T. KITANO
KODAI MATH. J.
39 (2016), 290–296

REIDEMEISTER TORSION OF A 3-MANIFOLD OBTAINED BY AN INTEGRAL DEHN-SURGERY ALONG THE FIGURE-EIGHT KNOT

Teruaki Kitano

Abstract

Let M be a 3-manifold obtained by a Dehn-surgery along the figure-eight knot. We give a formula of the Reidemeisiter torsion of M for any $SL(2;\mathbf{C})$-irreducible representation. It has a rational expression of the trace of the image of the meridian.

1. Introduction

Reidemeister torsion is a piecewise linear invariant for manifolds and originally defined by Reidemeister, Franz and de Rham in 1930's. In 1980's Johnson developed a theory of the Reidemeister torsion from the view point of certain relation to the Casson invariant of a homology 3-sphere. He also derived an explicit formula for the Reidemeister torsion of a homology 3-sphere obtained by a $1/n$-Dehn surgery along any torus knot for $SL(2;\mathbf{C})$-irreducible representations. We generalized the Johnson's formula for any Seifert fibered space [2] along his studies.

In this paper, we give a formula for 3-manifolds obtained by Dehn surgeries along the figure-eight knot. Let $K \subset S^3$ be the figure-eight knot. The knot group $\pi_1(S^3\backslash K)$ has the following presentation

$$\pi_1(S^3\backslash K) = \langle x, y \mid wx = yw \rangle$$

where $w = xy^{-1}x^{-1}y$. Now x is a meridian.

Let M be a 3-manifold obtained by a $1/n$-surgery along K. The fundamental group $\pi_1(M)$ admits a presentation as follows;

$$\pi_1(M) = \langle x, y \mid wx = yw, xl^n = 1 \rangle$$

where $l = w^{-1}\tilde{w}$ and $\tilde{w} = x^{-1}yxy^{-1}$. Now l is a longitude. Let $\rho : \pi_1(M) \to SL(2;\mathbf{C})$ be an irreducible representation. Assume the chain complex $C_*(M;\mathbf{C}^2_\rho)$

2010 *Mathematics Subject Classification*. 57M27.
Key words and phrases. Reidemeister torsion, Dehn surgery, figure eight knot.
Received June 2, 2015; revised July 31, 2015.

is acyclic. Then Reidemeister torsion $\tau_\rho(M) = \tau(C_*(M; \mathbf{C}^2_\rho))$ is given by the following.

THEOREM 1.1.

$$\tau_\rho(M) = \frac{2(u-1)}{u^2(u^2-5)}$$

where $u = \mathrm{tr}(\rho(x))$.

Remark 1.2.
(1) We remark the trace u cannot move freely on the complex plane in the above formula. The value u depends on the surgery coefficient n.
(2) Tran [9] discusses the generalization of the above formula for twist knots.

Acknowledgements. This research was supported by JSPS KAKENHI 25400101. The author thanks Michel Boileau, Michael Heusener and Takayuki Morifuji for helpful comments and discussions. He also thanks the referee for correcting some mistake.

2. Definition of Reidemeister torsion

First let us describe the definition of the Reidemeister torsion for $SL(2;\mathbf{C})$-representations. Since we do not give details of definitions and known results, please see Johnson [1], Milnor [5, 6, 7] and Kitano [2, 3] for details.

Let W be an n-dimensional vector space over $\mathbf{C}$ and let $\mathbf{b} = (b_1, \dots, b_n)$ and $\mathbf{c} = (c_1, \dots, c_n)$ be two bases for W. Setting $b_j = \sum p_{ji} c_i$, we obtain a nonsingular matrix $P = (p_{ij})$ with entries in $\mathbf{C}$. Let $[\mathbf{b}/\mathbf{c}]$ denote the determinant of P.

Suppose

$$C_* : 0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

is an acyclic chain complex of finite dimensional vector spaces over $\mathbf{C}$. We assume that a preferred basis $\mathbf{c}_i$ for C_i is given for each i. Choose some basis $\mathbf{b}_i$ for $B_i = \mathrm{Im}(\partial_{i+1})$ and take a lift of it in C_{i+1}, which we denote by $\tilde{\mathbf{b}}_i$. Since $B_i = Z_i = \mathrm{Ker}\ \partial_i$, the basis $\mathbf{b}_i$ can serve as a basis for Z_i. Furthermore since the sequence

$$0 \to Z_i \to C_i \to B_{i-1} \to 0$$

is exact, the vectors $\mathbf{b}_i \cup \tilde{\mathbf{b}}_{i-1}$ form a basis for C_i. Here $\tilde{\mathbf{b}}_{i-1}$ is a lift of $\mathbf{b}_{i-1}$ in C_i. It is easily shown that $[\mathbf{b}_i \cup \tilde{\mathbf{b}}_{i-1}/\mathbf{c}_i]$ does not depend on the choice of a lift $\tilde{\mathbf{b}}_{i-1}$. Hence we can simply denote it by $[\mathbf{b}_i \cup \mathbf{b}_{i-1}/\mathbf{c}_i]$.

DEFINITION 2.1. The torsion $\tau(C_*)$ is given by the alternating product

$$\prod_{i=0}^{m} [\mathbf{b}_i \cup \mathbf{b}_{i-1}/\mathbf{c}_i]^{(-1)^{i+1}}.$$

Remark 2.2. It is easy to see that $\tau(C_*)$ does not depend on the choices of the bases $\{\mathbf{b}_0, \dots, \mathbf{b}_m\}$.

Now we apply this torsion invariant of chain complexes to the following geometric situations. Let M be a finite CW-complex and $\tilde{M}$ a universal covering of M. The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ as deck transformations. Then the chain complex $C_*(\tilde{M}; \mathbf{Z})$ has the structure of a chain complex of free $\mathbf{Z}[\pi_1(M)]$-modules. We denote the 2-dimensional vector space $\mathbf{C}^2$ by V. Using a representation $\rho : \pi_1(M) \to SL(2; \mathbf{C})$, V has the structure of a $\mathbf{Z}[\pi_1(M)]$-module. Then we denote it by V_ρ and define the chain complex $C_*(M; V_\rho)$ by $C_*(\tilde{M}; \mathbf{Z}) \otimes_{\mathbf{Z}[\pi_1(M)]} V_\rho$. Here we choose a preferred basis

$$\{\tilde{u}_1 \otimes \mathbf{e}_1, \tilde{u}_1 \otimes \mathbf{e}_2, \dots, \tilde{u}_k \otimes \mathbf{e}_1, \tilde{u}_k \otimes \mathbf{e}_2\}$$

of $C_q(M; V_\rho)$ where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a canonical basis of $V = \mathbf{C}^2$ and $\tilde{u}_1, \dots, \tilde{u}_k$ are lifts of the q-cells giving the preferred basis of $C_q(M; \mathbf{Z})$.

We suppose that all homology groups $H_*(M; V_\rho)$ are vanishing. In this case we call ρ an acyclic representation.

Definition 2.3. Let $\rho : \pi_1(M) \to SL(2; \mathbf{C})$ be an acyclic representation. Then the Reidemeister torsion $\tau_\rho(M)$ is defined to be the torsion $\tau(C_*(M; V_\rho))$.

Remark 2.4.
(1) We define $\tau_\rho(M) = 0$ for a non-acyclic representation ρ.
(2) The Reidemeister torsion $\tau_\rho(M)$ depends on several choices. However it is well known that the Reidemeister torsion is a piecewise linear invariant. See Johnson [1] and Milnor [5, 6, 7].

Here we recall the Reidemeister torsion of the torus and the solid torus.

Proposition 2.5. *Let $\rho : \pi_1(T^2) \to SL(2; \mathbf{C})$ be a representation.*
(1) *This representation ρ is an acyclic representation if and only if there exists an element $z \in \pi_1(T^2)$ such that $\mathrm{tr}(\rho(z)) \neq 2$.*
(2) *If ρ is acyclic, then it holds $\tau_\rho(T^2) = 1$.*

Next we consider the solid torus $S^1 \times D^2$ with $\pi_1(S^1 \times D^2) \cong \mathbf{Z}$ generated by γ.

Proposition 2.6. *Let $\pi_1(S^1 \times D^2) \to SL(2; \mathbf{C})$ be a representation. Then it holds*

$$\tau_\rho(S^1 \times D^2) = \frac{1}{\det(\rho(\gamma) - E)} = \frac{1}{2 - \mathrm{tr}(\rho(\gamma))}$$

for a generator $\gamma \in \pi_1(S^1 \times D^2) \cong \mathbf{Z}$. Here E is the identity matrix in $SL(2; \mathbf{C})$.

From here we assume M is a compact 3-manifold with an acyclic representation $\rho : \pi_1(M) \to SL(2;\mathbf{C})$. Here we take a torus decomposition of $M = A \cup_{T^2} B$. For simplicity, we write the same symbol ρ for a restricted representation to subgroups $\pi_1(A)$, $\pi_1(B)$ and $\pi_1(T^2)$ of $\pi_1(M)$.

By this torus decomposition, we have the following exact sequence:

$$0 \to C_*(T^2; V_\rho) \to C_*(A; V_\rho) \oplus C_*(B; V_\rho) \to C_*(M; V_\rho) \to 0.$$

Proposition 2.7. *Let $\rho : \pi_1(M) \to SL(2;\mathbf{C})$ be a representation which restricted to $\pi_1(T^2)$ is acyclic. Then $H_*(M; V_\rho) = 0$ if and only if $H_*(A; V_\rho) = H_*(B; V_\rho) = 0$. In this case it holds*

$$\tau_\rho(M) = \tau_\rho(A)\tau_\rho(B).$$

We apply this proposition to any 3-manifold obtained by Dehn-surgery along a knot. Now let M be a closed 3-manifold obtained by a $1/n$-surgery along the figure eight knot K. We take an open tubular neighborhood $N(K)$ of K and its knot exterior $E(K) = S^3 \backslash N(K)$. Under the presentation

$$\pi_1(E(K)) = \langle x, y \mid wx = yw \rangle$$

where $w = xy^{-1}x^{-1}y$, $l = w^{-1}\tilde{w}$ and $\tilde{w} = x^{-1}yxy^{-1}$, x is a meridian and $l = w^{-1}\tilde{w}$ is a longitude.

We denote its closure of $N(K)$ by $\overline{N}$ which is homeomorphic to $S^1 \times D^2$. Since this 3-manifold M is obtained by Dehn-surgery along K, we have a torus decomposition

$$M = E(K) \cup \overline{N}.$$

Let $\rho : \pi_1(E(K)) = \pi_1(S^3 \backslash K) \to SL(2;\mathbf{C})$ be a representation which extends to $\pi_1(M)$.

Remark 2.8. We remark that $\gamma = l^{\pm 1}$ in $\pi_1(M)$ if and only if the surgery coefficient is $1/n$.

In this case it holds the following.

Proposition 2.9. *If ρ is acyclic on $\pi_1(T^2)$ and $\pi_1(M)$, then $\tau_\rho(M) = \tau_\rho(E(K))\tau_\rho(\overline{N})$. Further if all chain comeplexes are acyclic, then*

$$\tau_\rho(M) = \frac{\tau_\rho(E(K))}{2 - \operatorname{tr}(\rho(l))}.$$

3. Main result

Recall the following lemma, which is the fundamental way to study $SL(2;\mathbf{C})$-representations of a 2-bridge knot. Please see [8] as a reference.

Lemma 3.1. *Let $X, Y \in SL(2, \mathbf{C})$. If X and Y are conjugate and $XY \neq YX$, then there exists $P \in SL(2; \mathbf{C})$ such that*

$$PXP^{-1} = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix}, \quad PYP^{-1} = \begin{pmatrix} s & 0 \\ -t & 1/s \end{pmatrix}.$$

We apply this lemma to irreducible representations of $\pi_1(E(K))$. For any irreducible representation ρ, we may assume that the representative of this conjugacy class is given by

$$\rho_{s,t} : \pi_1(E(K)) \to SL(2; \mathbf{C}) \quad (s, t \in \mathbf{C}\backslash\{0\})$$

where

$$\rho_{s,t}(x) = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix}, \quad \rho_{s,t}(y) = \begin{pmatrix} s & 0 \\ -t & 1/s \end{pmatrix}$$

Simply we write ρ to $\rho_{s,t}$ for some s, t. We compute the matrix

$$R = \rho(w)\rho(x) - \rho(y)\rho(w) = (R_{ij})$$

to get the defining equations of the space of the conjugacy classes of the irreducible representations.

- $R_{11} = 0$,
- $R_{12} = 3 - \frac{1}{s^2} - s^2 + 3t - \frac{t}{s^2} - s^2 t + t^2$,
- $R_{21} = 3t - \frac{t}{s^2} - s^2 t + 3t^2 - \frac{t^2}{s^2} - s^2 t^2 - t^3 = tR_{12}$,
- $R_{22} = 0$.

Hence $R_{12} = 0$ is the equation defining the space of the conjugacy classes of the irreducible representations.

This equation

$$3 - \frac{1}{s^2} - s^2 + 3t - \frac{t}{s^2} - s^2 t + t^2 = 0$$

can be solved in t as

$$t = \frac{1 - 3s^2 + s^4 \pm \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^2}.$$

Here it can be seen that $L = \rho(l) = (l_{ij})$ is given by the followings:

Lemma 3.2.

$$l_{11} = 1 - \frac{t}{s^2} + s^2 t - t^2 + \frac{t^2}{s^4} - \frac{t^2}{s^2} + s^2 t^2 - t^3 - \frac{t^3}{s^2}$$

$$l_{12} = \frac{t}{s^3} + s^3 t - \frac{t^2}{s} - st^2$$

$$l_{21} = \frac{t^2}{s^3} - \frac{2t^2}{s} - 2st^2 + s^3 t^2 + \frac{t^3}{s^3} - \frac{2t^3}{s} - 2st^3 + s^3 t^3 - \frac{t^4}{s} - st^4$$

$$l_{22} = 1 + \frac{t}{s^2} - s^2 t - t^2 + \frac{t^2}{s^2} - s^2 t^2 + s^4 t^2 - t^3 - s^2 t^3$$

Here we get the trace of direct computation.

$$\operatorname{tr}(\rho(l)) = 2 - 2t^2 + \frac{t^2}{s^4} + s^4 t^2 - 2t^3 - \frac{t^3}{s^2} - s^2 t^3$$

It is easy to see that $\operatorname{tr}(\rho(l)) \neq 2$ if $u = s + \frac{1}{s} = 2$. Hence there exists an element $z \in \pi_1(T^2)$ such that $\operatorname{tr}(\rho(z)) \neq 2$. This means that ρ is always acyclic on T^2. Now we have

$$\tau_\rho(M) = \tau_\rho(E(K))\tau_\rho(\overline{N}).$$

Here we obtain the Reidemeister torsion of $E(K)$ as follows. See [3] for precise computation.

PROPOSITION 3.3.

$$\tau_\rho(E(K)) = -2(u-1)$$

where $u = s + \frac{1}{s}$.

By substituting

$$t = \frac{1 - 3s^2 + s^4 \pm \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^2}$$

in $\operatorname{tr}(\rho(l))$, we get the following proposition.

PROPOSITION 3.4.

$$\tau_\rho(\overline{N}) = -\frac{1}{u^2(u^2-5)}.$$

Therefore we obtain the following formula:

$$\begin{aligned}\tau_\rho(M) &= \tau_\rho(E(K))\tau_\rho(\overline{N}) \\ &= (-2(u-1))\left(-\frac{1}{u^2(u^2-5)}\right) \\ &= \frac{2(u-1)}{u^2(u^2-5)}.\end{aligned}$$

Remark 3.5. The representations for $u^2 - 5 = 0$ are degenerate into reducible representation from irreducible representations.

REFERENCES

[1] D. JOHNSON, A geometric form of Casson's invariant and its connection to Reidemeister torsion, unpublished lecture notes.

[2] T. Kitano, Reidemeister torsion of Seifert fibered spaces for $SL(2;\mathbf{C})$ representations, Tokyo J. Math. **17** (1994), 59–75.
[3] T. Kitano, Reidemeister torsion of the figure-eight knot exterior for $SL(2;\mathbf{C})$-representations, Osaka J. Math. **31** (1994), 523–532.
[4] T. Kitano, Reidemeister torsion of a homology 3-sphere obtained by a Dehn surgery along the $(2\alpha, \beta)$-torus knot, to appear in Tohoku Math. J.
[5] J. Milnor, Two complexes which are homeomorphic but combinatorially distinct, Ann. of Math. **74** (1961), 575–590.
[6] J. Milnor, A duality theorem for Reidemeister torsion, Ann. of Math. **76** (1962), 137–147.
[7] J. Milnor, Whitehead torsion, Bull. Amer. Math. Soc. **72** (1966), 358–426.
[8] R. Riley, Nonabelian representations of 2-bridge knot groups, Quart. J. Math. Oxford, Ser. (2) **35** (1984), 191–208.
[9] A. Tran, Reidemeister torsion and Dehn surgery on twist knots, arXiv:1506.02896.

Teruaki Kitano
Department of Information Systems Science
Faculty of Science and Engineering
Soka University
Tangi-cho 1-236, Hachioji
Tokyo 192-8577
Japan
E-mail: kitano@soka.ac.jp