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RELATION BETWEEN ALGEBRAIC AND GEOMETRIC VIEW ON NURBS TENSOR PRODUCT SURFACES*

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Abstract. NURBS (Non-Uniform Rational B-Splines) belong to special approximation curves and surfaces which are described by control points with weights and B-spline basis functions. They are often used in modern areas of computer graphics as free-form modelling, modelling of processes. In literature, NURBS surfaces are often called tensor product surfaces. In this article we try to explain the relationship between the classic algebraic point of view and the practical geometrical application on NURBS.

Keywords: tensor product surface, bilinear form, B-spline, NURBS

MSC 2010: 53A05

1. Introduction

NURBS have become a standard type of mathematical approximation of surfaces in modern computer graphics. The general NURBS surface is described by a net of control points with weights and by two knot vectors. Theory of NURBS is summarized in [9], for example. NURBS objects are often used for free-form modelling because of their good modification possibilities (e.g. technic FFD—see [12]). The NURBS are used in different branches, for example robotics [5], film industry [13], reverse engineering [10], GIS [14], physical computing [11], etc.

The basic B-spline theory was proposed by Carl de Boor in [1], where the tensor product is schematically described. Tensor calculus is described in [2] and [6]. Non-tensor product NURBS surfaces using the smoothing cofactor-conformality method are constructed in [7].

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In this paper, we try to explain the algebraic point of view on tensor product surfaces and to establish the connection between this theory and the geometrical and practical use of NURBS.

Section 2 briefly outlines the basic tensor theory. In Section 3 the algebraic approach to B-spline functions and curves is discussed. Section 4 discusses the projective extension of NURBS curves and defines the abstract curve as a set of curves which are invariant with each other.

In Section 5 we deal with the NURBS surfaces. Analogously to Section 4, we introduce an abstract surface based on the characteristic form. In the last Section 6, we discuss some results and practical examples of our theory.

2. Tensor calculus

Let U, V be vector spaces over a field $\mathbb{T}$. A bilinear form ω on $U \times V$ is a function $\omega\colon\ U \times V \to T$ which satisfies the well-known axioms. The vector space of all bilinear forms between spaces U and V is called the tensor product. These mappings can be written as

$$\omega(\mathbf{u},\mathbf{v}) = \sum_{i=0}^{m}\sum_{j=0}^{n} u_i v_j \omega(\mathbf{e}_i,\mathbf{f}_j) = \sum_{i=0}^{m}\sum_{j=0}^{n} u_i v_j a_{ij}, \quad a_{ij} = (e_i, f_j), \tag{2.1}$$

or in the matrix form

$$\omega(\mathbf{u},\mathbf{v}) = (u_0, u_1, \dots, u_m)\begin{pmatrix} a_{00} & a_{01} & \dots & a_{0n} \\ a_{10} & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m0} & a_{m1} & \dots & a_{mn} \end{pmatrix}\begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_n, \end{pmatrix}, \tag{2.2}$$

where u_i, $i = 0, \dots, m$, v_j, $j = 0, \dots, n$ are the coordinates of the vectors $\mathbf{u} \in U$, $\mathbf{v} \in V$ with bases $\{\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_m\}$, $\{\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_n\}$.

A surface is described by a function of two parameters as a mapping of a plane area to Euclidean 3-dimensional space. Formally

$$S(s,t) = (x(s,t), y(s,t), z(s,t)) = \sum_{i=0}^{m}\sum_{j=0}^{n} f_i(u) g_j(v) b_{ij} \tag{2.3}$$

where $b_{ij} = (x_{ij}, y_{ij}, z_{ij})$, $0 \leqslant s, t \leqslant 1$, $m = 2$, $n = 2$.

Eq. (2.1) is formally similar to Eq. (2.3). Therefore, NURBS surface defined by Eq. (2.3) is often called as the tensor product surface (see [9]).

Eq. (2.3) defines a mapping of the set $D = \langle s1, s2\rangle \times \langle t1, t2\rangle$ (the set of ordered pairs $[s,t]$—the so called parameters area) to the Euclidean point space E_3. Therefore, the point $S = [x, y, z]$ of the surface $\mathcal{S}$ is a function of two real variables s, t.

3. B-spline curves—algebraic point of view

Definition 3.1. Let $\boldsymbol{t} = (t_0, t_1, \dots t_n)$ be a non-decreasing sequence of positive real numbers. The sequence $\boldsymbol{t}$ is called a knot vector. The B-spline function of degree p is defined as

$$N_i^0(t) = \begin{cases} 1, & t \in \langle t_i, t_{i+1}), \\ 0, & \text{otherwise,} \end{cases}$$

$$N_i^p(t) = \frac{t - t_i}{t_{i+p} - t_i} N_i^{p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_i} N_{i+1}^{p-1}(t), \tag{3.1}$$

where $p > 0$, $0 \leqslant i \leqslant n - p - 1$, $i \leqslant p \leqslant n - 1$, $\frac{0}{0} = 0$.

E x a m p l e 3.1. The knot vector is $\mathbf{t} = (0, 0, 1, 1)$, the degree $p = 0, 1$, the basis functions are

$$N_0^0(t) = 0, \quad N_1^0(t) = 1, \quad N_2^0(t) = 0,$$
$$N_0^1(t) = 1 - t, \quad N_1^1(t) = t, \quad t \in \langle 0, 1).$$

E x a m p l e 3.2. The knot vector is $\mathbf{t} = (0, 0, 0, 1, 1, 1)$, the degree $p = 0, 1, 2$, the basis functions are

$$N_i^0(t) = 0 \quad \text{for } i = 0, 1, 3, 4, \quad N_2^0(t) = 1,$$
$$N_0^1(t) = N_3^1(t) = 0, \quad N_1^1(t) = 1 - t, \quad N_2^1(t) = t,$$
$$N_0^2(t) = (1 - t)^2, \quad N_1^2(t) = 2t(1 - t), \quad N_2^2(t) = t^2, \quad t \in \langle 0, 1).$$

We can see that the polynomials of degree two correspond to the Bezier coefficients, because the Bezier curve is a subset of NURBS curves for a special knot vector (see e.g. [4]).

An arbitrary B-spline curve can be defined as

Definition 3.2. Let $\mathbf{t} = (t_0, t_1, \dots, t_n)$ be a knot vector, let $p \geqslant 1$ be the degree and $\mathbf{P}_i \in E^d$, $i = 0, 1, \dots, m$, the control points. The B-spline curve is defined by

$$C(t) = \sum_{i=0}^{m} \mathbf{P}_i N_i^p(t), \tag{3.2}$$

where N_i^p are the B-spline functions.

Eq. (3.2) can be written in the vector form as

$$\mathbf{c}(t) = \sum_{i=0}^{m} \mathbf{P}_i N_i^p(t), \tag{3.3}$$

where $\mathbf{c}(t)$ is the varying location vector of the curve points.

4. Explanation of weights of non-uniform B-spline (NURBS)—geometric point of view

NURBS curves are defined similarly to the B-spline. Every point $\mathbf{P}_i \in E^d$ has its real weight ω_i, $i = 0, 1, \ldots, m$. So an expression similar to (3.2) can be written as

$$C(t) = \frac{\sum_{i=0}^{m} \omega_i \mathbf{P}_i N_i^p(t)}{\sum_{i=0}^{m} \omega_i N_i^p(t)} \tag{4.1}$$

and one coordinate can be written as

$$c_k(t) = \frac{\sum_{i=0}^{m} \omega_i p_{ik} N_i^p(t)}{\sum_{i=0}^{m} \omega_i N_i^p(t)}, \quad k = 1, 2, \ldots, d. \tag{4.2}$$

More information about the influence of the weights on the form of the NURBS curve is available in [8].

Coordinates $c_k(t)$ in Eq. (4.2) are not linear combinations of basic functions, therefore they cannot be used for tensor product construction. We will show that these coordinates can be written as linear combinations of B-spline functions.

Let $\mathbf{P}_i = [p_{i1}, p_{i2}, \ldots, p_{id}] \in E^d$ be a d-dimensional point of the Euclidean space E^d. Eq. (3.2) can be written as

$$c_k(t) = \sum_{i=0}^{m} p_{ik} N_i^p(t), \quad k = 1, 2, \ldots, d, \tag{4.3}$$

where $\mathbf{C} = [c_1, c_2, \ldots, c_d] \in E^d$. In the projective extension $\overline{E}^d$ of the space E^d there is a correspondence between points $\mathbf{C}, \mathbf{P}_i \in E^d$ and $\overline{\mathbf{C}}, \overline{\mathbf{P}}_i \in \overline{E}^d$. These points can be written as

$$\overline{\mathbf{P}}_i = (\overline{p}_{i1}, \ldots, \overline{p}_{id}, \overline{p}_{id+1}) = \omega_i(p_{i1}, \ldots, p_{id}, 1) = (\omega_i p_{i1}, \ldots, \omega_i p_{id}, \omega_i), \tag{4.4}$$

$$\overline{\mathbf{C}} = (\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_d, \overline{c}_{d+1}) = \omega(c_1, c_2, \ldots, c_d, 1) = (\omega c_1, \omega c_2, \ldots, \omega c_d, \omega), \tag{4.5}$$

where $\omega_i \neq 0$ ($\omega \neq 0$) generates the point $\overline{\mathbf{P}}_i$ or $\overline{\mathbf{C}}$ using its arithmetic representation $(p_{i1}, \dots, p_{id}; 1)$ or $(c_1, \dots, c_d; 1)$ respectively.

In the projective extension $\overline{E}^d$ of the space E^d, Eq. (4.3) can be written as

$$\text{(4.6)} \qquad \overline{c}_k(t) = \sum_{i=0}^{m} \overline{p}_{ik} N_i^p(t) = \sum_{i=0}^{m} \omega_i p_{ik} N_i^p(t), \quad k = 1, 2, \dots, d+1.$$

It is evident that the basis functions for the B-spline are the same as for NURBS. The difference between Eq. (4.2) and Eq. (4.6) is in the usage of non-homogeneous and homogeneous coordinates. An arbitrary point $\overline{C}$ is

$$\text{(4.7)} \qquad \begin{aligned} \overline{C} &= (\overline{c}_1, \overline{c}_2, \dots, \overline{c}_d, \overline{c}_{d+1}) \\ &= \Big(\sum_{i=0}^{m} \omega_i p_{i1} N_i^p(t), \sum_{i=0}^{m} \omega_i p_{i2} N_i^p(t), \dots, \sum_{i=0}^{m} \omega_i p_{id} N_i^p(t), \sum_{i=0}^{m} \omega_i N_i^p(t) \Big). \end{aligned}$$

We have $\overline{c}_{d+1} \neq 0$ for proper points in (4.7). We get the Cartesian coordinates of the corresponding euclidean point by dividing by the last number:

$$\text{(4.8)} \qquad \overline{C} = \Big(\frac{\overline{c}_1}{\overline{c}_{d+1}}, \frac{\overline{c}_2}{\overline{c}_{d+1}}, \dots, \frac{\overline{c}_d}{\overline{c}_{d+1}}, 1 \Big) \Rightarrow C = \Big(\frac{\overline{c}_1}{\overline{c}_{d+1}}, \frac{\overline{c}_2}{\overline{c}_{d+1}}, \dots, \frac{\overline{c}_d}{\overline{c}_{d+1}} \Big).$$

Eq. (4.8) corresponds to Eq. (4.1) of the rational expression of the NURBS curve.

Abstract curves

NURBS curves or surfaces are commonly used in various CAD/CAM systems. The work with weights is based on decreasing or increasing the number from the explicit value 1. It causes the different shape of the curve (surface). Weights are obviously interpreted as the weights of the control points. From the geometrical point of view, however, the situation is somewhat different.

Example 4.1. Let us have a NURBS curve with arbitrary three control points P_0, P_1, P_2, a knot vector $(0, 0, 0, 1, 1, 1)$ and a weight vector $(1, \omega_1, 1)$. The value of ω_1 changes the type of the curve:

- $\omega_1 < 1 \to$ ellipse,
- $\omega_1 = 1 \to$ parabola,
- $\omega_1 > 1 \to$ hyperbola.

We see that the position of the control points does not change the type of the curve. So the weights decide about the geometric behavior of the curve.

Definition 4.1. The set of all curves with the same weight vector $(\omega_0, \dots, \omega_m)$ over the same knot vector is called an abstract NURBS curve. The expression of the last term in Eq. (4.7)

$$\overline{c}_{d+1} = \sum_{i=0}^{m} \omega_i N_i^p \tag{4.9}$$

is called the equation of the abstract curve.

R e m a r k 4.1. Eq. (4.9) is not an equation of a concrete curve. A particular curve is determined by the control points chosen. In this way, a representation of the abstract curve is actually chosen.

R e m a r k 4.2. The concrete curves in the same abstract curve have the same geometrical behavior. It means that one curve can be transformed to another by projective transformations.

For $m = 2$, Eq. (4.9) is the equation of an abstract conic section. It describes the set of all conic sections which are projectively invariant. With a special choice of the control points we get the classical conic sections, i.e. hyperbola, parabola, ellipse. For example, the characteristic equation

$$\overline{c}_{d+1} = N_0^2 + \frac{\sqrt{2}}{2} N_1^2 + N_2^2 \tag{4.10}$$

or the weight vector

$$\left(1, \tfrac{1}{2}\sqrt{2}, 1\right) \tag{4.11}$$

determine the elliptic arc defined by the so-called polar conjugate diameters. If the control points P_0, P_1, P_2 satisfy

$$\overrightarrow{P_0P_1} \perp \overrightarrow{P_1P_2},$$

these diameters are the axes of the ellipse. If, in addition,

$$|P_0P_1| = |P_1P_2| = r,$$

then the arc is a quadrant. The abstract curve defined by the vector (4.11) is illustrated in Fig. 1. Axis affinity as an important example of projective transform is used.

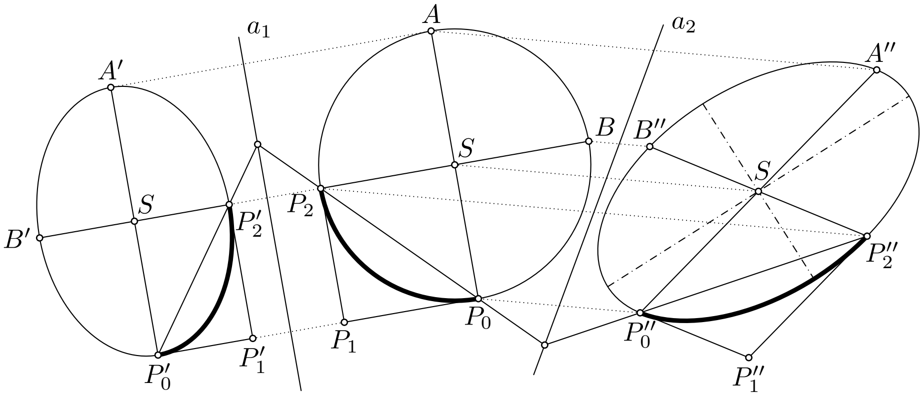

Figure 1. Different representants of abstract NURBs curve $(1, \sqrt{2}/2, 1)$.

R e m a r k 4.3. By means of suitable control points, four representations may be selected in (4.11) to create an ellipse.

R e m a r k 4.4. Substitute the vector $k\boldsymbol{\omega} = (k\omega_0, \ldots, k\omega_m)$, $k \neq 0$ for $\boldsymbol{\omega} = (\omega_0, \ldots, \omega_m)$ in (4.9). Then we get

$$
\overline{C} = \Bigl(k \sum_{i=0}^{m} \omega_i p_{i1} N_i^p(t), k \sum_{i=0}^{m} \omega_i p_{i2} N_i^p(t), \ldots, k \sum_{i=0}^{m} \omega_i p_{id} N_i^p(t), k \sum_{i=0}^{m} \omega_i N_i^p(t)\Bigr). \tag{4.12}
$$

Dividing by the last member we obtain the same homogeneous coordinates as for the vector $\boldsymbol{\omega} = (\omega_0, \ldots, \omega_m)$. It means that the weight vectors $\boldsymbol{\omega}, k\boldsymbol{\omega}$, $k \neq 0$, define the same abstract curve. This fact is used in particular for $k = -1$ in the next chapter.

5. NURBS surface as the tensor product of NURBS curves

NURBS surface is defined on a regular net of control points $\mathbf{P}_{ij} \in E^d$, $i = 0, \ldots, m$; $j = 0, \ldots n$ with weights ω_{ij} and with two knot vectors $\mathbf{u} = (u_0, u_1, \ldots, u_{m+p+1})$ and $\mathbf{v} = (v_0, v_1, \ldots, v_{n+q+1})$.

The equation of the NURBS surface can be written as

$$
C(u, v) = \frac{\sum\limits_{i=0}^{m} \sum\limits_{j=0}^{n} \omega_{ij} \mathbf{P}_{ij} N_i^p(u) N_j^q(v)}{\sum\limits_{i=0}^{m} \sum\limits_{j=0}^{n} N_i^p(u) N_j^q(v)}, \tag{5.1}
$$

where $(u, v) \in \langle u_0, u_{m+p+1}) \times \langle v_0, v_{n+q+1})$.

The kth coordinate of a point on a NURBS surface is

$$c_k(u,v) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{n} \omega_{ij} p_{ijk} N_i^p(u) N_j^q(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} N_i^p(u) N_j^q(v)}, \quad k = 0,1,\dots,d, \tag{5.2}$$

where p_{ijk} is the kth Cartesian coordinate of $\mathbf{P}_{ij}$.

In the projective extension $\overline{E}_d$ of the space E_d, NURBS surface has equations

$$\overline{c}_k(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \omega_{ij} \overline{p}_{ijk} N_i^p(u) N_j^q(v), \quad k = 0,1,\dots,d, \tag{5.3}$$

$$\overline{c}_{d+1}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \omega_{ij} N_i^p(u) N_j^q(v), \tag{5.4}$$

where p_{ijk} is the kth projective coordinate of the corresponding $\overline{P}_{ij}$.

The matrix notation of Eqs. (5.3) and (5.4) is

$$\overline{c}_k = (N_0^p, N_1^p, \dots, N_m^p) \begin{pmatrix} \omega_{00} p_{00k} & \omega_{01} p_{01k} & \dots & \omega_{0n} p_{0nk} \\ \omega_{10} p_{10k} & \omega_{11} p_{11k} & \dots & \omega_{1n} p_{1nk} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{m0} p_{m0k} & \omega_{m1} p_{m1k} & \dots & \omega_{mn} p_{mnk} \end{pmatrix} \begin{pmatrix} N_0^q \\ N_1^q \\ \vdots \\ N_n^q \end{pmatrix} \tag{5.5}$$

and

$$\overline{c}_{d+1} = (N_0^p, N_1^p, \dots, N_m^p) \begin{pmatrix} \omega_{00} & \omega_{01} & \dots & \omega_{0n} \\ \omega_{10} & \omega_{11} & \dots & \omega_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{m0} & \omega_{m1} & \dots & \omega_{mn} \end{pmatrix} \begin{pmatrix} N_0^q \\ N_1^q \\ \vdots \\ N_n^q \end{pmatrix}. \tag{5.6}$$

Definition 5.1. The matrix of type $(m+1) \times (n+1)$ in (5.6) is called the weight matrix. The set of all NURBS surfaces with the same weight matrix is called an abstract NURBS surface. Equation (5.4) or (5.6) is called the equation of the abstract NURBS surface (compare with Def. 4.1).

Remarks 4.2 and 4.1 hold for abstract and concrete NURBS surfaces by analogy.

Assume $\omega_{ij} \neq 0$ for all $i = 0,1,\dots,m$, $j = 0,1,\dots,n$ and choose $r = 0,\dots,m$ and $s = 0,\dots,n$ arbitrary but constant indexes. Let

$$\overline{\omega}_{rs} = \pm\sqrt{|\omega_{rs}|}. \tag{5.7}$$

If $\overline{\omega}_{rs} > 0$, let

$$\overline{\omega}_{is} = \frac{\omega_{is}}{\overline{\omega}_{rs}}; \quad \overline{\omega}_{rj} = \frac{\omega_{rj}}{\overline{\omega}_{rs}}; \quad \overline{\omega}_{ij} = \frac{\omega_{ij}}{\overline{\omega}_{is}\overline{\omega}_{rj}}.$$

If $\overline{\omega}_{rs} < 0$, let

$$\overline{\omega}_{is} = -\frac{\omega_{is}}{\overline{\omega}_{rs}}; \quad \overline{\omega}_{rj} = \frac{\omega_{rj}}{\overline{\omega}_{rs}}; \quad \overline{\omega}_{ij} = \frac{\omega_{ij}}{\overline{\omega}_{is}\overline{\omega}_{rj}}.$$

Equations (5.5) and (5.6) are transformed to equations

$$(5.8) \qquad \overline{c}_k = (\overline{\omega}_{0s}N_0^p, \dots, \overline{\omega}_{rs}N_r^p, \dots, \overline{\omega}_{ms}N_m^p) \times \begin{pmatrix} \overline{\omega}_{00}p_{00k} & \dots & p_{0sk} & \dots & \overline{\omega}_{0n}p_{0nk} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{r0k} & \dots & p_{rsk} & \dots & p_{rnk} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \overline{\omega}_{m0}p_{m0k} & \dots & p_{msk} & \dots & \overline{\omega}_{mn}p_{mnk} \end{pmatrix} \begin{pmatrix} \overline{\omega}_{r0}N_0^q \\ \vdots \\ \overline{\omega}_{rs}N_s^q \\ \vdots \\ \overline{\omega}_{rn}N_n^q \end{pmatrix}$$

and

$$(5.9) \qquad \overline{c}_{d+1} = (\overline{\omega}_{0s}N_0^p, \dots, \overline{\omega}_{rs}N_r^p, \dots, \overline{\omega}_{ms}N_m^p) \times \begin{pmatrix} \overline{\omega}_{00} & \dots & 1 & \dots & \overline{\omega}_{0n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \overline{\omega}_{m0} & \dots & 1 & \dots & \overline{\omega}_{mn} \end{pmatrix} \begin{pmatrix} \overline{\omega}_{r0}N_0^q \\ \vdots \\ \overline{\omega}_{rs}N_s^q \\ \vdots \\ \overline{\omega}_{rn}N_n^q \end{pmatrix}$$

by these substitutions.

To construct the tensor product surface, it is necessary to work with vector spaces $\mathbf{N}_m^p$ and $\mathbf{N}_n^q$ which are generated respectively by B-spline functions $[N_0^p, N_1^p, \dots, N_m^p]$ and $[N_0^q, N_1^q, \dots, N_n^q]$ with knot vectors $\mathbf{u}$ and $\mathbf{v}$.

The elements of these spaces are linear combinations of basis functions in the form (4.9)—sets of projective invariant curves (abstract curves). The coordinates of the abstract NURBS curve as a vector of the vector space N_m^p are the coordinates of its weight vector $(\omega_0, \omega_1, \dots, \omega_m)$.

The tensor product $\mathbf{N}_m^p \otimes \mathbf{N}_n^q$ of vector spaces $\mathbf{N}_m^p$ and $\mathbf{N}_n^q$ (sets of abstract curves, i.e. projectively invariant NURBS curves) is a set of bilinear forms

$$(5.10) \quad \omega(N_{\mathbf{u}}^p, N_{\mathbf{v}}^q) = \overline{c}_{d+1} = (\overline{\omega}_{0s}, \dots, \overline{\omega}_{rs}, \dots, \overline{\omega}_{ms}) \begin{pmatrix} \overline{\omega}_{00} & \dots & 1 & \dots & \overline{\omega}_{0n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \overline{\omega}_{m0} & \dots & 1 & \dots & \overline{\omega}_{mn} \end{pmatrix} \begin{pmatrix} \overline{\omega}_{r0} \\ \vdots \\ \overline{\omega}_{rs} \\ \vdots \\ \overline{\omega}_{rn} \end{pmatrix}.$$

Definition 5.2. The set of all bilinear forms (5.10) is called the r,s-partial tensor product of abstract curves given by vectors $(\overline{\omega}_{0s},\ldots,\overline{\omega}_{rs},\ldots,\overline{\omega}_{ms})$ and $(\overline{\omega}_{r0},\ldots,\overline{\omega}_{rs},\ldots,\overline{\omega}_{rn})$. If in the weight matrix (5.10), $\overline{\omega}_{ij}=1$ holds for all i, j; $i=0,1,\ldots,m$, $j=0,1,\ldots,n$ then (5.10) is called the total tensor product (or shortly the tensor product) of these curves. The bilinear form (5.10) is the weight form of the abstract surface.

A sign conversion of $\overline{\omega}_{rs}$ according to (5.7) results in the sign conversion of all ω_{is} or ω_{rj} in (5.8), (5.9), (5.10), i.e. in the sign conversion of the weight vectors of both the curves in the tensor product. However, the vectors $\overline{\omega}$ and $-\overline{\omega}$ define the same abstract curves (see Remark 4.4). The transform of the characteristic equation (5.6) to the r,s-partial tensor product (5.10) is unique.

The partial tensor product of abstract curves does not define the abstract surface uniquely. For example, all 0,0-partial products of the same NURBS abstract curves have identical two boundaries, but other rth and sth abstract curves may be different. The total tensor product defines the abstract surface uniquely which follows from the fact that all r-curves (s-curves) are identical.

6. Results and applications

The dimensions of the spaces $\mathbf{N}_m^p$ and $\mathbf{N}_n^q$ are $m+1$ or $n+1$ respectively. It means that they are isomorphic with linear spaces $\mathbb{R}^{m+1}$ and $\mathbb{R}^{n+1}$ respectively but the spaces $\mathbf{N}_m^p$ and $\mathbf{N}_n^q$ are more suitable for practical use.

Example 6.1. The weight form

$$\omega(N_2^2\,N_2^2)=\left(1,\frac{1}{2},1\right)\begin{pmatrix}1&1&1\\1&\overline{\omega}_{11}&\overline{\omega}_{12}\\1&\overline{\omega}_{21}&\overline{\omega}_{22}\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix}$$

describes the surface which is created by the 0,0-partial tensor product of the ellipse $\left(1,\frac{1}{2},1\right)$ and the hyperbola $(1,2,1)$. These two abstract curves represent the adjoining curves of the surface boundary. The next boundary curves are specified by characteristic vectors $\left(1,\frac{1}{2}\overline{\omega}_{12},\overline{\omega}_{22}\right)$ and $(1,2\overline{\omega}_{21},\overline{\omega}_{22})$. Other r-curves, s-curves are curves with characteristic vectors $\left(1,\frac{1}{2}\overline{\omega}_{11},\overline{\omega}_{21}\right)$ and $(1,2\overline{\omega}_{11},\overline{\omega}_{12})$.

Example 6.2. Consider an abstract surface with the weight form

$$\omega(N_2^2\,N_2^2)=\left(1,\frac{1}{2},1\right)\begin{pmatrix}1&1&1\\1&2&1\\1&1&1\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix}.$$

It is a partial tensor product of 0,0-type (or 0,2; 2,0; 2,2 types too) of the ellipse $\left(1, \frac{1}{2}, 1\right)$ and the hyperbola $(1, 2, 1)$. These curves form the boundary of the surface, the "central" curves are the parabola $(1, 1, 1)$ and the hyperbola $(1, 4, 1)$.

E x a m p l e 6.3. The characteristic form of the total tensor product of a parabola and a line is

$$\omega(N_1^1\, N_2^2) = (1, 1) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The resulting surface is a parabolic cylinder.

E x a m p l e 6.4. The characteristic form of the total tensor product of an elliptic arc with a hyperbolic arc is a part of an elliptical hyperboloid

$$\omega(N_2^2\, N_2^2) = \left(1, \frac{\sqrt{2}}{2}, 1\right) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}. \tag{6.1}$$

By using suitable control points, four representatives may be selected in (6.1) to create a unipartite elliptical hyperboloid (according to Remark 4.3).

7. Conclusion

In this paper, the relationship between the classical view on tensor product and application of NURBS surfaces was explained. Our idea is based on projective extension of surfaces and introducing the abstract curves and surfaces. These sets of projective invariant curves and surfaces demonstrate their projective properties clearly than the control points. Our approach was illustrated on some examples.

In our future work we are interested in studying abstract objects with singularities and their properties.

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