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REINVESTIGATION OF THE NONUNIQUENESS OF THE FLOW OF A VISCOELASTIC FLUID OVER A STRETCHING SHEET

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1. Background. Rajagopal, Na, and Gupta have studied the flow of a viscoelastic fluid over a stretching sheet [1]. The uniqueness of the solution of the problem has been examined by McLeod and Rajagopal [2] and by Troy et al. [3]. Wen-Dong Chang [4] recently claimed that the solution of the problem is not necessarily unique. Motivated by the work of the above-mentioned authors, here the solution of the problem is reinvestigated analytically through a simple mathematical procedure.

2. Mathematical procedure. The nondimensional equation with the boundary conditions for the boundary layer model developed by Rajagopal, Na, and Gupta [1], using similarity solution principles, is

$$(f')^2 - ff'' = f''' - k_1\{2f'f''' - (f'')^2 - ff''''\}, \tag{1}$$

$$f'(0) = 1, \qquad f(0) = 0, \qquad f'(\infty) = 0 \tag{2a,b,c}$$

where the nondimensional physical quantity k_1 is a positive constant and primes denote differentiation with respect to η. Equations (1) and (2) represent a two-point fourth-order nonlinear differential equation having only three boundary conditions. The fourth boundary condition is obtained by using Eqs. (2a) and (2b) in Eq. (1) as

$$(1 - 2k_1)f'''(0) + k_1\{f''(0)\}^2 = 1. \tag{3}$$

By differentiating Eq. (1) with respect to η and applying the boundary conditions (2a) and (2b), one gets

$$f''''(0) = \frac{f''(0)}{(1 - k_1)}. \tag{4}$$

It should be noted that $f''''(0)$ in Eq. (4) becomes infinity for $k_1 = 1$. Thus, the limit of the applicability of the solution of the problem with respect to the nondimensional positive physical quantity $k_1 < 1$.

By assuming

$$f = A - F \tag{5}$$

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in Eq. (1) and substituting $e^{m\eta}$ for F, one gets

$$m^2(k_1 A m^2 + m + A) = 0. \tag{6}$$

The nonzero roots of Eq. (6) are

$$m_{1,2} = \frac{-1 \pm \sqrt{1 - 4Ak_1^2}}{2Ak_1}. \tag{7}$$

The solution of Eq. (1) can be represented by

$$f = A - \{Be^{m_1\eta} + Ce^{m_2\eta}\}. \tag{8}$$

By using Eq. (8) in Eqs. (2a) and (2b), the constants B and C expressed in terms of A are

$$B = \frac{m_2 A + 1}{m_2 - m_1}, \qquad C = \frac{m_1 A + 1}{m_1 - m_2}. \tag{9a,b}$$

By using Eqs. (6)–(9), $f''(0)$ and $f'''(0)$ in terms of A can be obtained as

$$f''(0) = \frac{A^2 - 1}{Ak_1}, \quad f'''(0) = \frac{1 - (1 + k_1)A^2}{(Ak_1)^2}. \tag{10a,b}$$

From Eqs. (3) and (10), one gets

$$k_1 A^4 - (1 + k_1 - k_1^2)A^2 + (1 - k_1) = 0. \tag{11}$$

The roots of Eq. (11) are

$$A^2 = (1 - k_1), \ \frac{1}{k_1}. \tag{12a,b}$$

It is very interesting to note that for $k_1 = \frac{1}{2}$, the nondimensional velocity gradient at the wall, $f''(0)$ from the additional boundary condition (3), is found to be $\pm\sqrt{2}$. This observation might give a clue for finding out the second closed-form solution for $k_1 = \frac{1}{2}$ [4]. Since there are two values for A^2, two closed-form solutions are found here for all values of $k_1 \in (0, 1)$.

First solution. For $A^2 = (1 - k_1)$, Eq. (7) implies

$$m_1 = -\frac{1}{A}, \qquad m_2 = -\frac{(1 - k_1)}{Ak_1}, \tag{13a,b}$$

and Eq. (9) implies

$$B = A, \qquad C = 0. \tag{14a,b}$$

From Eqs. (8), (13), and (14), the first solution of the problem is obtained as

$$f = A\{1 - e^{-\eta/A}\} = \sqrt{(1 - k_1)}\{1 - e^{-\eta/\sqrt{(1-k_1)}}\}. \tag{15}$$

In order to satisfy Condition (2c), the positive value of A is considered in Eq. (15).

Second solution. For $A^2 = 1/k_1$, Eq. (7) implies

$$m_{1,2} = \frac{-1 \pm \sqrt{3}i}{2Ak_1} \tag{16a,b}$$

and Eq. (9) implies

$$B = \frac{A}{2}\left\{1 - \frac{(1-2k_1)}{\sqrt{3}} i\right\}, \qquad C = \frac{A}{2}\left\{1 + \frac{(1-2k_1)}{\sqrt{3}} i\right\}. \qquad (17a,b)$$

From Eqs. (8), (16), and (17), the second solution of the problem is found as

$$f = A\left(1 - e^{-\xi}\left\{\cos(\zeta) + \frac{(1-2k_1)}{\sqrt{3}}\sin(\zeta)\right\}\right), \qquad (18)$$

where

$$\xi = \frac{\eta}{2Ak_1} = \frac{\eta}{2\sqrt{k_1}}, \qquad \zeta = \frac{\sqrt{3}\eta}{2Ak_1} = \frac{1}{2}\frac{\sqrt{3}}{\sqrt{k_1}}\eta.$$

In Eq. (18) the value of A is considered to be positive so as to satisfy the boundary condition (2c).

These two solutions given in Eqs. (15) and (18) are found to be quite different. It can be verified from Eqs. (15) and (18) that for all $0 < k_1 < 1$, as $\eta \to \infty$, $f''(\eta) \to 0$, whereas at $\eta = 0$, $f''(0) = -1/\sqrt{(1-k_1)} < 0$ from Eq. (15), and $f''(0) = (1-k_1)/k_1^{3/2} > 0$ from Eq. (18). The values of $f''(0)$ can be obtained directly from Eq. (10a) by substituting the values of A as $\sqrt{(1-k_1)}$ and $1/\sqrt{k_1}$, respectively.

3. Validity of the solution. Equations (15) and (18) represent two solutions for Eq. (1) with the boundary conditions (2) when $k_1 = 0$. Troy et al. [3] have found the first solution of the problem as given in Eq. (15). Another solution of the problem for the case $k_1 = \frac{1}{2}$, obtained by Wen-Dong Chang [4], can be found from Eq. (18). It has been proved simultaneously by McLeod and Rajagopal [2] and by Troy et al. [3] that Eqs. (1) and (2) have a unique solution

$$f(\eta) = 1 - e^{-\eta} \quad \text{for } k_1 = 0,$$

in which $f''(\eta) < 0$ for all $0 < \eta < \infty$.

Though for $k_1 \not\equiv 0$, two solutions exist for the present problem, the important constraint needed to get the realistic solution of the physical problem, which was missed in [4], is

$$f''(\eta) \le 0 \quad \text{for all } 0 < \eta < \infty.$$

The requirement of $f''(\eta) < 0$ everywhere to get a realistic solution of the present physical problem is explained below.

By physical intuition, one should expect that a slightly elastic fluid will produce a boundary layer only slightly altered in its dimensions from a viscous one. For a small value of k_1 (say, 0.0001), the dimensionless velocity gradient at the wall from the first solution is $f''(0) = -1.00005$, and its value from the second solution is obtained as 999900. For $k_1 = 0$, $f''(0) = -1$. Such a drastic change in the value of $f''(0)$ for a small value of k_1 obtained from the second solution is not reasonable. Since the first solution gives insight into the boundary layer for weakly elastic fluids, in the sense that $k_1 \ll 1$, it is a realistic solution for $0 < k_1 < 1$.

Rajagopal et al. [1] used a perturbation analysis by expanding the solution in powers of k_1 and obtained numerical estimates on the behaviour of the solution of equations. The function suggested by Troy et al. [3] {which is nothing but the first solution of the problem given in Eq. (15)} is in exact agreement with the approximate solution of Rajagopal et al. [1]. Since the first solution gives the boundary-layer behaviour for $0 < k_1 < 1$, the velocity gradient $f'' < 0$ everywhere for the present problem.

Beard and Walters [5] have extended the Prandtl boundary layer theory for an idealized elastico-viscous liquid. The boundary layer equations are solved numerically for the case of two-dimensional flow near a stagnation point. It is demonstrated that the main effect of elasticity is to increase the velocity in the boundary layer and also to increase the stress on the solid boundary. It is noticed from the first solution that the magnitude of the velocity gradient at the wall increases with k_1. From the second solution, it is found that the velocity gradient at the wall decreases drastically with k_1.

Regarding the validity of small values of k_1, Surma Devi and Nath [6] and their quoted references pointed out that the second-order fluid (i.e., viscoelastic fluid) governed by Eq. (1) represents the behaviour of fluids with short memory and the characteristic time scale associated with the motion is large compared with time representing the memory of the fluid. Hence, the assumption of small values of k_1 is valid especially for dilute polymer solutions.

Based on the observations above, the first solution given in Eq. (15) represents a realistic solution for the present physical problem for all $0 \le k_1 < 1$.

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