Relational Understanding and Instrumental Understanding[1]

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Faux Amis

Faux amis is a term used by the French to describe words which are the same, or very alike, in two languages, but whose meanings are different. For example:

French word	*Meaning in English*
histoire	story, not history
libraire	bookshop, not library
chef	head of any organisation, not only chief cook
agrément	pleasure or amusement, not agreement
docteur	doctor (higher degree) not medical practitioner
médecin	medical practitioner, not medicine
parent	relations in general, including parents

One gets *faux amis* between English as spoken in different parts of the world. An Englishman asking in America for a biscuit would be given what we call a scone. To get what we call a biscuit, he would have to ask for a cookie. And between English as used in mathematics and in everyday life there are such words as field, group, ring, ideal.

A person who is unaware that the word he is using is a *faux ami* can make inconvenient mistakes. We expect history to be true, but not a story. We take books without paying from a library, but not from a bookshop; and so on. But in the foregoing examples there are cues which might put one on guard: difference of language, or of country, or of context.

If, however, the same word is used in the same language, country and context, with two meanings whose difference is non-trivial but as basic as the difference between the meaning of (say) 'histoire' and 'story', which is a difference between fact and fiction, one may expect serious confusion. Two such words can be identified in the context of mathematics; and it is the alternative meanings attached to these words,

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each by a large following, which in my belief are at the root of many of the difficulties in mathematics education today.

One of these is 'understanding'. It was brought to my attention some years ago by Stieg Mellin-Olsen, of Bergen University, that there are in current use two meanings of this word. These he distinguishes by calling them 'relational understanding' and 'instrumental understanding'. By the former is meant what I have always meant by understanding, and probably most readers of this article: knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have in the past described as 'rules without reasons', without realising that for many pupils *and their teachers* the possession of such a rule, and ability to use it, was why they meant by 'understanding'.

Suppose that a teacher reminds a class that the area of a rectangle is given by $A = L \times B$. A pupil who has been away says he does not understand, so the teacher gives him an explanation along these lines. "The formula tells you that to get the area of a rectangle, you multiply the length by the breadth." "Oh, I see," says the child, and gets on with the exercise. If we were now to say to him (in effect) "You may think you understand, but you don't really," he would not agree. "Of course I do. Look; I've got all these answers right." Nor would he be pleased at our devaluing of his achievement. And with his meaning of the word, he does understand.

We can all think of examples of this kind: 'borrowing' in subtraction, 'turn it upside down and multiply' for division by a fraction, 'take it over to the other side and change the sign', are obvious ones; but once the concept has been formed, other examples of instrumental explanations can be identified in abundance in many widely used texts. Here are two from a text used by a former direct-grant grammar school, now independent, with a high academic standard.

Multiplication of fractions To multiply a fraction by a fraction, multiply the two numerators together to make the numerator of the product, and the two denominators to makes its denominator.

E.g. $\frac{2}{3}$ of $\frac{4}{5} = \frac{2\times 4}{3\times 5} = \frac{8}{15}$

$\frac{3}{5} \times \frac{10}{13} = \frac{30}{65} = \frac{6}{13}$

The multiplication sign $\times$ is generally used instead of the word 'of'.

Circles The circumference of a circle (that is its perimeter, or the length of its boundary) is found by measurement to be a little more than three times the length of its diameter. In any circle the circumference is approximately 3.1416 times the diameter, which is roughly $3\frac{1}{7}$ times the diameter. Neither of these figures is exact, as the exact number cannot be expressed either as a fraction or a decimal. The number is represented by the Greek letter π.

$$\text{Circumference} = \pi d \text{ or } 2\pi r,$$
$$\text{Area} = \pi r^2.$$

The reader is urged to try for himself this exercise of looking for and identifying examples of instrumental explanations, both in texts and in the classroom. This will have three benefits. (i) For persons like the writer, and most readers of this article, it may be hard to realise how widespread is the instrumental approach. (ii) It will help, by repeated examples, to consolidate the two contrasting concepts. (iii) It is a good preparation for trying to formulate the difference in general terms. Result (i) is necessary for what follows in the rest of the present section, while (ii) and (iii) will be useful for the others.

If it is accepted that these two categories are both well-filled, by those pupils and teachers whose goals are respectively relational and instrumental understanding (by the pupil), two questions arise. First, does this matter? And second, is one kind better than the other? For years I have taken for granted the answers to both these questions: briefly, 'Yes; relational.' But the existence of a large body of experienced teachers and of a large number of texts belonging to the opposite camp has forced me to think more about why I hold this view. In the process of changing the judgement from an intuitive to a reflective one, I think I have learnt something useful. The two questions are not entirely separate, but in the present section I shall concentrate as far as possible on the first: does it matter?

The problem here is that of a mis-match, which arises automatically in any *faux ami* situation, and does not depend on whether A or B's meaning is 'the right one'. Let us imagine, if we can, that school A send a team to play school B at a game called 'football', but that neither team knows that there are two kinds (called 'association' and 'rugby'). School A plays soccer and has never heard of rugger, and vice versa for B. Each team will rapidly decide that the others are crazy, or a lot of foul players. Team A in particular will think that B uses a mis-shapen ball, and commit one foul after another. Unless the two sides stop and talk about what game they think they are playing at, long enough to gain some mutual understanding, the game will break up in disorder and the two teams will never want to meet again.

Though it may be hard to imagine such a situation arising on the football field, this is not a far-fetched analogy for what goes on in many mathematics lessons, even now. There is this important difference, that one side at least cannot refuse to play. The encounter is compulsory, on five days a week, for about 36 weeks a year, over 10 years or more of a child's life.

Leaving aside for the moment whether one kind is better than the other, there are two kinds of mathematical mis-matches which can occur.

1. Pupils whose goal is to understand instrumentally, taught by a teacher who wants them to understand relationally.
2. The other way about.

The first of these will cause fewer problems short-term to the pupils, though it will be frustrating to the teacher. The pupils just won't want to know all the careful groundwork he gives in preparation for whatever is to be learnt next, nor his careful explanations. All they want is some kind of rule for getting the answer. As soon as this is reached, they latch on to it and ignore the rest.

If the teacher asks a question that does not quite fit the rule, of course they will get it wrong. For the following example I have to thank Mr. Peter Burney, at that time a student at Coventry College of Education on teaching practice. While teaching area, he became suspicious that the children did not really understand what they were doing. So he asked them: "What is the area of a field 20 cms by 15 yards?" The reply was: "300 square centimetres". He asked: "Why not 300 square yards?" Answer: "Because area is always in square centimetres."

To prevent errors like the above the pupils need another rule (or, of course, relational understanding), that both dimensions must be in the same unit. This anticipates one of the arguments which I shall use against instrumental understanding, that it usually involves a multiplicity of rules rather than fewer principles of more general application.

There is of course always the chance that a few of the pupils will catch on to what the teacher is trying to do. If only for the sake of these, I think he should go on trying. By many, probably a majority, his attempts to convince them that being able to use the rule is not enough will not be well received. 'Well is the enemy of better,' and if pupils can get the right answers by the kind of thinking they are used to, they will not take kindly to suggestions that they should try for something beyond this.

The other mis-match, in which pupils are trying to understand relationally but the teaching makes this impossible, can be a more damaging one. An instance which stays in my memory is that of a neighbour's child, then seven years old. He was a very bright little boy, with an I.Q. of 140. At the age of five he could read *The Times,* but at seven he regularly cried over his mathematics homework. His misfortune was that he was trying to understand relationally teaching which could not be understood in this way. My evidence for this belief is that when I taught him relationally myself, with the help of Unifix, he caught on quickly and with real pleasure.

A less obvious mis-match is that which may occur between teacher and text. Suppose that we have a teacher whose conception of understanding is instrumental, who for one reason or other is using a text which aim is relational understanding by the pupil. It will take more than this to change his teaching style. I was in a school which was using my own text[1], and noticed (they were at Chapter 1 of *Book* 1) that some of the pupils were writing answers like

'the set of {flowers}'.

When I mentioned this to the teacher (he was head of mathematics) he asked the class to pay attention to him and said: "Some of you are not writing your answers properly. Look at the example in the book, at the beginning of the exercise, and be sure you write you answers exactly like that."

Much of what is being taught under the description of "modern mathematics" is being taught and learnt just as instrumentally as were the syllabi which have been replaced. This is predictable from the difficulty of accommodating (restructuring) our existing schemas[2]. To the extent that this is so, the innovations have probably done more harm than good, by introducing a mis-match between the teacher and the aims implicit in the new content. For the purpose of introducing ideas such as sets, mappings and variables is the help which, rightly used, they can give to relational understanding. If pupils are still being taught instrumentally, then a 'traditional' syllabus will probably benefit them more. They will at least acquire proficiency in a number of mathem-atical techniques which will be of use to them in other subjects, and whose lack has recently been the subject of complaints by teachers of science, employers and others.

Near the beginning I said that two *faux amis* could be identified in the context of mathematics. The second one is even more serious; it is the word 'mathematics' itself. For we are not talking about better and worse teaching of the same kind of mathematics. It is easy to think this, just as our imaginary soccer players who did not know that their opponents were playing a different game might think that the other side picked up the ball and ran with it because they could not kick properly, especially with such a mis-shapen ball. In which case they might kindly offer them a better ball and some lessons on dribbling.

It has taken me some time to realise that this is not the case. I used to think that maths teachers were all teaching the same subject, some doing it better than others.

I now believe that *there are two effectively different subjects being taught under the same name, 'mathematics'*. If this is true, then this difference matters beyond any of the differences in syllabi which are so widely debated. So I would like to try to emphasise the point with the help of another analogy.

Imagine that two groups of children are taught music as a pencil-and-paper subject. They are all shown the five-line stave, with the curly 'treble sign at the beginning; and taught that marks on the lines are called E, G, B, D, F. Marks between the lines are called F, A, C, E. They learn that a line with an open oval is called a minim, and is worth two with blacked-in ovals which are called crotchets, or four with blacked-in ovals and a tail which are called quavers, and so on – musical multiplication tables if you like. For one group of children, all their learning is of this kind and nothing beyond. If they have a music lesson a day, five days a week in school terms, and are told that it is important, these children could in time probably learn to write out the marks for simple melodies such as *God Save the Queen* and *Auld Lang Syne,* and to solve simple problems such as 'What time is this in?' and 'What key?', and even 'Transpose this melody from C major to A major.' They would find it boring, and the rules to be memorised would be so numerous that problems like 'Write a simple accompaniment for this melody' would be too difficult for most. They would give up the subject as soon as possible, and remember it with dislike.

The other group is taught to associate certain sounds with these marks on paper. For the first few years these are audible sounds, which they make themselves on simple instruments. After a time they can still imagine the sounds whenever they see or write the marks on paper. Associated with every sequence of marks is a melody, and with every vertical set a harmony. The keys C major and A major have an audible relationship, and a similar relationship can be found between certain other pairs of keys. And so on. Much less memory work is involved, and what has to be remembered is largely in the form of related wholes (such as melodies) which their minds easily retain. Exercises such as were mentioned earlier ('Write a simple accompaniment') would be within the ability of most. These children would also find their learning intrinsically pleasurable, and many would continue it voluntarily, even after O-level or C.S.E.

For the present purpose I have invented two non-existent kinds of 'music lesson', both pencil-and-paper exercises (in the second case, after the first year or two). But the difference between these imaginary activities is no greater than that between two activities which actually go on under the name of mathematics. (We can make the analogy closer, if we imagine that the first group of children were initially taught sounds for the notes in a rather half-hearted way, but that the associations were too ill-formed and unorganised to last.)

The above analogy is, clearly, heavily biased in favour of relational mathematics. This reflects my own viewpoint. To call it a viewpoint, however, implies that I no longer regard it as a self-evident truth which requires no justification: which it can hardly be if many experienced teachers continue to teach instrumental mathematics. The next step is to try to argue the merits of both points of view as clearly and fairly as possible; and especially of the point of view opposite to one's own. This is why the next section is called Devil's Advocate. In one way this only describes that part which puts the case for instrumental understanding. But it also justifies the other part, since an imaginary opponent who thinks differently from oneself is a good device for making clearer to oneself why one does think this way.

Devil's Advocate

Given that so many teachers teach instrumental mathematics, might this be because it does have certain advantages? I have been able to think of three advantages (as distinct from situational reasons for teaching this way, which will be discussed later).

1. Within its own context, instrumental mathematics is usually easier to understand; sometimes much easier. Some topics, such as multiplying two negative numbers together, or dividing by a fractional number, are difficult to understand relationally. "Minus times minus equals plus" and "to divide by a fraction you turn it upside down and multiply' are easily remembered rules. If what is wanted is a page of right answers, instrumental mathematics can provide this more quickly and easily.
2. So the rewards are more immediate, and more apparent. It is nice to get a page of right answers, and we must not underrate the importance of the feeling of success which pupils get from this. Recently I visited a school where some of the children describe themselves as 'thickos'. Their teachers use the term too. These children need success to restore their self-confidence, and it can be argued that they can achieve this more quickly and easily in instrumental mathematics than in relational.
3. Just because less knowledge is involved, one can often get the right answer more quickly and reliably by instrumental thinking than relational. This difference, is so marked that even relational mathematicians often use instrumental thinking. This is a point of much theoretical interest, which I hope to discuss more fully on a future occasion.

The above may well not do full justice to instrumental mathematics. I shall be glad to know of any further advantages which it may have.

There are four advantages (at least) in relational mathematics.

4. *It is more adaptable to new tasks.* Recently I was trying to help a boy who had learnt to multiply two decimal fractions together by dropping the decimal point, multiplying as for whole numbers, and re-inserting the

decimal point to give the same total number of digits after the decimal point as there were before. This is a handy method if you know why it works. Through no fault of his own, this child did not; and not unreasonably, applied it also to division of decimals. By this method $4.8 \div 0.6$ came to 0.08. The same pupil had also learnt that if you know two angles of a triangle, you can find the third by adding the two given angles together and subtracting from 180°. He got ten questions right this way (his teacher believed in plenty of practise), and went on to use the same method for finding the exterior angles. So he got the next five answers wrong.

I do not think he was being stupid in either of these cases. He was simply extrapolating from what he already knew. But relational understanding, by knowing not only what method worked but why, would have enabled him to relate the method to the problem, and possibly to adapt the method to new problems. Instrumental understanding necessitates memorising which problems a method works for and which not, and also learning a different method for each new class of problems. So the first advantage of relational mathematics leads to:

5. *It is easier to remember.* There is a seeming paradox here, in that it is certainly harder to learn. It is certainly easier for pupils to learn that 'area of a triangle $= \frac{1}{2}$ base $\times$ height' than to learn why this is so. But they then have to learn separate rules for triangles, rectangles, parallelograms, trapeziums; whereas relational understanding consists partly in seeing all these in relation to the area of a rectangle. It is still desirable to know the separate rules; one does not want to have to derive them afresh every time. But knowing also how they are inter-related enables one to remember them as parts of a connected whole, which is easier.

There is more to learn – the connections as well as the separate rules – but the result, once learnt, is more lasting. So there is less re-learning to do, and long-term the time taken may well be less altogether.

Teaching for relational understanding may also involve more actual content. Earlier, an instrumental explanation was quoted leading to the statement ‘Circumference = πd’. For relational understanding of this, the idea of a proportion would have to be taught first (among others), and this would make it a much longer job than simply teaching the rules as given. But proportionality has such a wide range of other applications that it is worth teaching on these grounds also. In relational mathematics this happens rather often. Ideas required for understanding a particular topic turn out to be basic for understanding many other topics too. Sets, mappings and equivalence are such ideas. Unfortunately the benefits which might come from teaching them are often lost by teaching them as separate topics, rather than as fundamental concepts by which whole areas of mathematics can be interrelated.

6. *Relational knowledge can be effective as a goal in itself.* This is an empiric fact, based on evidence from controlled experiments using non-mathematical material. The need for external rewards and punishments is greatly reduced, making what is often called the ‘motivational’ side of teacher’s job much easier. This is related to:
7. *Relational schemas are organic in quality.* This is the best way I have been able to formulate a quality by which they seem to act as an agent of their own growth. The connection with 3 is that if people get satisfaction from relational understanding, they may not only try to understand relationally new material which is put before them, but also actively seek out new material and explore new areas, very much like a tree extending its roots or an animal exploring new territory in search of nourishment. To develop this idea beyond the level of an analogy is beyond the scope of the present paper, but it is too important to leave out.

If the above is anything like a fair presentation of the cases for the two sides, it would appear that while a case might exist for instrumental mathematics short-term and within a limited context, long-term and in the context of a child’s whole education it does not. So why are so many children taught only instrumental mathematics throughout their school careers? Unless we can answer this, there is little hope of improving the situation.

An individual teacher might make a reasoned choice to teach for instrumental understanding on one or more of the following grounds.

1. That relational understanding would take too long to achieve, and to be able to use a particular technique is all that these pupils are likely to need.
2. That relational understanding of a particular topic is too difficult, but the pupils still need it for examination reasons.
3. That a skill is needed for use in another subject (e.g. science) before it can be understood relationally with the schemas presently available to the pupil.
4. That he is a junior teacher in a school where all the other mathematics teaching is instrumental.

All of these imply, as does the phrase ‘make a reasoned choice’, that he is able to consider the alternative goals of instrumental and relational understanding on their merits and in relation to a particular situation. To make an informed choice of this kind implies awareness of the distinction, and relational understanding of the mathematics itself. So nothing else but relational understanding can ever be adequate for a teacher. One has to face the fact that this is absent in many who teach mathematics; perhaps even a majority.

Situational factors which contribute to the difficulty include:

1. *The backwash effect of examinations.* In view of the importance of examinations for future employment, one can hardly blame pupils if success in these is one of their major aims. The way pupils work cannot but be influenced by the goal for which they are working, which is to answer correctly a sufficient number of questions.
2. *Over-burdened syllabi.* Part of the trouble here is the high concentration of the information content of mathematics. A mathematical statement may condense into a single line as much as in another subject might take over one or two paragraphs. By mathematicians accustomed to handling such concentrated ideas, this is often overlooked (which may be why most mathematics lecturers go too fast). Non-mathematicians do not realise it at all. Whatever the reason, almost all syllabi would be much better if much

reduced in amount so that there would be time to teach them better.

3. *Difficulty of assessment* of whether a person understands relationally or instrumentally. From the marks he makes on paper, it is very hard to make valid inference about the mental processes by which a pupil has been led to make them; hence the difficulty of sound examining in mathematics. In a teaching situation, talking with the pupil is almost certainly the best way to find out; but in a class of over 30, it may be difficult to find the time.
4. *The great psychological difficulty for teachers of accommodating (re-structuring) their existing and long-standing schemas,* even for the minority who know they need to, want to do so, and have time for study.

From a recent article[3] discussing the practical, intellectual and cultural value of a mathematics education (and I have no doubt that he means relational mathematics!) by Sir Hermann Bondi, I take these three paragraphs. (In the original, they are not consecutive.)

> So far my glowing tribute to mathematics has left out a vital point: the rejection of mathematics by so many, a rejection that in not a few cases turns to abject fright.
>
> The negative attitude to mathematics, unhappily so common, even among otherwise highly-educated people, is surely the greatest measure for our failure and a real danger to our society.
>
> This is perhaps the clearest indication that something is wrong, and indeed very wrong, with the situation. It is not hard to blame education for at least a share of the responsibility; it is harder to pinpoint the blame, and even more difficult to suggest new remedies.

If for 'blame' we may substitute 'cause', there can be small doubt that the widespread failure to teach relational mathematics – a failure to be found in primary, secondary and further education, and in 'modern' as well as 'traditional' courses – can be identified as a major cause. To suggest new remedies is indeed difficult, but it may be hoped that diagnosis is one good step towards a cure. Another step will be offered in the next section.

A Theoretical Formulation

There is nothing so powerful for directing one's actions in a complex situation, and for coordinating one's own efforts with those of others, as a good theory. All good teachers build up their own stores of empirical knowledge, and have abstracted from these some general principles on which they rely for guidance. But while their knowledge remains in this form it is largely still at the intuitive level within individuals, and cannot be communicated, both for this reason and because there is no shared conceptual structure (schema) in terms of which it can be formulated. Were this possible, individual efforts could be integrated into a unified body of knowledge which would be available for use by newcomers to the profession. At present most teachers have to learn from their own mistakes.

For some time my own comprehension of the difference between the two kinds of learning which lead respectively to relational and instrumental mathematics remained at the intuitive level, though I was personally convinced that the difference was one of great importance, and this view was shared by most of those with whom I discussed it. Awareness of the need for an explicit formulation was forced on me in the course of two parallel research projects; and insight came, quite suddenly, during a recent conference. Once seen it appears quite simple, and one wonders why I did not think of it before. But there are two kinds of simplicity: that of naivety; and that which, by penetrating beyond superficial differences, brings simplicity by unifying. It is the second kind which a good theory has to offer, and this is harder to achieve.

A concrete example is necessary to begin with. When I went to stay in a certain town for the first time, I quickly learnt several particular routes. I learnt to get between where I was staying and the office of the colleague with whom I was working; between where I was staying and the office of the colleague with whom I was working; between where I was staying and the university refectory where I ate; between my friend's office and the refectory; and two or three others. In brief, I learnt a limited number of fixed plans by which I could get from particular starting locations to particular goal locations.

As soon as I had some free time, I began to explore the town. Now I was not wanting to get anywhere specific, but to learn my way around, and in the process to see what I might come upon that was of interest. At this stage my goal was a different one: to construct in my mind a cognitive map of the town.

These two activities are quite different. Nevertheless they are, to an outside observer, difficult to distinguish. Anyone seeing me walk from A to B would have great difficulty in knowing (without asking me) which of the two I was engaged in. But the most important thing about an activity is its goal. In one case my goal was to get to B, which is a physical location. In the other it was to enlarge or consolidate my mental map of the town, which is a state of knowledge.

A person with a set of fixed plans can find his way from a certain set of starting points to a certain set of goals. The characteristic of a plan is that it tells him what to do at each choice point: turn right out of the door, go straight on past the church, and so on. But if at any stage he makes a mistake, he will be lost; and he will stay lost if he is not able to retrace his steps and get back on the right path.

In contrast, a person with a mental map of the town has something from which he can produce, when needed, an almost infinite number of plans by which he can guide his steps from any starting point to any finishing point, provided only that both can be imagined on his mental map. And if he does take a wrong turn, he will still know where he is, and thereby be able to correct his mistake without getting lost; even perhaps to learn from it.

The analogy between the foregoing and the learning of mathematics is close. The kind of learning which leads to instrumental mathematics consists of the learning of an increasing number of fixed plans, by which pupils can find their way from particular starting points (the data) to required finishing points (the answers to the questions). The plan tells them what to do at each choice point, as in the concrete example. And as in the concrete example, *what has to be done next is determined purely by the local situation.* (When you see the post office, turn left. When you have cleared brackets, collect like terms.) There is no awareness of the overall relationship between successive stages, and the final goal. And in both cases, the learner is dependent on outside guidance for learning each new ‘way to get there’.

In contrast, learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point. (I say 'in principle' because of course some of these paths will be much harder to construct than others.)

This kind of learning is different in several ways from instrumental learning.

1. The means become independent of particular ends to be reached thereby.
2. Building up a schema within a given area of knowledge becomes an intrinsically satisfying goal in itself.
3. The more complete a pupil's schema, the greater his feeling of confidence in his own ability to find new ways of 'getting there' without outside help.
4. But a schema is never complete. As our schemas enlarge, so our awareness of possibilities is thereby enlarged. Thus the process often becomes self-continuing, and (by virtue of 3) self-rewarding.

Taking again for a moment the role of devil's advocate, it is fair to ask whether we are indeed talking about two subjects, relational mathematics and instrumental mathematics, or just two ways of thinking about the same subject matter. Using the concrete analogy, the two processes described might be regarded as two different ways of knowing about the same town; in which case the distinction made between relational and instrumental understanding would be valid, but not between instrumental and relational mathematics.

But what constitutes mathematics is not the subject matter, but a particular kind of knowledge about it. The subject matter of relational and instrumental mathematics may be the same: cars travelling at uniform speeds between two towns, towers whose heights are to be found, bodies falling freely under gravity, etc. etc. But the two kinds of knowledge are so different that I think that there is a strong case for regarding them as different kinds of mathematics. If this distinction is accepted, then the word 'mathematics' is for many children indeed a false friend, as they find to their cost.

The State of Play

This is already a long article, yet it leaves many points awaiting further development. The applications of the theoretical formulation in the last section to the educational problems described in the first two have not been spelt out. One of these is the relationship between the goals of the teacher and those of the pupil. Another is the implications for a mathematical curriculum.

In the course of discussion of these ideas with teachers and lecturers in mathematical education, a number of other interesting points have been raised which also cannot be explored further here. One of these is whether the term ‘mathematics’ ought not to be used for relational mathematics only. I have much sympathy with this view, but the issue is not as simple as it may appear.

There is also research in progress. A pilot study aimed at developing a method (or methods) for evaluating the quality of children’s mathematical thinking has been finished, and has led to a more substantial study in collaboration with the N.F.E.R. as part of the TAMS continuation project. A higher degree thesis at Warwick University is nearly finished; and a research group of the Department of Mathematics at the University of Quebec in Montreal is investigating the problem with first and fourth grade children. All this will I hope be reported in due course.

The aims of the present paper are twofold. First, to make explicit the problem at an empiric level of thinking, and thereby to bring to the forefront of attention what some of us have known for a long time at the back of our minds. Second, to formulate this in such a way that it can be related to existing theoretical knowledge about the mathematical learning process, and further investigated at this level and with the power and generality which theory alone can provide.

References

1. R. R. Skemp: *Understanding Mathematics* (U.L.P.).
2. For a fuller discussion see R. R. Skemp: *The Psychology of Learning Mathematics* (Penguin 1972) pp. 43–46.
3. H. Bondi: *The Dangers of Rejecting Mathematics (Times Higher Education Supplement,* 26.3.76.