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RELATIONS BETWEEN THE s-SELFDECOMPOSABLE AND SELFDECOMPOSABLE MEASURES[1]

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The classes of the s-selfdecomposable and decomposable probability measures are related to the limit distributions of sequences of random variables deformed by some nonlinear or linear transformations respectively. Both are characterized in many different ways, among others as distributions of some random integrals. In particular we get that each selfdecomposable probability measure is s-selfdecomposable. This and other relations between these two classes seem to be rather unexpected.

0. Introduction. The class, L, of all selfdecomposable probability measures plays an important role in classical probability theory. It is well known that L coincides with the class of limit distributions of normed sums of sequences of independent random variables. Lévy (1937) characterized elements from L in terms of their Lévy spectral function, Kubik (1962) described L as the smallest closed semigroup containing some generators and finally Urbanik (1968), using Choquet's Theorem, found a general form of their characteristic functionals. Quite recently Wolfe (1982) (on the real line) and Jurek-Vervaat (1983) (on Banach spaces and with a completely different proof from Wolfe's) have proved that L coincides with probability distributions of the following random integrals

$$\int_0^\infty e^{-s}\,dY(s) = \int_0^1 s\,dY(-\ln s), \tag{0.1}$$

where Y is a $D[0, \infty)$-valued random variable with stationary independent increments such that $Y(1)$ has a finite logarithmic moment. Moreover from Sato-Yamazato (1984) we obtain a characterization of the infinitesimal generators of the Ornstein-Uhlenbeck type processes $Z(t){:}= \int_0^t e^{-s}\,dY(s)$ where limit distributions ($t \to \infty$) coincide with the class L.

The class L is naturally connected with the linear operators T_r ($T_r x{:} = rx$, $r \in \mathbb{R}$, $x \in \mathbb{R}$). Jurek (1977) introduced and examined the class, $\mathscr{U}$, of s-selfdecomposable probability measures, which in its definition uses the nonlinear shrinking operators U_r ($U_r x{:} = \max(0, |x| - r)x/|x|$, $x \in \mathbb{R}$, $r \in \mathbb{R}^+$). Recently we noticed that the class $\mathscr{U}$ was investigated by Medgyessy (1967) and by O'Connor (1979a) but from a quite different point of view.

In the present paper, in Section 2, we study the class $\mathscr{U}$ on Banach spaces.

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Among others we prove that $\mathscr{U}$ coincides with probability distributions of the following random integrals

$$\int_0^1 t\, dY(t), \tag{0.2}$$

where Y is $D[0, 1]$-valued random variable with stationary independent increments. Moreover we examine the mapping $\mathscr{J}$: $\mathscr{L}(Y(1)) \to \mathscr{L}(\int_0^1 t\, dY(t))$, the problem of generators and characteristic functionals for $\mathscr{U}$. In Section 3 we collect descriptions of L (on Banach spaces) and properties of the mapping $\mathscr{J}$: $\mathscr{L}(Y(1)) \to \mathscr{L}(\int_0^\infty e^{-s}\, dY(s))$ and in Section 4 we indicate the relations between the s-selfdecomposable and selfdecomposable probability measures on a Banach space, cf. formulae (0.1) and (0.2).

1. Notations and preliminaries. Let E be a real separable Banach space with topological dual $E^\star$ and Borel σ-field $\mathscr{B}(E)$ induced by a norm $\|\cdot\|$. By $\langle \cdot, \cdot \rangle$ we denote the dual pair between $E^\star$ and E. Further, $\mathscr{P}(E)$ denotes the topological semigroup of all Borel probability measures on E with the convolution "$\star$" and weak convergence "$\Rightarrow$". Given a Borel mapping f from E into E and a measure $\mu \in \mathscr{P}(E)$, we write $f\mu$ for the probability measure defined by means of the formula

$$(f\mu)(B) = \mu(f^{-1}(B)) \quad \text{for} \quad B \in \mathscr{B}(E). \tag{1.1}$$

If $\mu \in \mathscr{P}(E)$ and for every $n = 2, 3, \cdots$ there exists $\mu_n \in \mathscr{P}(E)$ such that $\mu_n^{\star n} = \mu$ then μ is said to be an *infinitely divisible measure.* The class of all infinitely divisible measures on E will be denoted by ID(E) and sometimes briefly by ID. Clearly ID(E) is a subsemigroup of $\mathscr{P}(E)$ closed in the weak topology. Moreover, it is known (cf. [1] Chapter 3, Theorem 6.2) that $\lambda \in$ ID(E) if and only if

$$\hat{\lambda}(x^\star) = \exp\left\{i\langle x^\star, x_0\rangle - \tfrac{1}{2}\langle x^\star, Rx^\star\rangle + \int_E K(x^\star, x)M(dx)\right\}, \quad x^\star \in E^\star. \tag{1.2}$$

Here $\hat{\lambda}$ denotes the *characteristic functional* of λ, $x_0 \in E$, R is a *Gaussian convariance operator,* K is the functions on $E^\star \times E$ given by

$$K(x^\star, x) = \exp i\langle x^\star, x\rangle - 1 - i\langle x^\star, x\rangle 1_B(x), \tag{1.3}$$

(1_B denoting the indicator function of the unit ball $B = \{x \in E\colon \|x\| \le 1\}$), and M a σ-finite measure on E which is finite on the complement of every neighbourhood of 0 and $M(\{0\}) = 0$; in the case of a Hilbert space the function K is usually expressed slightly differently. Since the representation (1.2) is unique we write $\lambda = [x_0, R, M]$, if $\hat{\lambda}$ is of the form (1.2). For $\lambda = [x_0, R, M]$ and $t \in \mathbb{R}^+$ (positive reals), by $\lambda^{\star t}$ we mean the infinitely divisible measure $[tx_0, tR, tM]$. The measure M in (1.2) is called *a Lévy measure* of λ and the function

$$L_M(A, r) := -M(\{x \in E\setminus\{0\}\colon x/\|x\| \in A,\ \|x\| > r\}) \tag{1.4}$$

a Lévy spectral function associated with M. Here $r \in \mathbb{R}^+$ and A is a Borel subset of the unit sphere $S = \{x \in E\colon \|x\| = 1\}$. $\mathscr{L}(X)$ denotes the probability distribution of a *random variable* (rv) X and $\mathbb{E}X$ its expected value (Bochner integral). Further, $D_E[a, b]$ denotes the set of E-valued functions on $[a, b]$ that are right-continuous on $[a, b)$, and have left-hand limits on $(a, b]$. We may assume that they are continuous at b; cf. [2] page 109 and [19] page 232. Further, $D_E[a, \infty)$ is defined as in [15]. For a given $D_E[a, b]$-valued rv Y and a real valued function f with bounded variation we define the random integral $\int_{(a,b]} f(t)\, dY(t, \omega)$ by formal integration by parts for each fixed ω, i.e.

$$\int_{(a,b]} f(t)\, dY(t, \omega) := f(b)Y(b) - f(a)Y(a) - \int_{(a,b]} Y(t, \omega)\, df(t). \tag{1.5}$$

The integrals on (a, ∞) are defined as limits in probability as $b \to \infty$.

The following easily follows from our pathwise definition of the random integral.

LEMMA 1.1. *If Y is a $D_E[a, b]$-valued rv with stationary independent increments, $Y(0) = 0$ a.s. and f has bounded variation in $[a, b]$ then*

(a) $$\hat{\mathscr{L}}\left(\int_{(a,b]} f(t)\, dY\right)(x^\star) = \exp \int_{(a,b]} [\log \hat{\mathscr{L}}(Y(1))(f(t)x^\star)]\, dt,$$

(b) $$\mathscr{L}\left(\int_{(a,b]} f(t)\, dY(t, \omega)\right)^{\star s} = \mathscr{L}\left(\int_{(a,b]} f(t)\, dY(st, \omega)\right) \quad \textit{for each} \quad s \in \mathbb{R}^+.$$

We end this introductory section with a weak convergence of measures theorem and a theorem on Lévy measures.

THEOREM 1.2. *Let A_n, A be linear bounded operators on E and $A_n \to A$ in the strong operator topology, i.e., $A_n x \to Ax$ for all $x \in E$. Further, let $\rho_n, \rho \in \mathrm{ID}(E)$, $c_n, c \in \mathbb{R}^+$ and $\rho_n \Rightarrow \rho$ and $c_n \to c$. Then*

$$A_n \rho_n^{\star c_n} \Rightarrow A\rho^{\star c}.$$

PROOF. Since the strong convergence of operators in E preserves the weak convergence of measures (cf. [5] Proposition 1.1) it is enough to show that $\rho_n^{\star c_n} \Rightarrow \rho^{\star c}$. To this end let us choose $D_E[0, \infty)$-valued rv's Y_n, Y with stationary independent increments such that $Y_n(0) = 0$ a.s., $Y(0) = 0$ a.s., and $\mathscr{L}(Y_n(1)) = \rho_n$, $\mathscr{L}(Y(1)) = \rho$, cf. [3] Theorem 14.20. Further let π_c be the natural projection from $D_E[0, \infty)$ to E defined as usual $\pi_c y := y(c)$. It is easy to see that π_c is continuous at y if and only if y is continuous at c, cf. [2] page 121, provided that $D_E[0, \infty)$ is endowed with the Skorohod topology. Theorem 2.7 in [21] implies that $Y_n \Rightarrow Y$ in $D_E[0, \infty)$. Since $P\{\omega\colon Y(c, \omega) \neq Y(c-, \omega)\} = 0$ we infer from Theorem 5.5 in [2] that $\mathscr{L}(\pi_{c_n} Y_n) \Rightarrow \mathscr{L}(\pi_c Y)$. But $\mathscr{L}(\pi_c Y) = \mathscr{L}(Y(c)) = \rho^{\star c}$, which completes the proof.

THEOREM 1.3. *Let G be a positive Borel measure on E such that $G(\{0\}) = 0$*

and

$$G'(A) = \int_{(0,1]} G(t^{-1}A)\,dt \quad \text{for all} \quad A \in \mathscr{B}(E\backslash\{0\}). \tag{1.6}$$

Then G' is a Lévy measure if and only if G is also.

PROOF. *The sufficiency.* Let Y be a $D_E[0, 1]$-valued rv with stationary independent increments such that G is a Lévy measure of $\mathscr{L}(Y(1))$. Then from Lemma 1.1(a) we get that G' is the Lévy measure of $\mathscr{L}(\int_{(0,1]} t\,dY(t))$.

The necessity. Note that G' and G are both finite or infinite measure and finite measures are Lévy ones. G is a Lévy measure, if and only if, its symmetrization also is and the symmetrization of G' corresponds to the symmetrization of G. Finally Lévy measures are finite on every complement of zero in E. Therefore we may assume and do that G' is symmetric and concentrated on the unit ball B in E. Consequently G is also concentrated on B.

Let $m_n(A)$: $= G(A \cap \{x \in E: (n+1)^{-1} < \|x\| \le n^{-1}\})$ for $n = 1, 2, \cdots$ and m_n' corresponds to m_n by (1.6). Let (ζ_n') be independent E-valued rv's such that $\mathscr{L}(\zeta_n') = [0, 0, m_n']$ and $S_n = \sum_{k=1}^n \zeta_k'$. From Theorem 4.7, page 119 in [1] we have $\mathscr{L}(S_n) \Rightarrow [0, 0, G']$ as $n \to \infty$, and Ito-Nisio Theorem (cf. [1] page 105, Theorem 2.10) gives that the series $\sum_{k=1}^\infty \zeta_k'$ converges a.s., say to the E-valued rv S and G' is its Lévy measure. Since G' has a bounded support S has all moments, in particular $E\|S\| < \infty$, cf. for example [10] Theorem 1. Further, from the Lévy inequality we have $P\{\sup_n \|S_n\| > t\} \le 2P\{\|S\| > t\}$ and hence $\mathbb{E}(\sup_n \|S_n\|) \le 2\mathbb{E}\|S\|$. Finally, Hoffmann-Jørgensen Theorem (cf. [1] page 106, Theorem 2.11) gives $S_n \to S$ in $L_1(E)$. Now let (ζ_n) be independent E-valued rv's such that $\mathscr{L}(\zeta_n) = [0, 0, m_n]$ and $T_n = \sum_{k=1}^n \zeta_k$. To prove that G is a Lévy measure we shall show that

$$\mathbb{E}\|T_{n-1} - T_{k-1}\| \le 2\mathbb{E}\|S_{n-1} - S_{k-1}\| \quad \text{for} \quad n > k. \tag{1.7}$$

In terms of measures it means that

$$\int_E \|x\|\, e(G\,|_{\{x: n^{-1} < \|x\| \le k^{-1}\}})(dx) \le 2\int_E \|x\|\, e(G'\,|_{\{x: n^{-1} < \|x\| \le k^{-1}\}})(dx), \tag{1.8}$$

where $\mu\,|_A$ denotes the restriction of a measure μ to a set A and for finite measure ρ on E

$$e(\rho) := e^{-\rho(E)} \sum_{k=0}^\infty \rho^{\star k}/k! = [x_\rho, 0, \rho]$$

with $x_\rho = \int_{\|x\| \le 1} x\rho(dx)$. If ρ and ρ' are connected by (1.6) then $\rho(E) = \rho'(E)$ and for $y \in E$

$$\int_E \|x + y\|\,\rho'(dx) = \int_E \int_0^1 \|tx + y\|\,dt\rho(dx) \ge \int_E \left\| \frac{1}{2}x + y \right\| \rho(dx).$$

Hence we get for $k \in \mathbb{N}$

$$\int_E \| x \| (\rho')^{\star k}(dx) = \int_E \cdots \int_E \| x_1 + x_2 + \cdots + x_k \| \rho'(dx_1) \cdots \rho'(dx_k)$$
$$\geq \frac{1}{2} \int_E \| x \| \rho^{\star k}(dx).$$

and this gives the following inequality

$$2 \int_E \| x \| e(\rho')(dx) \geq \int_E \| x \| e(\rho)(dx). \tag{1.9}$$

Thus, (1.9) with (1.7) implies that the series $\sum_{k=1}^{\infty} \zeta_k$ converges, say to the E-valued rv T, in $L_1(E)$. The Ito-Nisio Theorem implies that also $\mathscr{L}(T_n) \Rightarrow \mathscr{L}(T) = [0, 0, G]$ as $n \to \infty$, i.e., G is a Lévy measure, which completes the proof.

REMARK 1.4. The sufficiency in Theorem 1.3 can also be proved arguing as in the necessity. Note that if ρ is symmetric and a finite measure then

$$\int_E \| y \| \rho(dy) = \int_E \| \tfrac{1}{2}(x + y) + \tfrac{1}{2}(y - x) \| \rho(dy) \leq \int_E \| x + y \| \rho(dy)$$

and hence for $0 < t < 1$ and $x \in E$ we have

$$\int_E \| tx + y \| \rho(dy) \leq t \int_E \| x + y \| \rho(dy) + (1 - t) \int_E \| y \| \rho(dy)$$
$$\leq \int_E \| x + y \| \rho(dy).$$

Finally, for $k \in \mathbb{N}$

$$\int_E \| x \| (\rho')^{\star k}(dx) \leq \int_E \| x \| \rho^{\star k}(dx),$$

and consequently

$$\int_E \| x \| e(\rho')(dx) \leq \int_E \| x \| e(\rho)(dx).$$

REMARK 1.5. In the case of a Hilbert space Theorem 1.3 is obvious because G' and G both integrate $\| x \|^2$ over the unit ball.

2. The class $\mathscr{U}$. For arbitrary $r \in \mathbb{R}^+$ we define transformations T_r and U_r from E onto E by means of the formulas:

$$T_r x := rx \tag{2.1}$$

and

$$U_r x := \max(0, \| x \| - r)x / \| x \|, \quad U_r 0 = 0. \tag{2.2}$$

The set $\{U_r: r \in \mathbb{R}^+\}$ forms a one-parameter semigroup of nonlinear operations on E called the *shrinking operations* (for short: *s-operations*). An infinitely divisible measure $\nu = [a, R, M]$ is said to be *s-selfdecomposable* if

$$M \geq U_r M \quad \text{for all} \quad r \in \mathbb{R}^+. \tag{2.3}$$

The class of all s-selfdecomposable measures on E will be denoted by $\mathscr{U}(E)$ or simply by $\mathscr{U}$.

REMARK 2.1. In the case of a Hilbert space H the class $\mathscr{U}(H)$ coincides with the class of limit distributions of the following sequences

$$U_{r_n}\xi_1 + U_{r_n}\xi_2 + \cdots + U_{r_n}\xi_n + a_n,$$

where (ξ_n) are H-valued independent rv's and the triangular array $(U_{r_n}\xi_j)$, $j = 1, \cdots, n$; $n = 1, 2, \cdots$ is infinitesimal, cf. [4] Theorem 5.1.

THEOREM 2.2. *The following conditions for a Lévy measure M are equivalent*:

(a) $M \geq U_r M$, *for all* $r \in \mathbb{R}^+$;

(b) *for each* $A \in \mathscr{B}(S)$ *the Lévy spectral function* L_M *has right and left derivatives with respect to* r *such that* $dL_M(A, r)/dr$ *is nonincreasing on* $\mathbb{R}^+$;

(c) *there is a unique measure* F *on* $E\setminus\{0\}$ *such that for each* $\varepsilon > 0$ $\int_{\|x\| \geq \varepsilon} \|x\| F(dx) < \infty$ *and* $M(A) = \int_0^\infty (U_t F)(A)\, dt$, *for all* $A \in \mathscr{B}(E\setminus\{0\})$;

(d) $M \geq cT_c M$ *for all* $0 < c < 1$;

(e) $M(A) = \int_0^1 (T_t G)(A)\, dt$ *for all* $A \in \mathscr{B}(E\setminus\{0\})$, *where* G *is a unique Lévy measure connected with* F *in* (c) *by formula* $G(A) = \int_A \|x\| F(dx)$.

PROOF. Proposition 3.1 in [9] gives (a) $\Leftrightarrow$ (b) and Theorem 2 in [6] gives (a) $\Leftrightarrow$ (c). Since (b) is equivalent to the statement that for fixed $A \in \mathscr{B}(S)$ the function $r \to L_M(A, r)$ is concave, it is easy to check that (b) $\Leftrightarrow$ (d), cf. [7] or [17]. To prove that (c) $\Leftrightarrow$ (e) we show that for all $A \in \mathscr{B}(E\setminus\{0\})$

$$\left[\int_0^\infty (U_t F)(A)\, dt = \int_0^1 (T_t G)(A)\, dt\right] \text{ iff } \left[G(A) = \int_A \|x\| F(fx)\right]. \tag{2.4}$$

Putting $A = \langle B; r\rangle := \{x \in E: x/\|x\| \in B, \|x\| > r\}$, $(B \in \mathscr{B}(S), r \in \mathbb{R}^+)$, into the left-hand equality in (2.4) we get

$$\int_r^\infty F(\langle B; s\rangle)\, ds = r \int_r^\infty G(\langle B; s\rangle)/s^2\, ds,$$

and taking the right derivative with respect to r we obtain

$$\begin{aligned} G(\langle B; r\rangle) &= \int_r^\infty F(\langle B; s\rangle)\, ds + rF(\langle B; r\rangle) \\ &= -\int_r^\infty s\, dF(\langle B, s\rangle) = \int_{\langle B; r\rangle} \|x\| F(dx). \end{aligned}$$

Hence $G(A) = \int_A \|x\| F(dx)$ for all $A \in \mathscr{B}(E\backslash\{0\})$. Conversely, taking into account the last equality for $A = \langle B; r\rangle$ we have

$$\int_0^1 (T_tG)(\langle B; r\rangle)\,dt$$

$$= -\int_0^1 \int_{rt^{-1}}^{\infty} s\,dF(\langle B; s\rangle)\,dt = \int_0^{\infty}\int_0^1 s1_{(r,\infty)}(st)\,dt\,dF(\langle B; s\rangle)$$

$$= -\int_0^{\infty}\int_0^s 1_{(r,\infty)}(u)\,du\,dF(\langle B; s\rangle) = -\int_0^{\infty}\int_u^{\infty} 1_{(r,\infty)}(u)\,dF(\langle B; s\rangle)\,du$$

$$= \int_r^{\infty} F(\langle B; u\rangle)\,du = \int_0^{\infty} (U_tF)(\langle B; r\rangle)\,dt.$$

This shows that (2.4) holds true and Theorem 1.3 gives that G is a Lévy measure, which completes the proof of Theorem 2.2.

COROLLARY 2.3. *A measure $\mu \in \mathscr{U}(E)$ if and only if, for every $0 < c < 1$ there exists $\mu_c \in \mathrm{ID}(E)$ such that*

$$\mu = T_c\mu^{\star c} \star \mu_c \tag{2.5}$$

PROOF. From (1.2) we infer that (2.5) is equivalent to the condition $M \geq cT_cM$ for every $0 < c < 1$. Hence and from Theorem 2.2 we get that $\mu \in \mathscr{U}(E)$ which completes the proof.

COROLLARY 2.4. *$\mathscr{U}(E)$ with convolution and weak topology forms a closed subsemigroup of* $\mathrm{ID}(E)$.

THEOREM 2.5. *A measure $\mu \in \mathscr{U}(E)$ if and only if, there exists a sequence (ν_n) $\subseteq \mathrm{ID}(E)$ such that*

$$T_{1/n}(\nu_1^{\star 1} \star \nu_2^{\star 2} \star \cdots \star \nu_n^{\star n})^{\star 1/n} \Rightarrow \mu.$$

PROOF. *Sufficiency.* Let us denote

$$\rho_n := T_{1/n}(\nu_1^{\star 1} \star \nu_2^{\star 2} \star \cdots \star \nu_n^{\star n})^{\star 1/n}.$$

For every $0 < c < 1$ we choose a sequence (m_n) of natural numbers such that $m_n < n$ and $m_n/n \to c$. From the equality

$$\rho_n = T_{m_n/n}\rho_{m_n}^{\star m_n/n} \star T_{1/n}(\nu_{m_n+1}^{\star m_n+1} \star \cdots \star \nu_n^{\star n})^{\star 1/n}, \tag{2.6}$$

and Theorem 1.2 and Theorem 2.1 in [19], page 58, we infer that the second factor in (2.6) converges, say to μ_c. Consequently we get $\mu = T_c\mu^{\star c} \star \mu_c$, i.e., μ is s-selfdecomposable.

Necessity. Suppose for every $0 < c < 1$, $\mu = T_c\mu^{\star c} \star \mu_c$ for some $\mu_c \in \mathrm{ID}(E)$.

Putting $\nu_1 := \mu$, $\nu_k := T_k\mu_{(k-1)/k}$ for $k \geq 2$ we have $\nu_1^{\star 1} \star \nu_2^{\star 2} \star \cdots \star \nu_n^{\star n} = T_n\mu^{\star n}$ because $T_k\mu^{\star k} = T_{k-1}\mu^{\star(k-1)} \star \nu_k^{\star k}$ whenever $k \geq 2$. Therefore $\mu = T_{1/n}(\nu_1^{\star 1} \star \cdots \star \nu_n^{\star n})^{\star 1/n}$ which completes the proof.

For $\nu \in \mathrm{ID}(E)$ there is a $D_E[0, 1]$-valued rv Y with stationary independent increments such that $Y(0) = 0$ a.s. and $\mathscr{L}(Y(1)) = \nu$. Let us define the mapping $\mathscr{J}$ from $\mathrm{ID}(E)$ into itself by means of the formula

$$\mathscr{J}\nu := \mathscr{L}\left(\int_0^1 t\, dY(t)\right). \tag{2.7}$$

If $\nu = [a, R, M]$ and $\mathscr{J}\nu = [a', R', M']$ then from Lemma 1.1 we get

$$R' = \tfrac{1}{3}R, \tag{2.8}$$

$$M'(A) = \int_0^1 (T_tM)(A)\, dt \quad \text{for} \quad A \in \mathscr{B}(E\setminus\{0\}), \tag{2.9}$$

$$a' = \frac{1}{2}a + \int_0^1 t \int_{1<\|x\|\leq t^{-1}} xM(dx)\, dt. \tag{2.10}$$

THEOREM 2.6. *The mapping $\mathscr{J}$ is a continuous isomorphism between semigroups $ID(E)$ and $\mathscr{U}(E)$. Moreover we have*

(a) $\mathscr{J}(\mu^{\star c}) = (\mathscr{J}\mu)^{\star c}$ *for* $c \in \mathbb{R}^+$,

(b) $\mathscr{J}(V\mu) = V(\mathscr{J}\mu)$ *for a bounded linear operator V on E.*

PROOF. For fixed $x^\star \in E^\star$ let us put

$$g_{x^\star}(s) := \log(\mathscr{J}\hat{\nu})(sx^\star) \quad \text{for} \quad s \in \mathbb{R}.$$

From Lemma 1.1 we have

$$g_{x^\star}(s) = s^{-1}\int_0^s \log \hat{\nu}(rx^\star)\, dr, \quad g_{x^\star}(0) = 0$$

and hence

$$\log \hat{\nu}(sx^\star) = g_{x^\star}(s) + s\,dg_{x^\star}(s)/ds.$$

Consequently

$$\hat{\nu}(x^\star) = (\mathscr{J}\nu)\hat{}(x^\star)\exp d(\log(\mathscr{J}\nu)\hat{}(sx^\star))/ds\,|_{s=1} \tag{2.11}$$

and this implies that $\mathscr{J}$ is one-to-one. The formulas $\mathscr{J}(\mu \star \nu) = \mathscr{J}(\mu) \star \mathscr{J}(\nu)$ and (a) follow from (2.8)–(2.10). Further, $\mathscr{J}$ is onto $\mathscr{U}(E)$ because of Theorem 2.2 ((a) $\Leftrightarrow$ (e)) and the formula (2.9). The equality

$$V(\mathscr{J}\nu) = V\mathscr{L}\left(\int_0^1 t\, dY(t)\right) = \mathscr{L}\left(\int_0^1 t\, d(VY(t))\right) = \mathscr{J}(V\nu)$$

gives the property (b). It remains to show the continuity of $\mathscr{J}$. Suppose $\nu_n, \nu \in \mathrm{ID}(E)$ and $\nu_n \Rightarrow \nu$. As in the proof of the Theorem 1.2 we can choose $D_E[0, 1]$-

valued rv's Y_n and Y with stationary independent increments such that $Y_n(0) = Y(0) = 0$ a.s., $\mathscr{L}(Y_n(1)) = \nu_n$, $\mathscr{L}(Y(1)) = \nu$ and $Y_n \Rightarrow Y$ in $D_E[0, 1]$. Since the functional

$$\phi(y) = \int_0^1 t\, dy(t) := y(1) - \int_0^1 y(t)\, dt, \quad y \in D_E[0, 1],$$

is continuous in the Skorohod topology (cf. [2] page 121) we get $\mathscr{L}(\phi(Y_n)) \Rightarrow \mathscr{L}(\phi(Y))$ i.e. $\mathscr{J}\nu_n \Rightarrow \mathscr{J}\nu$ which completes the proof.

Recall that μ is a *stable measure* on E if μ is Guassian, i.e., $\mu = [a, R, 0]$ or $\mu = [a, 0, M]$ and

$$M(A) = \int_S \int_0^\infty 1_A(tx) t^{-(p+1)}\, dt \gamma(dx), \tag{2.12}$$

where $0 < p < 2$ and γ is a finite Borel measure on the unit sphere S.

THEOREM 2.7. *A measure $\mathscr{J}(\nu)$ is stable if and only if ν is also.*

PROOF. The sufficiency is obvious because of (2.8) and (2.9). Further, if $\mathscr{J}\nu$ is Gaussian measure then so is ν. Suppose $\mathscr{J}\nu$ is stable with the exponent $0 < p < 2$. Taking into account the fact that M' has the form (2.9) and (2.12) simultaneously we get that its Lévy spectral function $L_{M'}$ (cf. (1.4)) satisfies the equations

$$L_{M'}(A, r) = r \int_r^\infty L_M(A, s)/s^2\, ds = \frac{-\gamma(A) r^{-p}}{p}.$$

Hence we obtain

$$-L_M(A, r) = (p + 1)\gamma(A) r^{-p}/p$$

i.e., M is of the form (2.12) which completes the proof.

We will say that $\mu \in \mathrm{ID}(E)$ is *$\mathscr{J}$-invariant* if $\mathscr{J}\mu = \mu^{\star c} \star \delta_x$ for some $c \in \mathbb{R}^+$ and $x \in E$.

THEOREM 2.8. *A measure μ is $\mathscr{J}$-invariant if and only if μ is a stable one.*

PROOF. It is easy to check that stable measures are $\mathscr{J}$-invariant with $c = 1/(p + 1)$. Conversely, if $[a, R, M]$ is $\mathscr{J}$-invariant then $R' = cR$ and $M' = cM$. Hence and from (2.8) we have $c = 1/3$ whenever $R \neq 0$. Moreover using the Lévy spectral functions of M' and cM we obtain

$$r \int_r^\infty L_M(A, s)/s^2\, ds = cL_M(A, r) \quad \text{for} \quad r \in \mathbb{R}^+ \quad \text{and} \quad A \in \mathscr{B}(S). \tag{2.13}$$

Taking the right derivatives in (2.13) we get

$$(c - 1)/cL_M(A, r) = rdL_M(A, r)/dr$$

and hence

$$-L_M(A, r) = \gamma(A)r^{-(1-c)/c}$$

where $\gamma(A) \geq 0$ and $(1 - c)/c > 0$. Consequently M is a Lévy measure of a stable one with the exponent $p = (1 - c)/c$ and this implies $c > 1/3$. Therefore, $\mathcal{J}$-invariant measure ν is a Gaussian if $c = 1/3$ or stable measure with the exponent $p = (1 - c)/c$ if $c > 1/3$. Thus the theorem is completely proved.

The next theorem gives a characterization of s-selfdecomposable measures in terms of their characteristic functionals, cf. [7] Theorem 2.1. It is a simple consequence of Theorem 2.2 ((a) $\Leftrightarrow$ (e)).

THEOREM 2.9. *A complex valued function ϕ on $E^\star$ is a characteristic functional of a s-selfdecomposable measure if and only if there exists a unique measure $\nu = [a, R, M] \in ID(E)$ such that*

$$\begin{aligned}\phi(x^\star) &= \exp \int_0^1 \log \hat{\nu}(tx^\star)\, dt \\ &= \exp\Big\{ i\Big\langle x, \frac{1}{2} a\Big\rangle - \frac{1}{6}\langle x^\star, Rx^\star\rangle \\ &\quad + \int_{E\setminus\{0\}} \Big[\frac{\exp i\langle x, x\rangle - 1}{i\langle x^\star, x\rangle} - 1 - \frac{1}{2} i\langle x^\star, x\rangle 1_B(x)\Big] M(dx)\Big\}.\end{aligned}$$

Generators for the semigroup $\mathscr{U}(E)$ of s-selfdecomposable are given in the following.

THEOREM 2.10. *$\mathscr{U}(E)$ is the smallest closed subsemigroup of $ID(E)$ containing all Gaussian measures and all measures of the form $[a, 0, N_{\alpha,\beta,z}]$, where $\alpha, \beta \in \mathbb{R}^+$, $z \in S$ and*

$$N_{\alpha,\beta,z}(A) = \int_S \int_0^\alpha 1_A(tx)\beta\delta_z(dx), \quad A \in \mathscr{B}(E\setminus\{0\}).$$

PROOF. Note that (2.9) gives

$$(c\delta_x)'(A) = \frac{c}{\|x\|} \int_S \int_0^{\|x\|} 1_A(tz)\delta_{x/\|x\|}(dz),$$

where $c \in \mathbb{R}^+$ and $x \in E\setminus\{0\}$. Further, all measures of the form $[a, 0, c\delta_x]$ ($a \in E$, $c \in \mathbb{R}^+$, $x \in E\setminus\{0\}$) and all Gaussian ones generate the whole class ID(E) taking their finite convolutions and weak limits, cf. [1] Chapter III, Theorem 4.7. Since $\mathcal{J}$ is a continuous isomorphism between ID(E) and $\mathscr{U}(E)$ we infer that the Theorem 2.10 holds true.

REMARK 2.1. The generators of the class $\mathscr{U}$ described in Theorem 2.10 are simpler than those from Theorem 3.1 in [9] and the above proof is completely different from that in [9].

3. The class L. One of the main aims of the present paper is to indicate relations between the class, $\mathscr{U}$, of s-selfdecomposable measures and the class, L, of selfdecomposable ones. Therefore, in this section we collect properties of L analogous to that of $\mathscr{U}$ proved in Section 2. Recall that an infinitely divisible measure $[a, R, M]$ is called *selfdecomposable* if

$$M \geq T_c M \quad \text{for} \quad 0 < c < 1. \tag{3.1}$$

The mappings T_c, $c \in \mathbb{R}^+$, defined in (2.1) form a one-parameter group of linear continuous operators on E.

REMARK 3.1. The class L coincides with the class of limit distributions of the sequences

$$T_{c_n}(\xi_1 + \cdots + \xi_n) + a_n,$$

where (ξ_n) are E-valued independent rv's and the triangular array $(T_{c_n}\xi_j)$, $j = 1, 2, \cdots, n$; $n = 1, 2, \cdots$ is infinitesimal. In case (ξ_n) are identically distributed we obtain the class of stable measures; cf. [13] Theorem 2.5, [5] Theorem 3.1.

THEOREM 3.2. *The following conditions for a Lévy measure M, are equivalent*:

(a) $M \geq T_c M$ *for* $0 < c < 1$;

(b) *for each* $A \in \mathscr{B}(S)$ *the Lévy spectral function* L_M *has right and left derivatives with respect to* r *such that* $r dL_M(A, r)/dr$ *is nonincreasing on* $\mathbb{R}^+$;

(c) *there is a unique Lévy measure* F *such that* $\int_{\|x\|>1} \log(1 + \|x\|)F(dx) < \infty$ *and*

$$M(A) = \int_0^\infty (T_{e^{-t}}F)(A)\, dt \quad \text{for all} \quad A \in \mathscr{B}(E\backslash\{0\}).$$

This is a particular case ($Q = I$) of the results from [6]. Further, from Theorem 3.2(c) or from [13] Theorem 2.5 we have

COROLLARY 3.3. *A measure* $\mu \in L(E)$ *if and only if for every* $0 < c < 1$ *there exists* $\mu_c \in \mathrm{ID}(E)$ *such that*

$$\mu = T_c\mu \bigstar \mu_c.$$

From Theorem 3.2(c) we also have

COROLLARY 3.4. *There are no selfdecomposable measures with nonzero finite Lévy measures.*

COROLLARY 3.5. $L(E)$ *with convolution and weak convergence forms a closed subsemigroup of* $\mathrm{ID}(E)$.

Let $\mathrm{ID}_{\log}(E) := \{\mu \in \mathrm{ID}: \int_E \log(1 + \|x\|)\mu(dx) < \infty\}$ and for $\nu \in \mathrm{ID}_{\log}$ put

$$\mathscr{I}\nu := \mathscr{L}\left(\int_0^\infty e^{-t}\, dY(t)\right), \tag{3.2}$$

where Y is $D_E[0, \infty)$-valued rv with stationary independent increments such that $Y(0) = 0$ a.s. and $\mathscr{L}(Y(1)) = \nu$; cf. [11] Theorem 2.3. If $\nu = [a, R, M]$ and $\mathscr{I}\nu = [a^0, R^0, M^0]$ then we have

$$R^0 = \tfrac{1}{2}R, \tag{3.3}$$

$$M^0(A) = \int_0^\infty (T_{e^{-t}}M)(A)\, dt \quad \text{for} \quad A \in \mathscr{B}(E\setminus\{0\}), \tag{3.4}$$

$$a^0 = a + \int_0^\infty e^{-t} \int_{1<\|x\|\le e^t} xM(dx)\, dt. \tag{3.5}$$

Further, we have the following connection between L and $\mathrm{ID}_{\log}$ proved in [11] Lemma 4.2.

THEOREM 3.6. *The mapping $\mathscr{I}$ is an isomorphism between the semigroups $ID_{\log}$ and L.*

REMARK 3.7. The continuity of $\mathscr{I}$ for $E = \mathbb{R}^d$ is proved in [20] Theorem 6.1 (taking $Q = I$) and partial results for $E = H$ are given in [11] Section 6.

Since stable measures have finite logarithmic moment they can be transformed by the mapping $\mathscr{I}$. Moreover, they are characterized as follows.

THEOREM 3.8. *A measure $\mathscr{J}\mu$ is stable if and only if μ is also.*

The proof is analogous to that of Theorem 2.7. Finally, defining the $\mathscr{I}$-invariance similarly to that of $\mathscr{J}$ we have the following fact; cf. [11] Theorem 5.1.

THEOREM 3.9. *A measure μ is $\mathscr{I}$-invariant if and only if μ is a stable one.*

Finally we shall give characterization of the class L in terms of a characteristic functional, cf. [11] Theorem 4.3.

THEOREM 3.10. *A complex valued function ϕ on $E^\star$ is a characteristic functional of a selfdecomposable measure if and only if there is a unique measure $\nu = [a, R, M] \in \mathrm{ID}_{\log}$ such that*

$$\begin{aligned}\phi(x^\star) &= \exp \int_0^1 \frac{\log \hat{\nu}(sx^\star)}{s}\, ds \\ &= \exp\left\{ i\langle x^\star, a\rangle - \frac{1}{4}\langle x^\star, Rx^\star\rangle \right. \\ &\qquad \left. + \int_{E\setminus\{0\}} \left(\int_0^1 \frac{\exp is\langle x^\star, x\rangle - 1}{s}\, ds - i\langle x^\star, x\rangle 1_B(x) \right) M(dx) \right\}.\end{aligned}$$

Let us note that the first equality for $E = \mathbb{R}$ was briefly mentioned in [22], page 898 without specifying the moment condition for ν.

4. Relations between $\mathscr{U}$ and L. Although the notions of s-selfdecomposability and selfdecomposability are defined by nonlinear and linear operators respectively, from Theorems 2.2(b) and 3.2(b) or from Corollaries 2.3 and 3.3 we have the following inclusion.

COROLLARY 4.1. *Each selfdecomposable measure is also s-selfdecomposable.*

From the equality

$$\int_{\|x\|>1} \log \|x\| M'(dx) = \int_0^1 \int_{t^{-1}}^{\infty} \log(tu)\, dL_M(S, u)\, dt$$
$$= \int_1^{\infty} \int_{u^{-1}}^{1} \log(tu)\, dt\, dL_M(S, u)$$
$$= \int_{\|x\|>1} \log \|x\| M(dx) - \int_{\|x\|>1} (1 - \|x\|^{-1}) M(dx)$$

and Theorem 2 in [10] we infer that

$$\mu \in \mathrm{ID}_{\log} \quad \text{if and only if} \quad \mathscr{J}\mu \in \mathrm{ID}_{\log}. \tag{4.1}$$

This allows us to define $\mathscr{J}(\mathscr{I}\mu)$ and $\mathscr{I}(\mathscr{J}\mu)$ for $\mu \in \mathrm{ID}_{\log}$. Moreover, using the formulae (2.8)–(2.10) and (3.3)–(3.5) we get the following corollary.

COROLLARY 4.2. *The mappings $\mathscr{I}$ and $\mathscr{J}$ commute (on $\mathrm{ID}_{\log}$) and if $\mu = [a, R, M]$ and $\mathscr{I}(\mathscr{J}\mu) = [a^{\sim}, R^{\sim}, M^{\sim}]$ then*

$$R^{\sim} = \frac{1}{6} R, \quad M^{\sim}(A) = \int_0^{\infty} \int_0^1 (T_{e^{-t}s} M)(A)\, ds\, dt \quad \textit{for} \quad A \in \mathscr{B}(E \setminus \{0\})$$

and

$$a^{\sim} = \frac{1}{2} a + \int_0^1 \int_0^1 t \int_{1<\|x\|\le(ts)^{-1}} x M(dx)\, dt\, ds.$$

Further, let us note that for $A \in \mathscr{B}(S)$ and $r \in \mathbb{R}^+$

$$L_{M^{\sim}}(A, r) = \int_r^{\infty} \int_s^{\infty} \frac{L_M(A, u)}{u^2}\, du\, ds$$
$$= \int_r^{\infty} \frac{(u - r) L_M(A, u)}{u^2}\, du = L_{M^0}(A, r) - L_{M'}(A, r),$$

i.e. $M^{\sim} = M^0 - M'$. Similarly we have $a^{\sim} = a^0 - a'$. Therefore, we have proved the following fact.

COROLLARY 4.3. *If $[a, R, M] \in \mathrm{ID}_{\log}$ and $\mathscr{I}(\mathscr{J}[a, R, M]) = [a^{\sim}, R^{\sim}, M^{\sim}]$ then $a^{\sim} = a^0 - a'$, $R^{\sim} = R^0 - R'$, $M^{\sim} = M^0 - M'$.*

As a consequence of the proofs of Theorems 2.7, 2.8, 3.8, and 3.9 we get

COROLLARY 4.4. *A measure $\mu \in \mathrm{ID}_{\log}$ is $\mathcal{I} \cdot \mathcal{J}$-invariant or $\mathcal{J} \cdot \mathcal{I}$-invariant if any only if μ is a stable measure.*

Finally we shall show when s-selfdecomposable measures are selfdecomposable ones.

THEOREM 4.5. *A measure $\mathcal{J}\nu$ is selfdecomposable if and only if there is $\mu \in \mathrm{ID}_{\log}$ such that $\nu = (\mathcal{I}\mu)^{\star}\mu$.*

PROOF. The sufficiency follows from Corollary 4.3. Conversely, suppose that $\mathcal{J}\nu = \mathcal{I}\rho$ where $\nu = [a, R, M]$, $\rho = [b, T, N]$ and $\rho \in \mathrm{ID}_{\log}$. Hence we have $R' = T^0$, $M' = N^0$ and $a' = b^0$. The first equality gives $R = T^0 + T$ and the second one gives

$$r \int_r^\infty \frac{L_M(A, s)}{s^2}\, ds = \int_r^\infty \frac{L_N(A, s)}{s}\, ds$$

for all $r \in \mathbb{R}^+$ and $A \in \mathcal{B}(S)$. After taking the right derivatives we get

$$L_M(A, r) = \int_r^\infty \frac{L_N(A, s)}{s}\, ds + L_N(A, r),$$

i.e., $M = N^0 + N$. Finally, the third equality implies

$$a = b + \left[b + 2\left(\int_0^\infty e^{-t} g_N(e^t)\, dt - \int_0^1 t g_M(t^{-1})\, dt \right) \right], \tag{4.2}$$

where the vector valued function $g_Q(u)$ (Q is a Lévy measure and $u > 1$) is defined by the formula

$$g_Q(u) := \int_{1 < \|x\| \le u} x Q(dx).$$

To complete the proof we have to show that the expression in the square brackets in (4.2) is equal to b^0 or equivalently that

$$\int_0^\infty e^{-t} g_N(e^t)\, dt = 2 \int_0^1 t g_M(t^{-1})\, dt$$

whenever $M = N^0 + N$. To see this it is enough to note that

$$\int_0^1 t g_{N^0}(t^{-1})\, dt = \int_0^1 \left(\frac{1}{2} - t \right) g_N(t^{-1})\, dt,$$

which can be easily derived, by some computation, from (3.4). Therefore, we have proved $[a, R, M] = [b^0, T^0, N^0] \star [b, T, N]$, which completes the proof.

COROLLARY 4.6. *Let $\nu \in \mathrm{ID}$ and $\rho \in \mathrm{ID}_{\log}$. Then $\mathcal{J}\nu = \mathcal{I}\rho$ if and only if $\nu = \mathcal{I}\rho \star \rho$. In other words $\mathcal{J}^{-1}(\mathcal{I}\rho) = \mathcal{I}\rho \star \rho$.*

A probability measure μ is said to be *s-stable* if μ is Gaussian or $\mu = [a, 0, M]$ and

$$M(A) = \int_S \int_0^\infty 1_A(tx)e^{-\alpha t}\, dt\gamma(dx) \quad \text{for} \quad A \in \mathscr{B}(E\setminus\{0\}), \tag{4.3}$$

where $0 < \alpha < \infty$ and γ is a finite Borel measure on the unit sphere S in E. Of course, M in (4.3) is finite. Further let us note that s-stable measures are also s-selfdecomposable.

REMARK 4.7. In the case of a Hilbert space the class of all s-stable measures coincides with the limit distributions of the sequences described in Remark 2.1 when the (ξ_n)'s are identically distributed; cf. [4] Theorem 8.1.

Corollary 3.4 implies the following statement.

COROLLARY 4.8. *A measure is stable and s-stable simultaneously if and only if it is Gaussian.*

REMARK 4.9. Non-Gaussian s-stable measures are not $\mathscr{J}$-invariant, cf. Theorems 2.8, 3.9, and also the analogue of Theorem 2.7 does not hold for them, cf. Theorem 3.8.

5. Final remarks.

5.1. The measures $[a, R, M] \in \mathscr{U}(\mathbb{R})$ with symmetric Lévy spectral function were investigated by Medgyessy (1967). He proved that all of them are unimodal at zero. Corollary 4.1 with Theorem 2.2(b) implies that selfdecomposable measures with symmetric Lévy measures are also unimodal. Yamazato (1978) proved that all measures from the class $L(\mathbb{R})$ are unimodal.

5.2. s-selfdecomposable measures were described as some limit distributions by O'Connor (1979) and Jurek (1977). The first proved Theorem 2.5 for $E = \mathbb{R}$, with a completely different proof from ours, and the second the characterization mentioned in Remark 2.1 for $E = H$.

5.3. Using an extreme-point method Urbanik (1968) characterized the class $L(\mathbb{R})$ in terms of characteristic functions, cf. Theorem 3.10. The idea of Urbanik's proof was exploited many times by himself and other authors. In particular, Jurek (1977) describes characteristic functionals of measures from $\mathscr{U}(H)$ finding extreme points of some compact convex set, cf. Theorem 2.9. An alternative method of proofs avoiding the extreme-point one, is given in Jurek (1982a).

5.4. Random integral representations of some limit distributions were introduced by Wolfe (1982) and by Jurek and Vervaat (1983). They characterized in this manner the selfdecomposable measures on $\mathbb{R}$ and E respectively, cf. Theorem 3.6. The integral representation of measures from $\mathscr{U}(H)$ is given in Jurek (1982b) and from Urbanik's classes L_m in Jurek (1983a). Let us note that the classes investigated by O'Connor (1979b, Theorem 1) also admit random integral representation. It is enough to choose an appropriate time scale deformation in the $D_E[0, 1]$-valued process with stationary independent increments, cf. (0.1). All the above suggest the following hypothesis:

Each class of limit distributions, derived from sequences of independent random variables, is the image of some subset of ID by some mapping defined as a random integral.

Note also that from the integral representation, the description in terms of characteristic functionals easily follows, cf. Remark 5.3.

5.5. Generators for the class $L(\mathbb{R})$ were found by Kubik (1962) and for $L(H)$ and $\mathscr{U}(H)$ by Jurek (1983b). The proofs are based on theorems concerning the convergence of infinitely divisible measures. The idea of using the mapping $\mathscr{J}$, in finding generators for $\mathscr{U}(E)$ is new; cf. Theorem 2.10. It may have further applications for classes of limit distributions for which the hypothesis in Remark 5.4 holds true.

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