Bulletin of the **S**ection of **L**ogic
Volume 10/4 (1981), pp. 185–190
reedition 2009 [original edition, pp. 185–191]

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RELATIONS BETWEEN PARACONSISTENT LOGIC AND MANY-VALUED LOGIC

Let us consider informally a theory T (deductive system). We say that T is trivial (or overcomplete) if all formulas of T are theorems of T; otherwise, we say that T is non-trivial (or not overcomplete). T is inconsistent if it has a negation symbol and there are, at least, two theorems of T such that one is the negation of the other; if this is not the case, T is consistent.

If the underlying logic of T is the classical logic, T is inconsistent if and only if it is trivial. The same occurs with a great part of well-known systems of logic. So, if we intend to study inconsistent but non-trivial theories, we must construct new types of logic as a foundation to those theories. The logical systems constructed with this intention, have been called paraconsistent systems (see [1]).

In general, the systems of paraconsistent logic must satisfy the following conditions:

1) From two contradictory formulas, P and $\neg P$, it should not be possible in general to deduce an arbitrary formula.
2) The system should contain most of the schemata and deduction rules of the classical calculus that do not inference with the first condition.

The first logician to construct a propositional system of paraconsistent logic was S. Jaśkowski, who, in 1948, following a suggestion of Łukasiewicz, proposed a calculus to serve as a basis for inconsistent but non-trivial theories. The system was called the propositional discussive calculus (see [6]).

Independently of the work of Jaśkowski, N. C. A. da Costa formulated, in 1958, certain propositional calculi for the same purpose. He also constructed predicate calculi (with and without equality), and the corresponding calculi of descriptions. Moreover, he constructed, for the first time, set theories which are inconsistent and apparently non-trivial (references may be found, for instance, in [2] and [3]).

There are many reasons to study paraconsistent theories, as shown in the following cases: 1^o) We decide to attempt a formalization of dialectical logic, even only for a better understanding of such logic, because some specialists think that it is in principle not formalizable; 2^o) The possible formalization of Meinong theory of objects, extensively studied nowadays, specially by Richard Routley, among others (see [12]); 3^o) The study of the axiom of separation in set theory or in the theory of types, when we eliminate the usual restrictions. These restrictions arise, as it is well known, to eliminate the *existence of inconsistent classes or predicates* (as, e.g., the *Russell set*, which is the set of all sets that do not belong to themselves); 4^o) In general, the investigation of deductive theories that have been shown inconsistent, such as the naive set theory and the infinitesimal calculus, in its original and naive formulation.

After a great development on the syntactical level, there has appeared, in recent years, research dedicated to the study of paraconsistent logic also on the semantical level. For the calculi C_n $(1 \leq n \leq \omega)$ of da Costa, a semantics has been proposed, called the semantics of valuations, which constitutes a generalization of the two-valued semantics of classical logic. On the other hand, some authors, such a Routley and Meyer, have developed certain semantics for paraconsistent systems, related to relevant logic (see [13]). The system of Jaśkowski's discussive logic was syntactically formulated, but now at least two semantical analyses of it were proposed: a Kripke style semantics and a valuation semantics (see [4] and [8]).

Our aim, in this paper, is to illustrate the relations that there exist between paraconsistent logic and many-valued logic. In particular, we would like to show how it is possible to obtain a system of paraconsistent logic, from a three-valued system.

In what follows, we use the terminology and notion of Rosser-Turquette's book [11].

We will take, as a point of departure, the three-valued system of Łukasiewicz-Tarski [10], but modifying the set of designated values. Our basic connectives will be C and N. Their correspondent truth-functions are defined by the following tables, where 1 and 2 are designated values:

$P\ C\ Q$:

P^Q	1	2	3
1	1	2	3
2	1	1	2
3	1	1	1

P	NP
1	3
2	2
3	1

These functions satisfy hypotheses $H1$ to $H7$ in Rosser-Turquette's book (see [11], pp. 10–12). We have, then, a truth-value stipulation, which we can resume in the following matrix: $M = \langle\{1,2,3\},\{1,2\},C,N\rangle$.

The other operators of Łukasiewicz-Tarski are introduced by the usual definitions: $P\ A\ Q\ =_{df}\ (P\ C\ Q)\ C\ Q \qquad P\ K\ Q\ =_{df}\ N(NP\ A\ NQ)$.

We now introduce a new operator of conjunction and a new operator of negation: $P\ \&\ Q\ =_{df}\ N(P\ K\ Q)\ C\ (P\ K\ Q) \qquad \neg P\ =_{df}\ P\ C\ NP$.

By means of these operators, we can define the operators $J_k(P)$ $(1 \leq k \leq 3)$, which are such that $J_k(P)$ is assertable when and only when P takes the truth-value k. This is done as follows:

$J_1(P) =_{df} P\ \&\ \neg(P\ \&\ \neg P) \qquad J_2(P) =_{df} P\ \&\ \neg P$
$J_3(P) =_{df} \neg P\ \&\ \neg(P\ \&\ \neg P)$

Finally, for our own purposes, we introduce two more operators: an implication ($\rightarrow$) and a disjunction ($\vee$).

$$P \rightarrow Q =_{df}\ J_3(Q)\ C\ J_3(P) \qquad\qquad P \vee Q =_{df}\ J_3(P) \rightarrow (P\ A\ Q)$$

Observe that, in our case, the new implication is a generalization of two-valued implication, which does not occur with the operator C of Łukasiewicz-Tarski (see [11], p. 17).

Theorem 1: *The classical positive logic holds in S, relative to $\rightarrow$, & and $\vee$ (but not to C, & and $\vee$).*

Observe also that the function $J_3(P)$ plays a role that was played by usual negation of two-valued logic, as we can see by the following result:

Theorem 2: *In the system S, defined by matrix M, $J_3(P)$ has all the properties of classical negation, relative to $\rightarrow$, & and $\vee$.*

As S contains statement functions $P \rightarrow Q$ and $J_k(P)$ $(1 \leq k \leq 3)$, which are definable in terms of our primitive functions, it is possible to present an axiomatic stipulation for it, by means of axiom schemes $A1$ to $A7$, and rule $R1$ of Rosser-Turquette's book (cf. [11], pp. 33–34).

Observe that $\rightarrow (p, q)$ and $j_k(p)$ satisfy "standard conditions" (see [11], p. 26). So, as a consequence, we have the following:

THEOREM 3: *The axiomatic stipulation of S is equivalent to its truth-value stipulation.*

We now introduce the following definition: We say that a formula is *normal* if it contains only the operators $\rightarrow$, &, $\vee$ and $\neg$. The set of normal formulas which holds in S will be denoted by F. Then, F is a subsystem of S.

THEOREM 4: *The positive classical logic holds in F.*

THEOREM 5: *The following schemata hold in F:*

$(P \rightarrow Q) \rightarrow (\neg P \vee Q)$; $(P \rightarrow \neg P) \rightarrow \neg P$; $(\neg P \rightarrow P) \rightarrow P$; $P \vee \neg P$; $(P \rightarrow Q) \vee (P \rightarrow \neg Q)$; $(\neg\neg P \vee Q) \rightarrow (P \vee Q)$; $(P \vee \neg\neg Q) \rightarrow (P \vee Q)$; $((P \,\&\, \neg Q) \rightarrow R) \rightarrow (P \rightarrow (P \rightarrow (Q \vee R))$; $P \vee \neg(P \& \neg P)$; $\neg\neg P \rightarrow P$; $\neg(P \& \neg P) \vee (P \,\&\, \neg P)$; $\neg P \vee \neg(P \,\& \neg P)$; $\neg P \vee (P \,\&\, \neg(P \,\&\, \neg P))$; $(\neg P \,\& \neg(P \,\&\, \neg P)) \vee (P \,\&\, \neg P) \vee (P \,\&\, \neg(P \,\&\, \neg P))$.

THEOREM 6: *Among others, the following schemata are not valid in F:*

$\neg(P \,\&\, \neg P)$; $\neg P \rightarrow (P \rightarrow Q)$; $\neg P \rightarrow (P \rightarrow \neg Q)$; $(P \,\&\, \neg P) \rightarrow Q$; $P \rightarrow (\neg P \rightarrow Q)$; $P \rightarrow (\neg P \rightarrow \neg Q)$; $(P \rightarrow Q) \rightarrow ((P \rightarrow \neg Q) \rightarrow \neg P)$; $P \rightarrow \neg\neg P$; $((P \vee Q) \,\&\, \neg P) \rightarrow Q$; $(\neg P \,\&\, \neg Q) \rightarrow \neg(P \vee Q)$; $(\neg P \vee Q) \rightarrow (P \rightarrow Q)$; $(P \rightarrow Q) \rightarrow \neg(P \,\&\, \neg Q)$; $(P \rightarrow Q) \vee (\neg P \rightarrow Q)$; $(P \rightarrow (Q \vee R)) \rightarrow ((P \,\&\, \neg Q) \rightarrow R)$; $((P \,\&\, Q) \rightarrow R) \rightarrow ((P \,\&\, \neg R) \rightarrow \neg Q)$.

As consequence of the preceding theorems, it follows that F is a paraconsistent logic, that is, it satisfies conditions 1 and 2 presented in the beginning of this paper. Obviously, F is also a many-valued logic. Furthermore, we can demonstrate that F is precisely the system P_1 of paraconsistent logic, proposed by A. M. Sette in [14]: it is sufficient to verify that the tables which characterize F are exactly the same as those which characterize P_1.

It is possible, still, to prove that F is an extension of the system C_n $(1 \leq n \leq \omega)$ of da Costa, as well as many other systems of paraconsistent logic. Consequently, it is clear that there exists a very close relation between paraconsistent logic and many-valued logic.

The system S can be extended to a system S' of the predicate calculus, following the methods of Rosser-Turquette (see [11], pp. 63–64). Analogously to the case of S, we can define a system F', which is the extension of F to predicate calculus.

Evidently, all of our results can be extended to systems with more than three values, by adapting the methods of Rosser-Turquette. It is also clear that there exists a relation between the systems studied here and those proposed in [5] and [9].

Finally, we would like to consider the problem of an intuitive interpretation of the negation in system F. It is natural to understand value 1 as *true* and value 3 as *false*. Concerning value 2, it can be interpreted as *true and contradictory*, with the meaning that a statement with value 2 is true, but its negation is also true. Note that contradictory statements only can be atomic, because with molecular statements, the value 2 disappears. Within this interpretation, we are accepting that contradictions, if they arise, never do so *at the level of logic*. Obviously, if we do not consider contradictory statements, we fall back completely on classical two-valued logic.

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