Regulation and controlled synchronization for complex dynamical systems

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Abstract

In this paper we investigate the problem of controlled synchronization as a regulator problem. In controlled synchronization one is given autonomous transmitter dynamics and controlled receiver dynamics. The question is to find a (output) feedback controller that achieves matching between transmitter and controlled receiver. Several variants of the problem where the standard solvability assumptions for the regulator problem are not met turn out to have a solution. Simulations on two standard synchronization examples are also included.

1 Introduction

The regulator problem is a central problem in control theory and deals with the asymptotic tracking of certain classes of *prescribed trajectories* and asymptotic rejection of *undesired disturbances*. Over the years, the problem has received a lot of attention. For linear systems the regulator problem was extensively studied in [4] and [5]. For nonlinear systems, the problem was first studied in [6] and afterwards in [7] on the one hand and [8] on the other hand. An account of the state of the art on the problem, including both the linear and the nonlinear setting, is given in the book [3].
Essential in the regulator problem is that in any of the solutions to the problem the required feedback compensator incorporates an internal model of the exosystem that generates the command signals and the exogenous disturbances, cf. [6]. Typically, the solution to the regulator problem presented in [8],[3] is a local one around an equilibrium point of the exosystem. Further, an important hypothesis in [8],[3] is that the equilibrium point is stable, while all points in a neighborhood of the equilibrium point are Poisson stable. Recall that a point w_0 is called Poisson stable for a flow $\phi_t(\cdot)$ if $\phi_t(w_0)$ is defined for all $t \in \mathbb{R}$ and for every neighborhood U of w_0 and every $T > 0$ there exist $t_1 > T$, $t_2 < -T$ such that $\phi_{t_1}(w_0), \phi_{t_2}(w_0) \in U$. In particular, this hypothesis implies that the exosystem has a critically stable linearization. It is generally accepted that various controller design problems can be cast into an appropriate variant of the regulator problem.

The purpose of this paper is to show that *synchronization* of two systems can, under suitable hypotheses, be formulated as a regulator problem. This would imply that in principle a number of (controlled) synchronization problems become solvable through the application of results from [3]. An easy example of the same idea was dealt with in [10]. Unfortunately, in most cases where one seeks a controller that achieves synchronization, the more or less standard hypothesis of Poisson stability is not fulfilled. Thus, further investigation of the solvability of the corresponding regulator problem is needed. We will show that this research can be done successfully, although no complete characterization of the solvability is at hand.

This paper is organized as follows. In the next section we recapitulate essential background on synchronization and controlled synchronization. Afterwards, in Section 3, we recast this within the context of a regulator problem, and a general preliminary result is presented and illustrated by means of an example. As another example of controlled synchronization when the hypotheses in [8],[3] do not hold, we consider the controlled synchronization of two coupled Van der Pol oscillators in Section 4. In Section 5, some conclusions are drawn.

2 Controlled synchronization

Although different formulations of what is called synchronization exist, we adopt here a definition that captures the essential ideas of it, cf. [12],[2]. Suppose two systems are given:

$$\begin{cases} \dot{w} &= s(w) \\ y &= \phi(w) \end{cases} \tag{1}$$

and

$$\begin{cases} \dot{x} &= f(x,y) \\ \eta &= \phi(x) \end{cases} \tag{2}$$

where both x and w are in $\mathbb{R}^n$ (or, more generally, in the same Riemannian manifold). We will assume that the origin is one of the equilibrium points of (1), and that $\phi(0) = 0$. System (1) is the so-called *transmitter* (or *master*), and system (2) is the *receiver* (or *slave*). Synchronization of (1) and (2) occurs if, no matter how (1) and (2) are initialized, we have that asymptotically their states will match, i.e.,

$$\lim_{t \to +\infty} \|x(t) - w(t)\| = 0 \tag{3}$$

Typically, the receiver (2) depends on (1) via the drive signal $y = \phi(w)$, which explains the transmitter/receiver terminology. It is clear that synchronization will only occur in particular cases. In a previous paper, [12], it was shown that an interpretation is to view (2) as an observer for (1) given the output signal $\phi(w)$. So in those applications where one is able to design (2) freely, this provides a potential solution for the synchronization problem. In most of the synchronization literature (see e.g. [13]) however, the systems (1) and (2) are systems that are given beforehand, so that no synchronization will occur in general. However, and this is the viewpoint that we take in this paper, we may consider a controlled version of the problem, in that we allow the receiver dynamics also to depend on a control variable u, that is, we replace (2) by

$$\dot{x} = f(x, y, u) \tag{4}$$

The natural question now is to seek an appropriate dynamic feedback of the form

$$u = \alpha(z, \eta, y) \tag{5}$$

where z is driven by the dynamics

$$\dot{z} = k(z, \eta, y) \tag{6}$$

such that the resulting closed loop dynamics (4,5,6) synchronizes with (1), i.e., (3) holds for (1,4,5,6), and some appropriate stability requirements are fulfilled. Note that this definition in a more general form is contained in [2], and typically brings the problem of controlled synchronization within the scope of the regulator problem.

3 Controlled synchronization as a regulator problem

The controlled synchronization problem as formulated above can be viewed as a regulator problem. The *exosystem* is given as (1) and generates the to-be-tracked signals. (We require full state regulation of w, but obviously variants are possible.) The plant is given as (4), and we seek a compensator of the form (6,5) such that the error $e(t) := x(t) - w(t)$ satisfies

$$\lim_{t \to +\infty} e(t) = 0 \tag{7}$$

Additionally, we require the unforced closed loop dynamics

$$\begin{cases} \dot{x} &= f(x, 0, \alpha(z, \eta, 0)) \\ \dot{z} &= k(z, \eta, 0) \end{cases} \tag{8}$$

to be asymptotically stable. Note that usually this requirement is not *explicitly* included in the synchronization problem, but it is very natural in our context. Also for synchronization purposes, this property seems quite natural (but not absolutely necessary!). In particular, in

the secure communication context this requirement implies that the receiver becomes silent when the transmitter is silent.
It is clear that with the above conditions we have brought the problem of controlled synchronization in the framework of a regulator problem. Unfortunately, in most applications of chaos synchronization (like e.g. secure communication ([13])) the system (1) possesses a chaotic attractor in which several equilibrium points with un unstable linearization are embedded. This means that one of the standard hypotheses from [3] on the exosystem (Poisson stability of the equilibrium) is *not* met. Therefore, the results from [3] are not directly applicable, and this makes the controlled synchronization problem difficult.
In the following we want to show that despite the fact that the transmitter/receiver dynamics do not satisfy the standard Poisson stability requirements, it is possible to solve the corresponding regulator problem in several cases. Our work can thus be viewed as an extension of the standard solutions of the regulator problem that have been given in [5],[3],[7]. The first variant we treat consists of solving the regulator problem for Lur'e-like systems, i.e., linear systems with a nonlinear output dependent feedback loop.

Theorem 3.1 *Consider the transmitter*

$$\left\{ \begin{array}{rcl} \dot{w} & = & Aw + \Psi(y) \\ y & = & Cw \end{array} \right. \tag{9}$$

and receiver

$$\left\{ \begin{array}{rcl} \dot{x} & = & Ax + \Psi(y) + Bu \\ \eta & = & Cx \end{array} \right. \tag{10}$$

with $w, x \in I\!R^n$ $u \in I\!R^m$, and Ψ a mapping of appropriate dimensions, and A, B, C matrices of appropriate dimensions. Under the assumptions that (C, A) is detectable and (A, B) is stabilizable, the controlled synchronization problem is solvable.

Proof Since (C, A) is detectable, there exists a matrix K such that all eigenvalues of $A+KC$ are in the open left half plane. Further, the fact that (A, B) is stabilizable implies that there exists a matrix F such that all eigenvalues of $A + BF$ are in the open left half plane. It is then readily checked that the following dynamic feedback solves the controlled synchronization problem:

$$\left\{ \begin{array}{rcl} \dot{\hat{w}} & = & A\hat{w} + K(\hat{y} - y) + \Psi(y) \\ \dot{\hat{x}} & = & A\hat{x} + K(\hat{\eta} - \eta) + \Psi(y) + Bu \\ \hat{y} & = & C\hat{w} \\ \hat{\eta} & = & C\hat{x} \\ u & = & F(\hat{x} - \hat{w}) \end{array} \right. \tag{11}$$

■

Below, we give an example of a chaotic transmitter and receiver where the result can be applied.

Example 3.2 In this example, we take as the transmitter the Chua circuit, which in dimen-

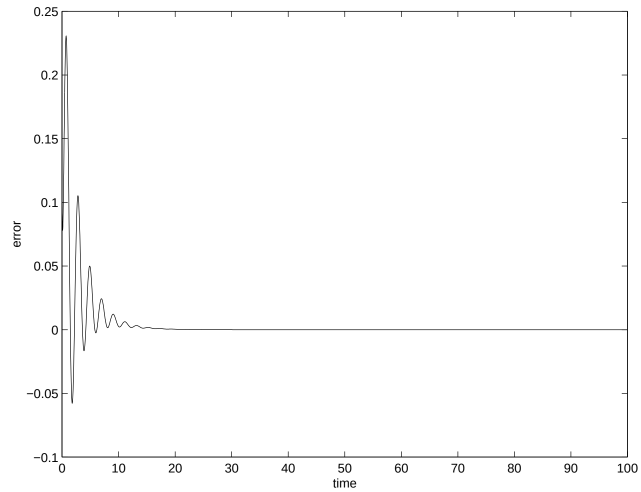

Figure 1: $x_1 - w_1$ for Example 3.2

sionless form is described by the equations

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -\alpha(m_1+1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}}_{A} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \underbrace{\begin{pmatrix} -\alpha(m_0 - m_1)\mathrm{sat}(w_1) \\ 0 \\ 0 \end{pmatrix}}_{\Psi(w_1)} \tag{12}$$

where sat$(\cdot)$ is the saturation function given by $\mathrm{sat}(w_1) = \frac{1}{2}(\mid w_1 + 1 \mid - \mid w_1 - 1 \mid)$. For the parameter values $\alpha = 15.6$, $m_0 = -\frac{8}{7}$, $m_1 = -\frac{5}{7}$, $\beta = 25$, this system is known to have a so called double scroll chaotic attractor, in which three unstable equilibrium points are embedded (see e.g. [1]). We assume that $y = w_1$ is the transmitted signal, so that (12) takes the form (9). Note that with this choice of y the pair (C, A) is observable. As the receiver, we take the system

$$\dot{x} = Ax + \Psi(w_1) + bu \tag{13}$$

where $u \in I\!R$ and $b \in I\!R^n$. Note that we now have that for a generic choice of b the pair (A, b) is stabilizable. Thus, all conditions of Theorem 3.1 are satisfied for a generic choice of b. In Figures 1,2,3 a simulation is given for (12),(13),(11) with $b = (1\ 1\ 1)^T$, $F = (-1\ \ -15.6\ 0)^T$, $K = (4.36\ 0\ 0)$.

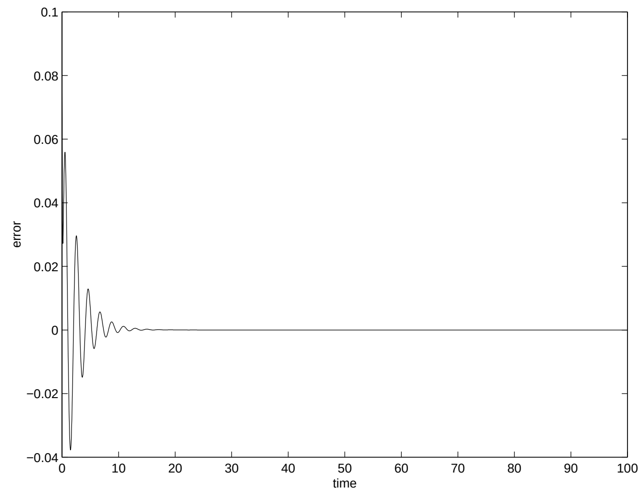

Figure 2: $x_2 - w_2$ for Example 3.2

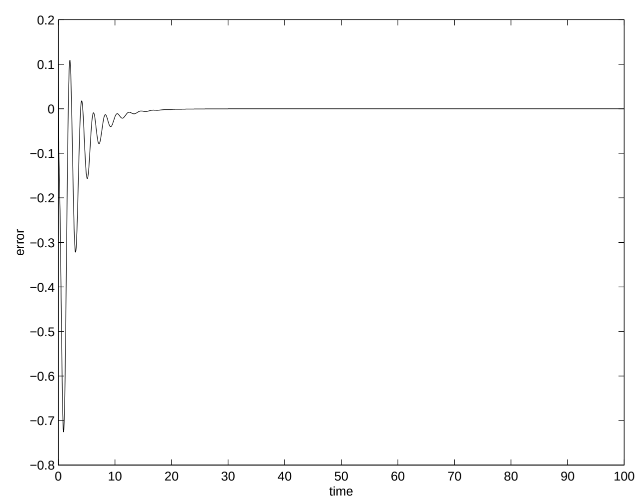

Figure 3: $x_3 - w_3$ for Example 3.2

4 Controlled synchronization of coupled Van der Pol systems

In this section we discuss the controlled synchronization problem for a (controlled) Van der Pol equation, i.e., as transmitter dynamics we take a Van der Pol system of the form

$$\begin{cases} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -w_1 - (w_1^2 - 1)w_2 \\ y &= w_1 \end{cases} \tag{14}$$

As receiver dynamics, we take the following controlled "copy" of (14):

$$\begin{cases} \dot{x}_1 &= x_2 + \alpha u \\ \dot{x}_2 &= -y - (y^2 - 1)x_2 + \beta u \end{cases} \tag{15}$$

where $u \in I\!R$ denotes the control signal and $\alpha, \beta \in I\!R$. We are now interested in the question whether or not both systems will synchronize by applying a static (high-gain) error feedback

$$u = -c(x_1 - w_1), \quad c \in I\!R \tag{16}$$

Defining the error signals $e_i := x_i - w_i$ $(i = 1, 2)$, and applying the controller (16), the error dynamics have the following form:

$$\dot{e} = \begin{pmatrix} -\alpha c & 1 \\ \beta c & -(y^2 - 1) \end{pmatrix} e \tag{17}$$

Thus, in order to solve the controlled synchronization problem for (14,15,16), we need to investigate whether there exists a $c \in I\!R$ such that the linear time-varying differential equation (17) is asymptotically stable. This will be done for different values of α, β in Subsection 4.2, after some properties of the Van der Pol differential equation (14) have been established in Subsection 4.1.

4.1 Properties of the Van der Pol differential equation

In this subsection, we establish some properties of the Van der Pol differential equation (14). We start with two well known properties.

Proposition 4.1 *Consider the differential equation (14). Then:*

(i) The origin is the only equilibrium point of (14), and it is an unstable focus.

(ii) The differential equation (14) has a unique limit cycle C that is (not uniformly) exponentially attracting for all initial conditions $w(0) \in I\!R^2 - \{0\}$, i.e.,

$$(\forall w(0) \in I\!R^2 - \{0\})(\exists \gamma, \lambda > 0)(\forall t \geq 0)(d(C, \phi_t(w(0))) \leq \gamma e^{-\lambda t})$$

where $d(C, w)$ denotes the distance between C and the point $w \in I\!R^2$, and $\phi_t(\cdot)$ denotes the flow of (14).

Proof See e.g. [1]. ■

The following result, that may be less well known, will be useful in the sequel. The proof of this result is based on modification of an argument in [14].

Proposition 4.2 *Consider the differential equation (14), and let C denote its limit cycle. Let $\tilde{w}(t)$ be a periodic solution that starts on C, and let T denote its period. Define*

$$p(t) := \tilde{w}_1(t)^2 - 1 \tag{18}$$

Then

$$\bar{p} := \frac{1}{T}\int_0^T p(\tau)d\tau > 0 \tag{19}$$

Proof See Appendix. ■

4.2 Controlled synchronization

In this section, we investigate the asymptotic stability of the differential equation (17). Note that this differential equaton may be interpreted as a linear time-varying differential equation, where the time-dependence is determined by the signal $y(t)$. By Proposition 4.1.*(ii)*, the solutions of (14) converge exponentially to the limit cycle C for all $w(0) \in I\!R^2 - \{0\}$. By invoking for example Theorem 6.1 in [16], this gives that (17) is asymptotically stable if and only if the following linear periodic differential equation is uniformly exponentially stable:

$$\dot{e} = \begin{pmatrix} -\alpha c & 1 \\ -\beta c & -p(t) \end{pmatrix} e \tag{20}$$

Here, $(\tilde{w}_1(t), \tilde{w}_2(t))$ is a(ny) T-periodic solution of (14) starting on C, and p is defined by (18). We will now investigate the uniform exponential stability of (20) for diffferent values of α, β, c.

4.2.1 Case 1: $\beta = 0, \alpha \neq 0$

Note that one may assume without loss of generality that $\alpha = 1$. The fundamental matrix $\Phi(t, t_0)$ of (20) is then given by

$$\Phi(t, t_0) = \begin{pmatrix} \exp(-c(t - t_0)) & \psi(t, t_0) \\ 0 & \exp(-\int_{t_0}^t p(\tau)d\tau) \end{pmatrix}$$

where $\psi(t, t_0) := \int_{t_0}^t \exp(-c(t - \tau) - \int_{t_0}^\tau p(\sigma)d\sigma)d\tau$. From this form and Proposition 4.2, it is readily seen that (20) is (uniformly) exponentially stable if and only if $c > 0$. Thus, in this case synchronization of (14) and (15) may be achieved by an error feedback $u = -c(x_1 - w_1)$, where $c > 0$. Note, however, that in this case the unforced closed loop dynamics (8) will have the form (20) with $\alpha = 1$, $\beta = 0$, $p(t) = -1$. From this it is easily seen that the unforced closed loop dynamics will be unstable, no matter how c is chosen.

4.2.2 Case 2: $\alpha = 0, \beta \neq 0$

One may now assume without loss of generality that $\beta = 1$. The unforced closed loop dynamics now have the form (20) with $\alpha = 0$, $\beta = 1$, $p(t) = -1$. This gives that also in this case the unforced closed loop dynamics are unstable, no matter how c is chosen.

We now derive a lower bound c_* so that (20) is uniformly exponentially stable for all $c > c_*$. Noting that $p(t)$ as defined in (18) is differentiable, we define

$$q(t) := \frac{1}{4}p(t)^2 + \frac{1}{2}\dot{p}(t) \tag{21}$$

and

$$\bar{q} := \frac{1}{T}\int_0^T |q(\tau)|d\tau \tag{22}$$

We then have the following result.

Theorem 4.3 *The differential equation (20) with p as defined in (18) is uniformly exponentially stable for all*

$$c > c_* := \left(\frac{\bar{q}}{\bar{p}}\right)^2 \tag{23}$$

Proof Define

$$\xi_1(t) := \exp\left(\frac{1}{2}\int_0^t p(\tau)d\tau\right) e_1(t) \tag{24}$$

and

$$\xi_2(t) := \exp\left(\frac{1}{2}\int_0^t p(\tau)d\tau\right)\left(\frac{1}{2}p(t)e_1(t) + e_2(t)\right) \tag{25}$$

We then have that ξ_1, ξ_2 satisfy the following Hill equation:

$$\left\{\begin{array}{rcl} \dot{\xi}_1 &=& \xi_2 \\ \dot{\xi}_2 &=& -(c - q(t))\xi_1 \end{array}\right. \tag{26}$$

where $q(t)$ is defined in (21). From [11] it then follows that there exists a $\gamma > 0$ such that the solutions of (26) satisfy

$$\|\xi(t)\| \leq \gamma\|\xi(0)\|\exp\left(\frac{1}{2\sqrt{c}}\int_0^t |q(\tau)|d\tau\right) \tag{27}$$

Combining (24),(25),(27), we then obtain that there exists a $\tilde{\gamma} > 0$ such that the solutions of (20) satisfy

$$\|e(t)\| \leq \tilde{\gamma}\|e(0)\|\exp\left(\frac{1}{2\sqrt{c}}\int_0^t |q(\tau)|d\tau - \frac{1}{2}\int_0^t p(\tau)d\tau\right) \tag{28}$$

Since $p(t)$ and $q(t)$ are T-periodic, it follows from (28) that (20) is uniformly exponentially stable if

$$\frac{1}{\sqrt{c}}\bar{q} - \bar{p} < 0$$

which establishes (23). ■.

Numerical approximations give that for (14) we have $\bar{p} \approx 1.06$, $\bar{q} \approx 1.45$, and thus $c_* \approx 1.87$. Further simulations indicate that in fact (14) and (15) synchronize for all $c > 1.39$, as is illustrated in Figures 4,5. Thus, the bound c_* given above may be quite conservative.

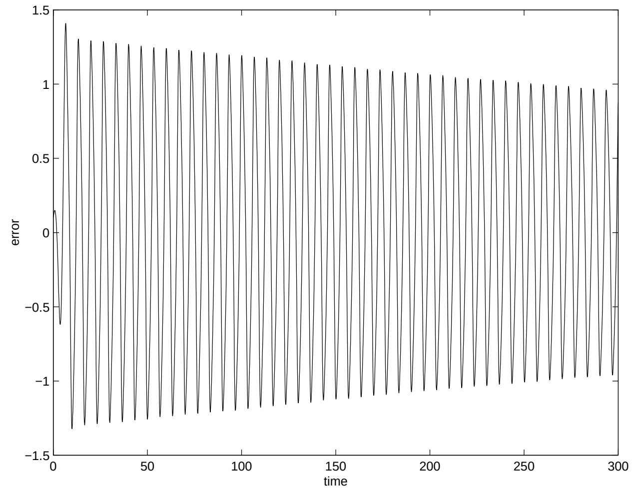

Figure 4: $x_1 - w_1$ for Case 2 with $c = 1.4$

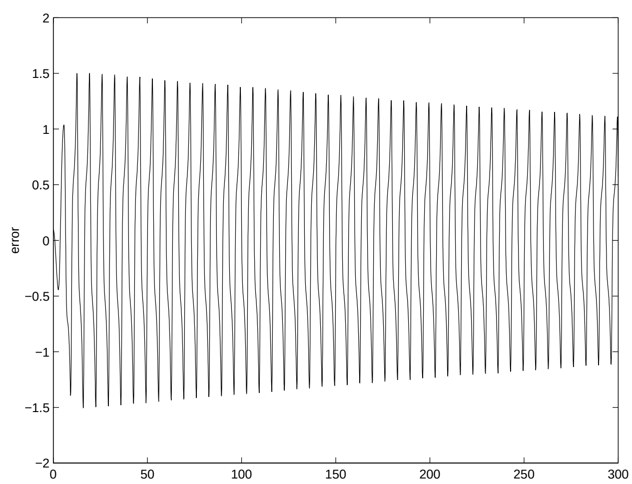

Figure 5: $x_2 - w_2$ for Case 2 with $c = 1.4$

4.2.3 Case 3: $\alpha \neq 0,\ \beta \neq 0$

Without loss of generality, we may assume that $\alpha = 1$.

Defining $\tilde{e}_1 := e_1$, $\tilde{e}_2 := e_2/c$, $\epsilon := 1/c$, we obtain the following differential equation:

$$\left\{ \begin{array}{rcl} \epsilon\dot{\tilde{e}}_1 & = & -\tilde{e}_1 + \tilde{e}_2 \\ \dot{\tilde{e}}_2 & = & -\beta\tilde{e}_1 - p(t)\tilde{e}_2 \end{array} \right. \tag{29}$$

If we apply a high-gain error feedback (i.e., $c \to +\infty$ in (16)), we will have that $\epsilon \downarrow 0$, and thus (29) becomes a singularly perturbed differential equation. Application of Tikhonov's Theorem (see e.g. [15]) then gives that (29) is uniformly exponentially stable when $\epsilon \downarrow 0$ if and only if the following differential equation is uniformly exponentially stable:

$$\dot{\xi} = -(\beta + p(t))\xi \tag{30}$$

It is readily checked that this is the case when

$$\beta > -\bar{p} \tag{31}$$

Thus, in this case synchronization of (14) and (15) may be achieved by high-gain error feedback when (31) holds.

The unforced closed loop dynamics now have the form (20) with $\alpha = -1$, $p(t) = -1$. This gives that the unforced closed loop dynamics are asymptotically stable if and only if $\beta, c > 1$.

For the case that $\beta > 1$, we next derive a lower bound c_* for c so that (20) is uniformly exponentially stable for all $c > c_*$. To this end, define

$$p_{\max} := \max\{p(t) \mid t \in [0,T]\} \tag{32}$$

and

$$p_{\min} := \min\{p(t) \mid t \in [0,T]\} \tag{33}$$

Note that from (18) and the fact that there exists a $t \in [0,T]$ such that $\tilde{w}_1(t) = 0$, it follows that $p_{\min} = -1$. Furthermore, it then follows from Proposition 4.2 that $p_{\max} > 0$. We now have the following result.

Theorem 4.4 *Consider the differential equation (20), and assume that $\beta > 1$. Then (20) is uniformly exponentially stable for*

$$c > c_* = p_{\max} + 2\beta + \sqrt{2\beta p_{\max} + 4\beta^2} \tag{34}$$

Proof Define

$$q_{\max} := \frac{1}{2}\left(-(c - p_{\max}) - \sqrt{(c - p_{\max})^2 - 4\beta c}\right) \tag{35}$$

and

$$q_{\min} := \frac{1}{2}\left(-(c - p_{\min}) - \sqrt{(c - p_{\min})^2 - 4\beta c}\right) \tag{36}$$

It is then straightforwardly checked that for $\beta > 1$, $c > c_*$ we have that $q_{\max}$ and $q_{\min}$ are real, and satisfy $q_{\min} < q_{\max} < 0$. Further, consider the Riccati differential equation

$$\dot{q} = q^2 + (c - p(t))q + \beta c \tag{37}$$

Note that it follows from (35) and (36) that

$$q_{\max}^2 + (c - p_{\max})q_{\max} + \beta c = 0 \tag{38}$$

and

$$q_{\min}^2 + (c - p_{\min})q_{\min} + \beta c = 0 \tag{39}$$

Denoting the right hand side of (37) by $F(t, q)$, this gives that

$$\begin{aligned} F(t, q_{\min}) = \quad & q_{\min}^2 + (c - p(t))q_{\min} + \beta c = \\ & q_{\min}^2 + (c - p_{\min})q_{\min} + \beta c + (p_{\min} - p(t))q_{\min} = \\ & (p_{\min} - p(t))q_{\min} \geq 0 \quad (\forall t) \end{aligned} \tag{40}$$

and, similarly,

$$F(t, q_{\max}) = \leq 0 \quad (\forall t) \tag{41}$$

The inequalities (40),(41) then imply that the interval $[q_{\min}, q_{\max}]$ is an invariant set for (37). Now let $q(t)$ be a solution of (37) satisfying $q(0) \in [q_{\min}, q_{\max}]$. Since we then have that $q(t) \in [q_{\min}, q_{\max}]$ $(\forall t \geq 0)$, we will have that the matrix

$$Q(t) := \begin{pmatrix} q(t) & 1 \\ 0 & \frac{1}{q(t)} \end{pmatrix}$$

is uniformly bounded and invertible for all $t \geq 0$. Thus we have that the time-varying coordinate change $\xi = Q(t)e$ is well-defined for all $t \geq 0$. It is straightforwardly checked that ξ satisfies the linear time-varying differential equation in triangular form:

$$\dot{\xi} = \begin{pmatrix} q(t) - p(t) & 0 \\ -\frac{\beta c}{q(t)^2} & -(q(t) + c) \end{pmatrix} \xi \tag{42}$$

It may now be shown that when $c > c_*$, we have that there exists a $\delta > 0$ such that $(q(t) - p(t)) < -\delta$ and $q(t) + c > \delta$ for all $t \geq 0$. This gives that (42) is uniformly exponentially stable. Since $Q(t)$ is uniformly bounded, this immediately gives that also (20) is uniformly exponentially stable. ■

In Figure 6, the numerically determined regions in the (β, c)-plane where (14) and (15) synchronize are indicated. From this figure, we draw the conclusion that (14) and (15) synchronize for all values of β and c for which the unforced closed loop dynamics are uniformly exponentially stable. Further, when comparing Figure 6 to the lower bound c_* given in (34), we see that this lower bound is very conservative.

Remark 4.5 Note that in fact all results obtained in this subsection may be applied to every linear time-varying differential equation of the form (20) for which $p(t)$ satisfies (19).

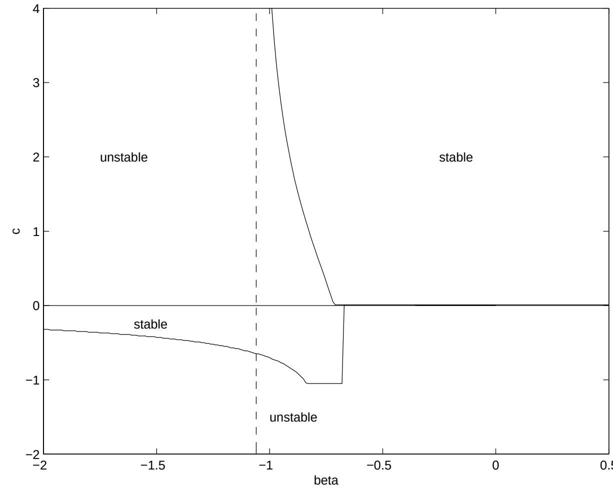

Figure 6: Stability regions for (20) in (β, c)-plane for Case 3

5 Conclusions

We have shown that the controlled synchronization problem can be treated as a regulator problem. However, in most applications where synchronization plays a role some of the standard assumptions for the solvability of the regulator problem are not fulfilled, and we are thus asked to find separate solvability conditions. In a few case we have established that it is possible to achieve such controlled synchronization and, in particular, we have shown that a few standard examples from the synchronization literature admit a solution. Simulations support our findings, and, in fact, suggest that for the synchronization in the Van der Pol example in Section 4 the bounds obtained are relatively conservative. It is therefore interesting to continue this research and to see whether a more general set of solvability conditions can be derived. Extensions to more general transmitter/receiver structures will also be a topic for future research.

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Appendix

Let $s(w)$ denote the vector field governing (14):

$$s(w) := \begin{pmatrix} w_2 \\ -w_1 - (w_1^2 - 1)w_2 \end{pmatrix} \tag{43}$$

We first establish some facts about the location of the curve C in the (w_1, w_2)-plane. First, note that s satisfies $s(-w) = -s(w)$, which establishes

Fact 1 The curve C is point symmetric with respect to the origin.

For (14), the horizontal isocline is given by

$$I_0 = \{w \in \mathbb{R}^2 \mid w_2 = \frac{-w_1}{w_1^2 - 1}\} \tag{44}$$

Now note that on the curve $I_0^- := I_0 \cap \{w \in \mathbb{R}^2 \mid w_1 < -1\}$ we have that $\dot{w}_1 > 0$, while on the curve $I_0^+ := I_0 \cap \{w \in \mathbb{R}^2 \mid w_1 > 1\}$ we have that $\dot{w}_1 < 0$. Since by Poincaré's Index Criterion (see e.g. [9]) the Jordan curve C should have an equilibrium point in its interior, we obtain with Proposition 4.1.*(i)*:

Fact 2 The curve C is located between the curves I_0^- and I_0^+, and has the origin in its interior.

Denote the interior of C by G. By Bendixson's Criterion (see e.g. [9]), we should have that

$$0 = \iint_G \text{div}s(w)dw_1dw_2 = \iint_G -(w_1^2 - 1)dw_1dw_2 \tag{45}$$

Together with Fact 1, and taking account of the direction field of (14), this gives:

Fact 3 The curve C crosses each of the straight lines $w_1 = \pm 1$ exactly twice.

The Facts 1,2,3 are illustrated by Figure 7.

Having established these facts, we consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(w) := \frac{1}{2}(w_1^2 + w_2^2) \tag{46}$$

and the function $\tilde{g} : [0, T] \to \mathbb{R}$ defined by

$$\tilde{g}(t) := g \circ \tilde{w}(t) \tag{47}$$

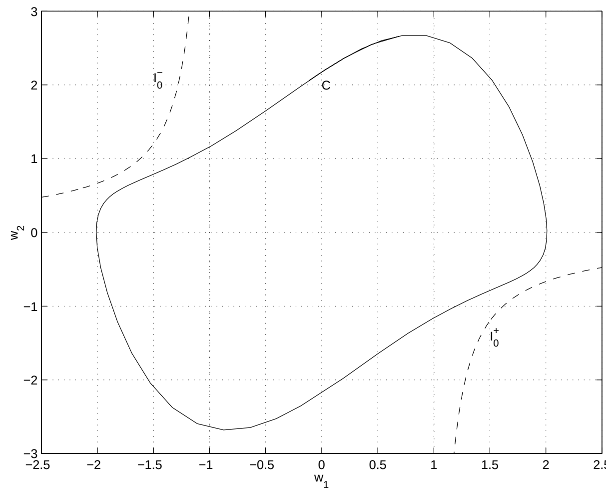

Figure 7: Location of the limit cycle C of the Van der Pol differential equation

If we are then able to show that

$$(\forall t \in [0,T])(\tilde{g}(t) > \frac{1}{2}) \tag{48}$$

our claim may be established in the following way. Note that

$$\begin{aligned} \dot{\tilde{g}} = \quad & -(\tilde{w}_1^2 - 1)\tilde{w}_2^2 = -(\tilde{w}_1^2 - 1)(\tilde{w}_1^2 + \tilde{w}_2^2 - 1) + (\tilde{w}_1^2 - 1)^2 = \\ & -(\tilde{w}_1^2 - 1)(2\tilde{g} - 1) + (\tilde{w}_1^2 - 1)^2 \end{aligned} \tag{49}$$

and thus p defined in (18) satisfies

$$p = (\tilde{w}_1^2 - 1) = -\frac{\dot{\tilde{g}}}{2\tilde{g} - 1} + \frac{(\tilde{w}_1^2 - 1)^2}{2\tilde{g} - 1} \tag{50}$$

Hence, if (48) holds, we obtain that

$$\begin{aligned} \int_0^T p(\tau)d\tau = \quad & \int_0^T \left(-\tfrac{\dot{\tilde{g}}}{2\tilde{g}-1} + \tfrac{(\tilde{w}_1^2-1)^2}{2\tilde{g}-1}\right) d\tau > \\ & \int_0^T \left(-\tfrac{\dot{\tilde{g}}}{2\tilde{g}-1}\right) d\tau = -\tfrac{1}{2}\log(2\tilde{g} - 1) \mid_0^T = 0 \end{aligned} \tag{51}$$

where the last equality follows from the fact that $\tilde{g}$ is T-periodic by definition.

Thus, what remains to be done is to show that indeed (48) holds. To this end, we investigate the extreme values of g on the curve C. Note that C is parametrized by $\tilde{w}(t)$ $(t \in [0,T])$. Thus, we can use $\tilde{g}(t)$ $(t \in [0,T])$ to investigate the extreme values of g on C. By (49), the only points where g can have an extreme value on C are the points on C where either $\tilde{w}_1 = \pm 1$ or $\tilde{w}_2 = 0$. Calculating $\ddot{\tilde{g}}$, we obtain

$$\ddot{\tilde{g}} = -2\tilde{w}_1\tilde{w}_2^3 - 2\tilde{w}_2(\tilde{w}_1^2 - 1)(-\tilde{w}_1 - (\tilde{w}_1^2 - 1)\tilde{w}_2) \tag{52}$$

This gives that g has minima on C at the points where $\tilde{w}_1 = \pm 1$, $\tilde{w}_2 < 0$, while it has maxima at the points where $\tilde{w}_1 = \pm 1$, $\tilde{w}_2 > 0$. Further, it follows from the location of these points and Facts 1,2,3 that g does not have extreme values on C at the points where $\tilde{w}_2 = 0$. From the point symmetry of C, the symmetry of g and the Weierstrass Theorem it then follows that the extrema found above are in fact global extrema. Thus, we obtain that on C we have

$$g(w) \geq g(w) \mid_{w_1 = \pm 1, w_2 < 0} = \frac{1}{2}(1 + w_2^2) \mid_{w_2 < 0} > \frac{1}{2} \tag{53}$$

■