Relations for Moments of Dual Generalized Order Statistics for a New Inverse Kumaraswamy Distribution

Abdulkader Monier Daghistani
Department of Statistics
King Abdulaziz University
Jeddah, Saudi Arabia
amadaghistani@kau.edu.sa

Bander Al-Zahrani
Department of Statistics
King Abdulaziz University
Jeddah, Saudi Arabia
bmalzahrani@kau.edu.sa

Muhammad Qaiser Shahbaz
Department of Statistics
King Abdulaziz University
Jeddah, Saudi Arabia
qshahbaz@gmail.com

Abstract

In this paper a new inverse Kumaraswamy distribution has been proposed. The recurrence relation for moments of dual generalized order statistics has been presented for the new inverse Kumaraswamy distribution. These include the recurrence relations for single, inverse, product and ratio moments of dual generalized order statistics for the new inverse Kumaraswamy distribution. Special cases of the recurrence relations have also been obtained.

Keywords: Dual Generalized Order Statistics, Inverse Kumaraswamy Distribution, Recurrence Relations, Moments.

1. Introduction

The dual generalized order statistics ($DGOS$) has been introduced by Kamps et al. (2003) as a unified model for descendingly arranged random variables. The joint distribution of $DGOS$ from some distribution $F(x)$ is given by Kamps et al. (2003) as

$$f_{1,\dots,n:n,m,k}(x_1,\dots,x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[F\left(x_n\right)\right]^{k-1}f(x_n) \times\left(\prod_{i=1}^{n-1}\left[F\left(x_i\right)\right]^{m}f(x_i)\right), \quad (1)$$

where n, m and k are parameters of the model. The quantities γ_j are given as $\gamma_j = k + (n-r)(m+1)$. Kamps et al. (2003) has shown that the density function of a single *DGOS* is

$$f_{r(d):n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x) \left[F(x)\right]^{\gamma_r - 1} g_{m(d)}^{r-1} \left[F(x)\right] \quad (2)$$

where $C_{r-1} = \prod_{j=1}^{r} \gamma_j$, $r = 1, 2, \cdots, n$ and

$$g_{m(d)}(x) = h_{m(d)}(1) - h_{m(d)}(x) = \begin{cases} \frac{1-x^{m+1}}{m+1}; & m \neq -1 \\ -\ln x; & m = -1 \end{cases}$$

We also have

$$h_{m(d)}(x) = \begin{cases} \frac{x^{m+1}}{m+1}; & m \neq -1 \\ \ln x; & m = -1 \end{cases}$$

Kamps et al. (2003) has further shown the joint density function of two *DGOS* $X_{r(d):n,m,k}$ and $X_{s(d):n,m,k}$ for $r < s$ is given as

$$\begin{aligned} f_{r(d),s(d):n,m,k}\left(x_1, x_2\right) = & \frac{C_{s-1}}{(r-1)!\,(s-r-1)!} f\left(x_1\right) f\left(x_2\right) \left\{F\left(x_1\right)\right\}^{m} \\ & \times g_{m(d)}^{r-1} \left\{F\left(x_1\right)\right\} \left\{F\left(x_2\right)\right\}^{\gamma_s - 1} \\ & \times \left[h_{m(d)}\left\{F\left(x_1\right)\right\} - h_{m(d)}\left\{F\left(x_2\right)\right\}\right]^{s-r-1} \\ & ; -\infty \leq x_1 \leq x_2 \leq \infty \end{aligned} \quad (3)$$

The density functions of *DGOS* given in (2) and (3) provide several models of ordered random variables as special case. Specifically for $m = 0$ and $k = 1$ the model reduces to *Reversed Order Statistics*. Also for $m = 0$ and $k = \alpha - n + 1$ the model reduces to *Reversed Order Statistics with non-integral sample size*. For $m = (n-i+1)\alpha_i - (n-i)\alpha_{i+1} - 1, i = 1, 2, ..., n-1$ and $k = \alpha_n$ the model reduces to *Lower Sequential Order Statistics based on the arbitrary distribution function* $F_r(t) = 1 - [F(t)]^{\alpha_r}; 1 \leq r \leq n$. Also for $m = -1$ and $k \in N$: k^{th} the model reduces to k^{th} *Lower record values*. For $m = \beta_i - \beta_{i+1}^{-1}$ and $k = \beta_n$ the model reduces to *Pfrifrer's lower record values from non-identically distributed random variables based upon* $F_r = 1 - [F(t)]^{\beta_r}; 1 \leq r \leq n$. More details about *DGOS* can be found in Ahsanullah and Nevzorov (2001), Shahbaz et al. (2016) and Shahbaz et al. (2017).

Since the emergence of *DGOS* several authors have studied the properties of *DGOS* for various base distributions. Obtaining recurrence relations for moments of generalized order statistics (*GOS*) and *DGOS* has been an area of study by several authors. Recurrence relations for moments of *GOS* from Pareto, generalized Pareto, and Burr distributions has been obtained by Pawlas and Szynal (2001). Recurrence relations for moments for *DGOS* for a general class of distributions; which provide inverse Weibull and inverse Exponential distribution as special case; have been obtained by Kotb et al. (2013) among others. The Kumaraswamy (1980) distribution is a useful bounded distribution with support over $[0, 1]$. The distribution has density function

$$f\left(x; \alpha, \beta\right) = \alpha\beta x^{\alpha-1} \left(1 - x^{\alpha}\right)^{\beta-1} \;;\; 0 < x < 1 \;;\; \alpha, \beta > 0. \quad (4)$$

The Kumaraswamy distribution is a useful alternative to the Beta distribution. Cordeiro and Castro (2010) have proposed a family of distributions by using Kumaraswamy distribution and is named as the Kumaraswamy family of distributions. Recently, AL-Fattah et al. (2017) have proposed an Inverted Kumaraswamy distribution by transforming the distribution (4) as $Y = X^{-1} - 1$. The density function of the inverse Kumaraswamy distribution, proposed by AL-Fattah et al. (2017) is

$$f(y;\alpha,\beta) = \alpha\beta(1+y)^{-(\alpha+1)}(1-(1+y)^{-\alpha})^{\beta-1}\ ;\ y>0\ ;\ \alpha,\beta>0. \tag{5}$$

The inverted Kumaraswamy distribution, given in (5), in not a proper inversion of the Kumaraswamy (1980) distribution although this can be used in studies related to life length of the components. In the following section we have given a new inverted Kumaraswamy distribution.

2. New Inverse Kumaraswamy Distribution

AL-Fattah et al. (2017) have proposed an inverted Kumaraswamy distribution (5) by using the transformation $Y = X^{-1} - 1$ in (4). We have proposed another version of the inverse Kumaraswamy distribution by using the transformation $Y = X^{-1}$ in (4). The density function and distribution function of our proposed inverse Kumaraswamy distribution are

$$f(y;\alpha,\beta) = \alpha\beta y^{-(\alpha+1)}(1-y^{-\alpha})^{\beta-1}\ ;\quad y>1\ ;\ \alpha,\beta>0, \tag{6}$$

and

$$F(y;\alpha,\beta) = (1-y^{-\alpha})^{\beta};\quad y>1\ ;\ \alpha,\beta>0. \tag{7}$$

We can see that the density and distribution functions are related as

$$F(y) = \frac{(1-y^{-\alpha})}{\alpha\beta y^{-(\alpha+1)}} f(x) \tag{8}$$

The relation given in (8) is useful in driving the recurrence relations for moments of the new inverse Kumaraswamy distribution. Will will derive these relations in the following sections.

3. Relation for Single Moments

Shahbaz et al. (2016) have provided the following relation between single moments of *DGOS* from any distribution by using a general result given by Khan et al. (2008)

$$\begin{aligned}\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} = & -\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\left[F(x)\right]^{\gamma_r} \\ & \times g^{r-1}_{m(d)}\left[F(x)\right]dx\end{aligned} \tag{9}$$

where $\mu^{p}_{r(d):n,m,k} = E\left(X^{p}_{r(d):n,m,k}\right)$ etc. The relaton (9) is useful in deriving recurrence relation for moments of any distribution. Now writting (9) as

$$\mu^p_{r(d):n,m,k} - \mu^p_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\left[F(x)\right]\left[F(x)\right]^{\gamma_r-1} \times g^{r-1}_{m(d)}\left[F(x)\right]dx \tag{10}$$

The recurrence relation for single moments of the new inverse Kumaraswamy distribution is obtained by using (8) in (10) as

$$\begin{aligned}\mu^p_{r(d):n,m,k} - \mu^p_{r-1(d):n,m,k} &= -\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{1}^{\infty} y^{p-1}\frac{(1-y^{-\alpha})}{\alpha\beta y^{-(\alpha+1)}}f(x)\left[F(y)\right]^{\gamma_r-1} \\ &\quad \times g^{r-1}_{m(d)}\left[F(y)\right]dy \\ &= -\frac{pC_{r-1}}{\alpha\beta\gamma_r(r-1)!}\int_{1}^{\infty}\left(y^{p+\alpha}-y^p\right)f(x)\left[F(y)\right]^{\gamma_r-1} \\ &\quad \times g^{r-1}_{m(d)}\left[F(y)\right]dy\end{aligned}$$

Simplifying, we have following relation for single moments of $DGOS$ for the new inverse Kumaraswamy distribution

$$\mu^p_{r(d):n,m,k} - \mu^p_{r-1(d):n,m,k} = -\frac{p}{\alpha\beta\gamma_r}\left[\mu^{p+\alpha}_{r(d):n,m,k} - \mu^p_{r(d):n,m,k}\right]$$

or

$$\mu^p_{r(d):n,m,k} = \mu^p_{r-1(d):n,m,k} - \frac{p}{\alpha\beta\gamma_r}\mu^{p+\alpha}_{r(d):n,m,k} + \frac{p}{\alpha\beta\gamma_r}\mu^p_{r(d):n,m,k} \tag{11}$$

The recurrence relation for special cases can be easily obtained. Using $m=0$ and $k=1$ in (11) we have following recurrence relations for single moments of reversed order statistics for the new inverse Kumaraswamy distribution

$$\mu^p_{r(d):n} = \mu^p_{r-1(d):n} - \frac{p}{\alpha\beta(n-r+1)}\mu^{p+\alpha}_{r(d):n} + \frac{p}{\alpha\beta(n-r+1)}\mu^p_{r(d):n} \tag{12}$$

Again, using $m=-1$, the recurrence relations for single moments of k^{th} lower record values for the new inverse Kumaraswamy distribution is

$$\mu^p_{K(r)} = \mu^p_{K(r)} - \frac{p}{\alpha\beta k}\mu^{p+\alpha}_{K(r)} + \frac{p}{\alpha\beta k}\mu^p_{K(r)} \tag{13}$$

The recurrence relations for other special cases can be obtained from (11).

4. Relation for Inverse Moments

The inverse moments of $DGOS$ are defined as

$$\mu^{-p}_{r(d):n,m,k} = E\left(X^{-p}_{r(d):n,m,k}\right) = \frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty} x^{-p}f(x)\left[F(x)\right]^{\gamma_r-1} \times g^{r-1}_{m(d)}\left[F(x)\right]dx$$

We know that the inverse moments of $DGOS$ for any distribution are related as

$$\begin{aligned}\mu^{-p}_{r(d):n,m,k}-\mu^{-p}_{r-1(d):n,m,k} &= \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{-p-1}\left[F(x)\right]^{\gamma_r} \\ &\times g^{r-1}_{m(d)}\left[F(x)\right]dx \\ &= \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{-p-1}\left[F(x)\right]\left[F(x)\right]^{\gamma_r-1} \\ &\times g^{r-1}_{m(d)}\left[F(x)\right]dx \end{aligned} \tag{14}$$

The recurrence relation for inverse moments of the new inverse Kumaraswamy distribution is obtained by usng (8) in (14) as

$$\begin{aligned}\mu^{-p}_{r(d):n,m,k}-\mu^{-p}_{r-1(d):n,m,k} &= \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{1}^{\infty} y^{-p-1}\frac{(1-y^{-\alpha})}{\alpha\beta y^{-(\alpha+1)}}f(x)\left[F(y)\right]^{\gamma_r-1} \\ &\times g^{r-1}_{m(d)}\left[F(y)\right]dy \\ &= \frac{pC_{r-1}}{\alpha\beta\gamma_r(r-1)!}\int_{1}^{\infty}\left(y^{-p+\alpha}-y^{-p}\right)f(x)\left[F(y)\right]^{\gamma_r-1} \\ &\times g^{r-1}_{m(d)}\left[F(y)\right]dy\end{aligned}$$

Simplifying, we have the following relation for inverse moments of $DGOS$ for the new inverse Kumaraswamy distribution

$$\mu^{-p}_{r(d):n,m,k}-\mu^{-p}_{r-1(d):n,m,k}=\frac{p}{\alpha\beta\gamma_r}\left[\mu^{-p+\alpha}_{r(d):n,m,k}-\mu^{-p}_{r(d):n,m,k}\right]$$

or

$$\mu^{p}_{r(d):n,m,k}=\mu^{p}_{r-1(d):n,m,k}+\frac{p}{\alpha\beta\gamma_r}\mu^{-p+\alpha}_{r(d):n,m,k}-\frac{p}{\alpha\beta\gamma_r}\mu^{-p}_{r(d):n,m,k} \tag{15}$$

The recurrence relation for special cases can be easily obtained from (15). Using $m=0$ and $k=1$ in (15) we have following recurrence relations for inverse moments of reversed order statistics for the new inverse Kumaraswamy distribution

$$\mu^{-p}_{r(d):n}=\mu^{-p}_{r-1(d):n}+\frac{p}{\alpha\beta(n-r+1)}\mu^{-p+\alpha}_{r(d):n}-\frac{p}{\alpha\beta(n-r+1)}\mu^{-p}_{r(d):n} \tag{16}$$

Again, using $m=-1$, the recurrence relation for inverse moments of k^{th} lower record values for the new inverse Kumaraswamy distribution is

$$\mu^{-p}_{K(r)}=\mu^{-p}_{K(r)}+\frac{p}{\alpha\beta k}\mu^{-p+\alpha}_{K(r)}-\frac{p}{\alpha\beta k}\mu^{-p}_{K(r)} \tag{17}$$

The recurrence relations for other special cases can be obtained from (15).

5. Relation for Product Moments

The product moments are useful in studying the joint variability between two *DGOS*. Shahbaz et al. (2016) have given the following relation for the product moments of *DGOS* for any distribution

$$\begin{aligned}&\mu^{p,q}_{r(d),s(d):n,m,k}-\mu^{p,q}_{r(d),s-1(d):n,m,k}\\&=-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}x_1^p x_2^{q-1}f(x_1)F(x_2)\\&\times\left[F(x_1)\right]^m g^{r-1}_{m(d)}\left[F(x_1)\right]\left[F(x_2)\right]^{\gamma_s-1}\\&\times\left\{h_{m(d)}\left[F(x_1)\right]-h_{m(d)}\left[F(x_2)\right]\right\}^{s-r-1}dx_2dx_1\end{aligned} \tag{18}$$

Now, using (8) in (18), the relation for product moments of *DGOS* for the new inverse Kumaraswamy distribution is obtained as

$$\begin{aligned}&\mu^{p,q}_{r(d),s(d):n,m,k}-\mu^{p,q}_{r(d),s-1(d):n,m,k}\\&=-\frac{qC_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!}\int_{1}^{\infty}\int_{1}^{y_1}y_1^p y_2^{q-1}\frac{(1-y_2^{-\alpha})}{y_2^{-(\alpha+1)}}f(y_1)f(y_2)\\&\times\left[F(y_1)\right]^m g^{r-1}_{m(d)}\left[F(y_1)\right]\left[F(y_2)\right]^{\gamma_s-1}\\&\times\left\{h_{m(d)}\left[F(y_1)\right]-h_{m(d)}\left[F(y_2)\right]\right\}^{s-r-1}dy_2dy_1\end{aligned}$$

or

$$\begin{aligned}&\mu^{p,q}_{r(d),s(d):n,m,k}-\mu^{p,q}_{r(d),s-1(d):n,m,k}\\&=-\frac{qC_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!}\int_{1}^{\infty}\int_{1}^{y_1}y_1^p\left(y_2^{q+\alpha}-y_2^{q}\right)f(y_1)f(y_2)\\&\times\left[F(y_1)\right]^m g^{r-1}_{m(d)}\left[F(y_1)\right]\left[F(y_2)\right]^{\gamma_s-1}\\&\times\left\{h_{m(d)}\left[F(y_1)\right]-h_{m(d)}\left[F(y_2)\right]\right\}^{s-r-1}dy_2dy_1\end{aligned}$$

or

$$\begin{aligned}&\mu^{p,q}_{r(d),s(d):n,m,k}-\mu^{p,q}_{r(d),s-1(d):n,m,k}\\&=-\frac{q}{\alpha\beta\gamma_s}\left[\mu^{p,q+\alpha}_{r(d),s(d):n,m,k}-\mu^{p,q}_{r(d),s(d):n,m,k}\right]\end{aligned}$$

or

$$\begin{aligned}\mu^{p,q}_{r(d),s(d):n,m,k}=&\,\mu^{p,q}_{r(d),s-1(d):n,m,k}-\frac{q}{\alpha\beta\gamma_s}\mu^{p,q+\alpha}_{r(d),s(d):n,m,k}\\&+\frac{q}{\alpha\beta\gamma_s}\mu^{p,q}_{r(d),s(d):n,m,k}\end{aligned} \tag{19}$$

The relation for product moments for special cases can be easily obtained from (19) by using different choices of constants involved. For example using $m=-1$, the relation for product moments of k^{th} lower record values for the new inverse Kumaraswamy distribution is

$$\mu^{p,q}_{K(r,s)} = \mu^{p,q}_{K(r,s-1)} - \frac{q}{\alpha\beta k}\mu^{p,q+\alpha}_{K(r,s)} + \frac{q}{\alpha\beta k}\mu^{p,q}_{K(r,s)} \tag{20}$$

The relation for product moments of lower record values can be easily obtained by sitting $k = 1$ in (20).

6. Relation for Ratio Moments

The ratio moments of $DGOS$ from any distribution $F(x)$ satisfies following relation

$$\begin{aligned}
&\mu^{p,-q}_{r(d),s(d):n,m,k} - \mu^{p,-q}_{r(d),s-1(d):n,m,k}\\
&= \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{-q-1} f(x_1)F(x_2)\\
&\times [F(x_1)]^m\, g^{r-1}_{m(d)}[F(x_1)]\,[F(x_2)]^{\gamma_s-1}\\
&\times \left\{h_{m(d)}[F(x_1)] - h_{m(d)}[F(x_2)]\right\}^{s-r-1} dx_2 dx_1
\end{aligned} \tag{21}$$

Now, using (8) in (21), the relation for ratio moments of $DGOS$ for the new inverse Kumaraswamy distribution is obtained as

$$\begin{aligned}
&\mu^{p,-q}_{r(d),s(d):n,m,k} - \mu^{p,-q}_{r(d),s-1(d):n,m,k}\\
&= \frac{qC_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!}\int_{1}^{\infty}\int_{1}^{y_1} y_1^p y_2^{-q-1}\frac{(1-y_2^{-\alpha})}{y_2^{-(\alpha+1)}} f(y_1)f(y_2)\\
&\times [F(y_1)]^m\, g^{r-1}_{m(d)}[F(y_1)]\,[F(y_2)]^{\gamma_s-1}\\
&\times \left\{h_{m(d)}[F(y_1)] - h_{m(d)}[F(y_2)]\right\}^{s-r-1} dy_2 dy_1
\end{aligned}$$

or

$$\begin{aligned}
&\mu^{p,-q}_{r(d),s(d):n,m,k} - \mu^{p,-q}_{r(d),s-1(d):n,m,k}\\
&= \frac{qC_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!}\int_{1}^{\infty}\int_{1}^{y_1} y_1^p\left(y_2^{-q+\alpha} - y_2^{-q}\right) f(y_1)f(y_2)\\
&\times [F(y_1)]^m\, g^{r-1}_{m(d)}[F(y_1)]\,[F(y_2)]^{\gamma_s-1}\\
&\times \left\{h_{m(d)}[F(y_1)] - h_{m(d)}[F(y_2)]\right\}^{s-r-1} dy_2 dy_1
\end{aligned}$$

or

$$\begin{aligned}
&\mu^{p,-q}_{r(d),s(d):n,m,k} - \mu^{p,-q}_{r(d),s-1(d):n,m,k}\\
&= \frac{q}{\alpha\beta\gamma_s}\left[\mu^{p,-q+\alpha}_{r(d),s(d):n,m,k} - \mu^{p,-q}_{r(d),s(d):n,m,k}\right]
\end{aligned}$$

or

$$\mu^{p,-q}_{r(d),s(d):n,m,k} = \mu^{p,-q}_{r(d),s-1(d):n,m,k} + \frac{q}{\alpha\beta\gamma_s}\mu^{p,-q+\alpha}_{r(d),s(d):n,m,k} - \frac{q}{\alpha\beta\gamma_s}\mu^{p,-q}_{r(d),s(d):n,m,k} \tag{22}$$

Using $m = -1$ in (22), we have the following relation for ratio moments of k^{th} lower record values for the new inverse Kumaraswamy distribution

$$\mu^{p,-q}_{K(r,s)} = \mu^{p,-q}_{K(r,s-1)} + \frac{q}{\alpha\beta k}\mu^{p,-q+\alpha}_{K(r,s)} - \frac{q}{\alpha\beta k}\mu^{p,-q}_{K(r,s)} \tag{23}$$

The relation for ratio moments of lower record values can be easily obtained by sitting $k = 1$ in (23).

7. Conclusions

In this paper we have obtained the recurrence relations for moments of *DGOS* for the new inverse Kumaraswamy distribution. The relations have been obtained for single, inverse, product and ratio moments of *DGOS*. These relations are useful in obtaining higher order moments from lower order moments.

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