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Related G-metrics and Fixed Points

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Abstract. For a single valued mapping T in a G-complete G-metric space (X,d), we show that if T^n, for some $n > 1$, is a contraction, then T itself is a contraction under another related G-metric d'. We establish moreover that if T is uniformly continuous, then d' is G-complete.

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1 Introduction and Preliminaries

After disproving most of the claims about the topology of D-metrics (see [12]), Mustafa and Sim introduced a more appropriate notion of generalized metrics, called G-metrics. In his PhD thesis (see [10]), Mustafa provided many examples of G-metric spaces and developed some of their properties. For instance, he proved that G-metric spaces are provided with a T_2 topology which makes them a convenient framework for topological notions like convergent sequences, Cauchy sequences, continuous mappings, completeness, etc. We also know from Mustafa [12, Proposition 5] that every G-metric space is topologically equivalent to a metric space but G-metric spaces and

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metric spaces are "isometrically" distinct. As the theory around G-metrics unfolds, the natural direction was to look at fixed points for mappings defined on these type of spaces. The Banach contractive mapping principle is the most celebrated result in fixed point theory and therefore represented the default starting point for fixed point theory in G-metrics. Hence we read the following result in [10] as:

Theorem 1.1. *([10])*
Let (X, G) be a G-complete G-metric space and let $T : X \to X$ be a mapping such that there exists $\lambda \in [0, 1)$ satisfying

$$G(Tx, Ty, Tz) \leq \lambda\, G(x, y, z), \tag{1.1}$$

whenever $x, y, z \in X$. Then T has a unique fixed point. In fact, T is a Picard operator.

We give the following corollary, extension to the G-metric case, of a result by Bryant[2], and which seems to appear nowhere in the literature.

Corollary 1.2. *(Compare[2])*
Let (X, G) be a G-complete G-metric space and let $T : X \to X$ be a mapping such that there exists $\lambda \in [0, 1)$ satisfying

$$G(T^n x, T^n y, T^n z) \leq \lambda\, G(x, y, z),$$

for some $n > 1$, whenever $x, y, z \in X$. Then T has a unique fixed point.

Proof. By Theorem 1.1, T^n has a unique fixed point, say $x \in X$ with $T^n(x) = x$. Since

$$T^{n+1}x = T(T^n x) = Tx = T^n(Tx),$$

it follows that $T(x)$ is a fixed point of T^n, and thus, by the uniqueness of x, we have $Tx = x$, that is, T has a fixed point. Since, the fixed point of T is necessarily a fixed point of T^n, so it is unique. □

A mapping T, satisfying condition (1.1) is called a contraction with contraction constant λ. The result of Corollary 1.2, for which the proof is quite trivial, establishes that if a power of a map T is a contraction, then T has a unique fixed point. Moreover observe, in the formulation, that the mapping T is not even assumed to be continuous[3]. The aim of this paper is to show

[3] In some applications, it is often the case that the mapping T is Lipschitzian, and therefore does not need to be a contraction, whereas some power of T^n is a contraction mapping.

that, if a mapping T defined on a G-metric space (X, G) is not a contraction but admits a power, say n, for which T^n is a contraction, therefore there exists a related G-metric G' such that T is a contraction on (X, G').

The elementary facts about G-metric spaces can be found in [12]. We give here a shortened form of these prerequisites.

Definition 1.3. (Compare [12, Definition 3]) Let X be a nonempty set, and let the function $G : X \times X \times X \to [0, \infty)$ satisfy the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$ whenever $x, y, z \in X$;

(G2) $G(x, x, y) > 0$ whenever $x, y \in X$ with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$ whenever $x, y, z \in X$ with $z \neq y$;

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in all three variables);

(G5)
$$G(x, y, z) \leq [G(x, a, a) + G(a, y, z)]$$
for any points $x, y, z, a \in X$.

Then (X, G) is called a G**-metric space**.

Definition 1.4. (Compare [13, Definition 1.4])

Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. We say $\{x_n\}$ is G-convergent to $x \in X$ if $\lim\limits_{n,m\to\infty} G(x, x_n, x_m) = 0$, that is for any each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all $n, m \geq N$. We call x the limit of the sequence $\{x_n\}$ and we write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$.

Proposition 1.5. *(Compare* [12, Proposition 6]*) Let (X, G) be a G-metric space. Define on X the metric d_G by $d_G(x, y) = G(x, y, y) + G(x, x, y)$ whenever $x, y \in X$. Then for a sequence $(x_n) \subseteq X$, the following are equivalent*

(i) (x_n) *is G-convergent to $x \in X$.*

(ii) $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$.

(iii) $\lim_{n\to\infty} d_G(x_n, x) = 0$.

(iv) $\lim_{n\to\infty} G(x, x_n, x_n) = 0$.

(v) $\lim_{n\to\infty} G(x_n, x, x) = 0$.

Definition 1.6. (Compare [13, Defintion 1.4]) Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if for any each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 1.7. *(Compare* [12, Proposition 9]*)*
In a G-metric space (X, G), the following are equivalent

(i) The sequence $(x_n) \subseteq X$ is G-Cauchy.

(ii) For each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geq N$.

Definition 1.8. (Compare [12, Definition 9]) A G-metric space (X, G) is G-complete if every G-Cauchy sequence of elements of (X, G) is G-convergent in (X, G).

Definition 1.9. Two G-metrics d_1 and d_2 on a set X are said to be equivalent if there exist $\alpha, \beta \geq 0$ such that

$$\alpha d_1(x, y, z) \leq d_2(x, y, z) \leq \beta d_1(x, y, z), \text{ for all } x, y, z \in X.$$

Definition 1.10. Given G-metric spaces (X, d_1) and (Y, d_2), a function $T : X \to Y$ is called uniformly continuous if for every real number $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y, z \in X$ with $d_1(x, y, z) < \delta$, we have that $d_2(Tx, Ty, Tz) < \varepsilon$.

Finally, we point out that many results about the fixed point theory in G-metric spaces can be read in [1–9, 11–15].

2 The results

Theorem 2.1. *Let d be a G-metric on a space X and $T : (X, d) \to (X, d)$ a self mapping such that:*

$$d(T^n x, T^n y, T^n z) \leq K d(x, y, z),$$

for some $n > 1$ and $0 < K < 1$, whenever $x, y, z \in X$. If λ is a nonnegative real such that

$$K^{\frac{1}{n}} < \frac{1}{\lambda} < 1,$$

then the application $d' : X^3 \to [0,\infty)$ *defined by :*

$$d'(x,y,z) = \sum_{i=0}^{n-1} \lambda^i d(T^i x, T^i y, T^i z), \ \textit{whenever } x,y,z \in X,$$

satisfies:

i) d' *is a* G*-metric on the space* X*;*

ii) $T : (X,d') \to (X,d')$ *a self mapping such that:*

$$d'(Tx,Ty,Tz) \leq \frac{1}{\lambda} d'(x,y,z).$$[4]

Proof. We first prove that d' is a G-metric:

(G1) Indeed for $x,y,z \in X$, if $x = y = z$, then

$$d'(x,y,z) = 0.$$

(G2) For all $x,y \in X$ with $x \neq y$, it is clear that

$$0 < d(x,x,y) \leq d'(x,x,y).$$

(G3) For all $x,y,z \in X$ with $z \neq y$, since

$$0 < \lambda^i d(T^i x, T^i x, T^i y) \leq \lambda^i d(T^i x, T^i y, T^i z), \text{then}$$

$$\begin{aligned} 0 < d'(x,x,y) &= \sum_{i=0}^{n-1} \lambda^i d(T^i x, T^i x, T^i y) \\ &\leq \sum_{i=0}^{n-1} \lambda^i d(T^i x, T^i y, T^i z) \\ &= d'(x,y,z). \end{aligned}$$

(G4) Trivially $d'(x,y,z) = d'(x,z,y) = d'(y,z,x) = \ldots$, (symmetry in all three variables).

[4]i.e. T is a contraction with constant $\frac{1}{\lambda}$ with respect to d'.

(G5) For all $x, y, z, a \in X$, since

$$\lambda^i d(T^i x, T^i y, T^i z) \le \lambda^i [d(T^i x, T^i a, T^i a) + d(T^i a, T^i y, T^i z)],$$

we get

$$\begin{aligned} d'(x,y,z) &= \sum_{i=0}^{n-1} \lambda^i d(T^i x, T^i y, T^i z) \\ &\le \sum_{i=0}^{n-1} \lambda^i [d(T^i x, T^i a, T^i a) + d(T^i a, T^i y, T^i z)] \\ &= \sum_{i=0}^{n-1} \lambda^i d(T^i x, T^i a, T^i a) + \sum_{i=0}^{n-1} \lambda^i d(T^i a, T^i y, T^i z) \\ &= d'(x,a,a) + d'(a,y,z). \end{aligned}$$

Hence, d' is a G-metric on X.
We now prove that $T : (X, d') \to (X, d')$ is a contraction with constant $\frac{1}{\lambda}$.
It is readily seen, by a simple computation, that

$$d'(Tx, Ty, Tz) = \frac{1}{\lambda}[d'(x,y,z) - d(x,y,z)] + \lambda^{n-1} d(T^n x, T^n y, T^n z).$$

Since $T^n : (X, d) \to (X, d)$ is a contraction with constant K, it follows that

$$\begin{aligned} d'(Tx, Ty, Tz) &\le \frac{1}{\lambda}[d'(x,y,z) - d(x,y,z)] + K\lambda^{n-1} d(x,y,z) \\ &= \frac{1}{\lambda} d'(x,y,z) + \left(K - \frac{1}{\lambda^n}\right) \lambda^{n-1} d(x,y,z) \\ &\le \frac{1}{\lambda} d'(x,y,z), \end{aligned}$$

because of the choice $K^{\frac{1}{n}} < \frac{1}{\lambda}$. This completes the proof. □

Remark 2.2. The map d' can be thought of as an approximation of order $n-1$ of a certain G-metric h, equivalent to d'. Indeed, under the assumptions of Theorem 2.1, it is readily seen that

$$
\begin{aligned}
d'(x,y,z) &\leq \sum_{i=0}^{\infty} \lambda^i d(T^i x, T^i y, T^i z) \\
&\leq d'(x,y,z) + \lambda^n K d'(x,y,z) + \lambda^{2n} K^2 d'(x,y,z) + \cdots \\
&= \frac{1}{1-\lambda^n K} d'(x,y,z).
\end{aligned}
$$

The sum $h(x,y,z) := \sum_{i=0}^{\infty} \lambda^i d(T^i x, T^i y, T^i z)$ therefore defines a G-metric h, equivalent to d', as long as the series happen to converge for some $\lambda > 1$. Moreover, whenever h is finite, the map $T : (X,h) \to (X,h)$ is a contraction with contraction constant $\frac{1}{\lambda}$.

Next, we establish that whenever the mapping $T : (X,d) \to (X,d)$ is uniformly continuous and the G-metric d is G-complete, then the G-metric d' is also G-complete.

Theorem 2.3. *We repeat the assumptions of Theorem 2.1. If T is uniformly continuous and the G-metric d is G-complete, then so is the G-metric d'.*

Proof. Since $d(x,y,z) \leq d'(x,y,z)$ for any $x,y,z \in X$, any G-Cauchy sequence in (X,d') is also a G-Cauchy sequence in (X,d). It is therefore enough to prove that, under uniform continuiy of T in (X,d), any G-convergent sequence in (X,d) is also G-convergent in (X,d').

So let $\{x_n\}$ be a sequence in the G-metric space (X,d) such that $\{x_n\}$ G-converges to some $\xi \in (X,d)$. Set $M = \max\{\lambda^i, i = 1, \cdots, n-1\}$ and observe that

$$M \geq \lambda > 1.$$

Since all the powers of T are also uniformly continuous in (X,d), we can write that, for any $\varepsilon > 0$, there exists $\eta > 0$ such that for all $x,y,z \in X$, and $i = 1, \cdots, n-1$

$$d(x,y,z) < \eta \Longrightarrow d(T^i x, T^i y, T^i z) < \frac{\varepsilon}{Mn}.$$

Since $\{x_n\}$ G-converges to some $\xi \in (X,d)$, there exists $n_0 \in \mathbb{N}$ such that

$$p \geq n_0 \Longrightarrow d(\xi, x_p, x_p) < \eta.$$

Then

$$p > n_0 \Longrightarrow d(T^i \xi, T^i x_p, T^i x_p) < \frac{\varepsilon}{Mn} \quad \text{for } i = 1, \cdots, n-1,$$

i.e.

$$d'(\xi, x_p, x_p) < \frac{\varepsilon}{n}\left[\frac{1}{M} + \frac{\lambda}{M} + \cdots + \frac{\lambda^{n-1}}{M}\right] < \varepsilon.$$

Thus $\{x_n\}$ G-converges to ξ with respect to the G-metric d'. □

We conclude this paper by an example, illustrating the importance of the requirement for T to be uniformly continuous.

Example 2.4. Let $X = [0, 3]$ be endowed with the G-metric d, defined as

$$d(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\} \text{ for all } x, y, z \in [0, 3].$$

Define $T : [0, 3] \to [0, 3]$ by

$$Tx = \begin{cases} 1, & \text{if } 0 \le x \le 2 \\ 2, & \text{if } 2 < x \le 3. \end{cases}$$

The mapping T is discontinuous at $x = 2$ but $T^2 x = 1$ for all $x, y, z \in [0, 3]$, i.e. T^2 is a contraction and the unique fixed point is $x = 1$. Moreover, any real $K \in (0, 1)$ can be used as contraction constant for T^2. We can then apply Theorem 2.1 with any λ such that $\lambda > 1$. From Theorem 2.1, the G-metric d' is given by

$$d'(x, y, z) = \begin{cases} d(x, y, z), & \text{if } x, y, z \le 2 \text{ or } x, y, z > 2, \\ d(x, y, z) + \lambda, & \text{otherwise.} \end{cases}$$

The G-metric d' is not G-complete. Indeed the sequence $\{x_n\}$ given by $x_n = 2 + \frac{1}{n}$ is G-Cauchy in (X, d') but does not G-converge; as the only candidate for a limit is 2 but

$$d'(x_n, x_n, 2) = \frac{1}{n} + \lambda > \lambda \text{ for all } n.$$

The above example shows that, unless one assumes at least pointwise continuity, the conclusion of Theorem 2.3 can fail.

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