}),\ 1 < p < \infty\ , \\ \text{(ii)} \ \ u_0 \text{ is Lipschitz continuous when } X = BU(\mathbb{R})\ , \\ \text{(iii)} \ \ u_0 \text{ is of essentially bounded variation on } \mathbb{R} \\ \qquad\qquad \text{when } X = L^1(\mathbb{R})\ . \end{cases}
$$

Moreover, the speed is $\|Bu_0\| = \|u_0'\|$ in case (i), the least Lipschitz constant in case (ii) and the variation of u_0 in case (iii). There is no regularizing in this example.

The differentiability and speed of u are independent of t. An estimate on the speed does imply some regularity in x as described above, but it does not imply $u(t) \in D(B)$.

The nonlinear problems $(2)_\alpha$, $\alpha \neq 1$, corresponds to the m-dissipative operators

$$(37) \qquad \begin{cases} D(B_\alpha) = \{v \in L^\infty(\mathbb{R}) : (|v|^{\alpha-1}v)' \in L^1(\mathbb{R})\} , \\ B_\alpha v = (|v|^{\alpha-1}v)' \end{cases}$$

in $L^1(\mathbb{R})$ and the operators S_α to which they give rise respect the order of $L^1(\mathbb{R})$ ([10], [4]). It is not true here that solutions of $(1)_\alpha$ in the sense of distributions are unique and extra conditions must be laid upon solutions - so called entropy conditions. See [20], [21], [28], which further explain other approaches to $(1)_\alpha$. The entropy solutions of $(1)_\alpha$ are given by S_α ([10]). Simple analyses by the method of characteristics shows that even if u_0 is smooth and compactly supported, the solution of $(2)_\alpha$ must become discontinuous as t increases - i.e. "shocks form". This is reflected in the S_α and in general $S_\alpha(t)u_0 \notin D(B_\alpha)$ large t and $u_0 \neq 0$. Here we have "regularizing" in that Theorem 1 estimates the speed of a solution $u(t)$ in the form c/t and additional considerations explained above then estimate the variation in x of $|u(t)|^\alpha$ sign $u(t)$ by the same quantity, but we also have "roughing" in that $u(t)$ need not be smooth in x (or even lie in $D(B_\alpha)$) even if u_0 is smooth.

Estimates on the variation of solutions of $\partial u/\partial t + \partial f(u)/\partial x = 0$ which decay like c/t are classical for convex functions f ([21]). Our estimates for, e.g., $\partial u/\partial t + \partial u^5/\partial x = 0$ are perhaps new, as are the pointwise estimates (35) for nonnegative solutions and the estimates for the perturbed and forced equations. See also [15]. Concerning generalizations of $(26)_\alpha$ see [19].

A final point of interest here is that Theorem 1 did not capture the regularizing present in the linear heat equation $(1)_1$ and, indeed, it could not for Theorem 1 uses only properties shared by $(2)_1$ for which there is no regularizing.

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It is well-known that solving the initial-value problem for the heat equation forward in time takes a "rough" initial temperature into a temperature which is smooth at later times $t > 0$. One aspect of this is the validity of certain estimates on tu_t when u is a solution of the heat equation. In this paper we prove related estimates on nonlinear evolution equations which are governed by homogeneous nonlinearities. The results apply to classes of nonlinear diffusion equations and to conservation laws. The results are interesting from the point (continued)

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ABSTRACT (continued)

of view of identifying a new "regularization" mechanism and the estimates themselves cast new light on the nature of the solutions of some initial-value problems with rough initial data.