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Abstract

The Sarkisov program studies birational maps between varieties that are end products of the Minimal Model Program (MMP) on nonsingular uniruled varieties. If X and Y are terminal $\mathbb{Q}$-factorial projective varieties endowed with a structure of Mori fibre space, a birational map $f: X \dashrightarrow Y$ is the composition of a finite number of *elementary Sarkisov links*. This decomposition is in general not unique: two such define a *relation in the Sarkisov program*. I define *elementary relations*, and show they generate relations in the Sarkisov program. Roughly speaking, elementary relations are the relations among the end products of suitable relative MMPs of Z over W with $\rho(Z/W) = 3$.

1. Introduction

Let Z be a nonsingular projective variety. Conjecturally, the *Minimal Model Program (MMP) terminates* for Z, and there is a finite sequence of elementary birational operations *directed by* K_Z

$$\varphi : Z = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n = X,$$

where X is terminal and $\mathbb{Q}$-factorial, and one of the following holds.

(i) X *is a minimal model*: K_X is numerically effective (nef); or

(ii) X *is a Mori fibre space*: X is endowed with a fibration morphism $p: X \to S$ such that $\rho(X/S) = 1$, and $-K_X$ is p-ample.

The map φ is *the result of a* K_Z-MMP, and X is a *distinguished representative* in the birational equivalence class of Z. However, neither φ nor X is unique. Since the birational classification of varieties is one of the primary goals of the MMP, it is natural to study birational maps between distinguished representatives that are results of the K_Z-MMP. The type of X (minimal model or Mori fibre space) depends on whether Z is uniruled or not, so that one only needs to consider birational maps between minimal models or between Mori fibre spaces. The *Sarkisov program* studies birational Mori fibre spaces. In fact, if $f: X/S \dashrightarrow Y/T$ is a birational map between Mori fibre spaces and $(\varphi_X, \varphi_Y): Z \to X \times Y$ is a resolution of f, then both $Z \to X$ and $Z \to Y$ are the results of K_Z-MMPs, so that the Sarkisov program studies birational maps between distinguished models produced by the MMP on nonsingular uniruled varieties.

More generally, one can investigate the structure of birational maps between distinguished representatives produced by the MMP on a nonsingular variety Z (or by the log-MMP on a Kawamata log terminal (klt) pair (Z, Δ)). If $\varphi_X : Z \dashrightarrow X$ and $\varphi_Y : Z \dashrightarrow Y$ are the results of K_Z-MMPs, φ_X and φ_Y can be decomposed into finitely many elementary steps of the MMP, the contractions of extremal rays. It follows that $X \dashrightarrow Y$ is a composition of maps $\varphi_k : X_k \dashrightarrow X_{k+1}$

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for $k=0,\dots,N-1$, where $X_0\simeq X$, $X_N\simeq Y$ and for each k, φ_k or φ_k^{-1} is the contraction of an extremal ray. A drawback of this tautological decomposition is that the intermediate varieties X_k are not in general distinguished representatives. The following two theorems give a decomposition where each intermediate step is a birational map between minimal models or between Mori fibre spaces.

Theorem 1.1 [Kaw08]. *A birational map $f: X \dashrightarrow Y$ between minimal models is the composition of a finite number of flops.*

Theorem 1.2 [Cor95, HM13]. *A birational map $f: X/S \dashrightarrow Y/T$ between Mori fibre spaces is the composition of a finite number of elementary Sarkisov links.*

While flops are elementary steps of the MMP itself, Sarkisov links are 'composite' operations. An *elementary Sarkisov link* $L_{i,j}: X_i/S_i \dashrightarrow X_j/S_j$ is the birational map between the two Mori fibre spaces produced by the K_Z-MMP over W, where Z is terminal, and $Z \longrightarrow W$ is a fibration with uniruled positive dimensional fibers and $\rho(Z/W)=2$ (see Definition 2.14). Corti proved Theorem 1.2 directly in [Cor95] for 3-folds, by the process of *untwisting* birational maps between Mori fibre spaces, which I now explain in a few words. Any birational map $f: X \dashrightarrow Y$ is the map $\Phi_{|\mathcal{L}|}$ determined by a suitable linear system $\mathcal{L}=f_*^{-1}\mathcal{L}_Y$, where $\mathcal{L}_Y$ is very ample on Y. In this formulation, f is an isomorphism precisely when $\mathcal{L}$ is basepoint free. In general, if U/R is a Mori fibre space, any $\mathbb{R}$-divisor (respectively ample divisor) on U can be written $-\lambda K_U + D_R$, for $\lambda\in\mathbb{R}$ (respectively $\lambda>0$) and D_R the pullback of a divisor (respectively of a sufficiently ample divisor) on R. Corti uses the form of divisors on X and Y to identify a Sarkisov link $L_{1,2}: X=X_1 \dashrightarrow X_2$ that partially resolves the highest multiplicity locus of $\operatorname{Bs}\mathcal{L}$. Replacing X by X_2, f by $f\circ L_{1,2}^{-1}$ and $\mathcal{L}$ by its image on X_2 is called untwisting f. After a finite number of untwistings, the image of $\mathcal{L}$ is basepoint free, so that f has a decomposition into elementary links. The proof in arbitrary dimension uses different ideas: both Theorems 1.1 and 1.2 follow from the study of the *geography of ample models* of *effective* adjoint divisors $K_Z+\Theta$, as Θ varies in a suitable region of the space of $\mathbb{R}$-divisors $\operatorname{Div}_{\mathbb{R}}(Z)$ as in [KKL13, SC11].

This paper studies *relations in the Sarkisov program*. Given a birational map $f: X \dashrightarrow Y$ between Mori fibre spaces, the decomposition of f into elementary Sarkisov links in Theorem 1.2 is not unique. A relation in the Sarkisov program is a (nontrivial) decomposition of an automorphism of a Mori fibre space X/S into Sarkisov links; equivalently it is a composition of Sarkisov links of the form $f_1\circ f_2^{-1}$, where f_1 and f_2 are distinct decompositions into links of the same birational map $X/S \dashrightarrow Y/T$. Relations arise naturally among Sarkisov links between Mori fibre spaces produced by the MMP on suitable varieties Z with $\rho(Z)\geqslant 3$.

As an example, let S be the del Pezzo surface obtained by blowing up two points P_1 and P_2 on $\mathbb{P}^2$. Let E_1, E_2 be the (-1)-curves on S and L the proper transform of the line through P_1 and P_2. Then, $\overline{\operatorname{Eff}}(S)$, the pseudoeffective cone of S, is the cone over a polytope $\mathcal{P}$ as in Figure 1. There is a fan of $\overline{\operatorname{Eff}}\,S=\cup\mathcal{C}_i$, where $\mathcal{C}_i$ are rational polyhedral cones, that determines a *geography of ample models*. Mori fibre spaces produced by K_S-MMPs are in one-to-one correspondence with those facets of the $\mathcal{C}_i$ that consist of nonbig divisors. In Figure 1, these are the facets of the $\mathcal{C}_i$ supported on the sides of the triangle, e.g. $[L+E_1, L]$ represents $\mathbb{P}^1_a\times\mathbb{P}^1_b\to\mathbb{P}^1_a$ (the indices a, b distinguish between the two fibrations $S\to\mathbb{P}^1$), $[L+E_2, L]$ represents $\mathbb{P}^1_a\times\mathbb{P}^1_b\to\mathbb{P}^1_b$ and $[E_1, L+E_1]$ represents $\mathbb{F}_{1a}\to\mathbb{P}^1_a$ (see Example 3.12 for details). In [HM13], the authors show that if two of these line segments have nonempty intersection (here at the points $L, L+E_1, L+E_2, E_1$ and E_2), there is an elementary Sarkisov link between the corresponding Mori fibre spaces. Further, a decomposition into elementary links of any birational map $f: X/S \dashrightarrow Y/T$ can be

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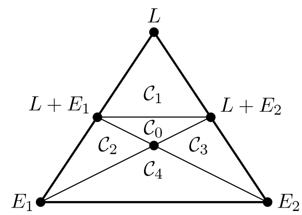

Figure 1. Geography of ample models of $S = \mathrm{Bl}_{\{P_1,P_2\}}\mathbb{P}^2$.

obtained by tracing a path on $\partial\mathcal{P}$ from a point in the interior of the line segment corresponding to X/S to a point in the interior of the line segment corresponding to Y/T. In the same way, a relation in the Sarkisov program *dominated by* S (among the end products of the MMP on S) is represented by a loop on $\partial\mathcal{P}$, i.e. by an element of the fundamental group $\pi_1(\mathcal{P}) = \pi_1(\partial\,\overline{\mathrm{Eff}}(S) \smallsetminus \{0\})$. Here, $\pi_1(\mathcal{P}) = \mathbb{Z}\cdot\gamma$, where γ represents the following relation.

$$\begin{array}{cccccccccccc}
S & = & S & & & & S & = & S & & \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\mathbb{P}^1_a\times\mathbb{P}^1_b & & \mathbb{F}_{1b} & = & \mathbb{F}_{1b},\mathbb{F}_{1a} & = & \mathbb{F}_{1a} & & \mathbb{P}^1_a\times\mathbb{P}^1_b & = & \mathbb{P}^1_a\times\mathbb{P}^1_b \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{P}^1_b & = & \mathbb{P}^1_b & & \mathbb{P}^2 & & \mathbb{P}^1_a & = & \mathbb{P}^1_a & & \mathbb{P}^1_b \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\{P\} & & \{P\} & = & \{P\} & = & \{P\} & & \{P\} & = & \{P\}
\end{array} \qquad (1)$$

I call a relation *elementary* if there is a variety W and morphisms $X_i \to S_i \longrightarrow W$ that commute with the links $L_{i,i+1}$ with $\rho(X_i/W) \leqslant 3$ for all i (see § 4 for a precise definition). If Z is a common resolution of all Sarkisov links $L_{i,i+1}$, an elementary relation corresponds to a simplicial loop on the boundary of the pseudoeffective cone of Z and can be thought of as a *relative Picard rank* 3 *MMP*. It is naturally dual to the boundary of a suitable two-dimensional polytope as in Figure 1.

I show that relations in the Sarkisov program correspond to simplicial loops on suitable polyhedral complexes supported on the locus of strictly pseudoeffective (effective nonbig) divisors of nonsingular uniruled varieties. The following theorem (Theorem 4.11) is then a straightforward application of van Kampen's theorem.

Theorem 1.3. *A relation in the Sarkisov program is the composition of a finite number of elementary relations.*

There are four *types* of elementary Sarkisov links that correspond to possible configurations in the *two-ray game*, or, loosely speaking, to configurations of extremal rays on a (weak) Fano variety Z of Picard rank 2. There is no such result when $\rho(Z) = 3$, and an explicit *classification* of elementary relations is therefore not a reasonable goal. However, one can define two types (A and B) of elementary relations consisting of an arbitrary number of elementary links (see Definition 4.6). For example, (1) is of type A. In Example 4.9, I determine all elementary relations that arise among the Mori fibre spaces that are end products of the MMP on Fano 3-folds Z with $\rho(Z) = 3$, and show that both types of relations occur. In fact, both types of

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relation are realised among the end products of the MMP on Picard rank 3 toric Fano 3-folds, or in the Cremona group of $\mathbb{P}^3$.

Outline

I sketch the main idea underlying the results in [HM13] and in this paper. For now, I do not distinguish between divisors and their numerical equivalence classes. Let $f : X \dashrightarrow Y$ be a birational map, and assume that $\varphi_X : Z \longrightarrow X$ and $\varphi_Y : Z \longrightarrow Y$ are results of the MMP on a nonsingular uniruled variety Z. Denote by $p_X : X \to S$ and $p_Y : Y \to T$ the Mori fibrations on X and Y. If D is an effective $\mathbb{Q}$-divisor on Z, $Z \dashrightarrow Z_D \simeq \operatorname{Proj} R(Z, D)$ (when it makes sense) is the *ample model of* D; the precise definition is recalled in § 2. Define klt pairs $(Z, \Theta_X), (Z, \Theta_Y)$, where Θ_X, Θ_Y are ample divisors such that φ_X (respectively φ_Y) is the ample model of $K_Z + (1+\varepsilon)\Theta_X$ (respectively $K_Z + (1+\varepsilon)\Theta_Y$) for $0 < \varepsilon \ll 1$ and $p_X \circ \varphi_X$ (respectively $p_Y \circ \varphi_Y$) that of $K_Z + \Theta_X$ (respectively $K_Z + \Theta_Y$). Then, for suitable Θ_X and Θ_Y, there is a rational polyhedral cone $\mathcal{C} \subset \overline{\operatorname{Eff}}_{\mathbb{R}}(Z)$, such that the following holds. Every divisor $D \in \mathcal{C}$ is effective, $K_Z + \Theta_X$ and $K_Z + \Theta_Y$ lie on $\partial^+\mathcal{C} = \mathcal{C} \cap (\overline{\operatorname{Eff}}_{\mathbb{R}}(Z) \smallsetminus \operatorname{Big}_{\mathbb{R}}(Z))$, while $K_Z + (1+\varepsilon)\Theta_X$ and $K_Z + (1+\varepsilon)\Theta_Y$ lie in the interior of $\mathcal{C}$. Section 2 recalls general results on cones such as $\mathcal{C}$. In particular, there is a decomposition of $\mathcal{C} = \coprod \mathcal{A}_i$ into relatively open rational polyhedral cones that determine a *geography of ample models*. Namely, all $\mathbb{Q}$-divisors $D \in \mathcal{A}_i$ have the same ample model and birational maps between these ample models are determined by incidence relations between the closed cones $\overline{\mathcal{A}_i}$. The Sarkisov program studies incidence relations between cones $\overline{\mathcal{A}_i}$ that intersect the locus of nonbig divisors $\partial^+\mathcal{C}$.

One can associate to a relation in the Sarkisov program a cone $\mathcal{C} \subset \operatorname{Div}_{\mathbb{R}}(Z)$, where Z is a common resolution of all the links in the relation, and a polyhedral complex $\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$ that reflects the geography of ample models on $\mathcal{C}$; this is done in § 3. There is a simplicial complex $\mathcal{N}_{\mathfrak{R}}$, the *nerve of* $\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$, that encodes the *Sarkisov program dominated by* Z. Explicitly, there is a one-to-one correspondence between vertices of $\mathcal{N}_{\mathfrak{R}}$ and Mori fibre spaces X_i/S_i, and between edges of $\mathcal{N}_{\mathfrak{R}}$ and elementary Sarkisov links between the X_i/S_i *dominated* by Z. Given a face $\mathcal{A} \subseteq \partial^+\mathcal{C}$ in the decomposition, one can construct a *residual cone* $\operatorname{resi}_{\mathcal{A}} \mathcal{C}$ of dimension $d = \dim \mathcal{C} - \dim \mathcal{A}$. There is an induced geography of ample models on $\operatorname{resi}_{\mathcal{A}} \mathcal{C}$, and an induced complex $\operatorname{resi}_{\mathcal{A}} \mathcal{N}_{\mathfrak{R}}$. If $\varphi : Z \longrightarrow W$ is the ample model of divisors $D \in \mathcal{A}$, the geography encodes the birational geometry of Z over W, and $\operatorname{resi}_{\mathcal{A}} \mathcal{N}_{\mathfrak{R}}$ the Sarkisov program among results of the MMP over W.

In § 4, I show that relations in the Sarkisov program correspond to edge-loops on $\mathcal{N}_{\mathfrak{R}}$, i.e. elements of $\pi_1(\mathcal{N}_{\mathfrak{R}})$, or, equivalently, of $\pi_1(\partial^+\mathcal{C}\backslash\{0\})$. Theorem 1.3 is proved by constructing a cover of $\mathcal{N}_{\mathfrak{R}}$ by residual complexes $\operatorname{resi}_{\mathcal{A}} \mathcal{N}_{\mathfrak{R}}$.

2. Preliminary results

I consider varieties defined over $\mathbb{C}$, and denote by $\mathbb{R}_+$ and $\mathbb{Q}_+$ the sets of nonnegative real and nonnegative rational numbers.

2.1 Convex geometry and polyhedral complexes

The topological closure of $\mathcal{S} \subset \mathbb{R}^N$ is denoted by $\overline{\mathcal{S}}$, and the boundary of a closed $\mathcal{C} \subset \mathbb{R}^N$ by $\partial\mathcal{C}$.

A *convex polyhedron* in $\mathbb{R}^N$ is defined by a finite number of (strict or not) linear inequalities in $\mathbb{R}^N$. A *rational polytope* in $\mathbb{R}^N$ is a compact set which is the convex hull of finitely many rational points in $\mathbb{R}^N$. A *rational polyhedral cone* in $\mathbb{R}^N$ is a convex cone spanned by finitely many rational vectors. The dimension of a cone in $\mathbb{R}^N$ is the dimension of the minimal $\mathbb{R}$-vector space containing it.

I recall some standard properties of convex sets which will be used repeatedly (see [Grü67] for references). Let $K \subset \mathbb{R}^N$ be a closed convex set. There is a unique subspace $L \subset \mathbb{R}^N$ of maximal dimension such that K contains a translate of L. If $L^* \subset \mathbb{R}^N$ is a subspace complementary to L, then $K = L + (L^* \cap K)$ and $L^* \cap K$ contains no line.

If $K \subset \mathbb{R}^N$ is a closed convex subset that contains no line, there is a hyperplane H such that $H \cap K$ is compact and has dimension $\dim K - 1$. In particular, if $K \subset \mathbb{R}^N$ is a closed cone with apex x_0, $K = \operatorname{Cone}_{x_0}(H \cap K)$.

Definition 2.1. A *polyhedral complex* $\mathcal{C}$ in $\mathbb{R}^N$ is a small category whose objects are convex polyhedra in $\mathbb{R}^N$, whose morphisms are isometric morphisms and that satisfies the following.

(i) If $c_1 \in \operatorname{Ob} \mathcal{C}$ and c_2 is a face of c_1, then $c_2 \in \operatorname{Ob} \mathcal{C}$, and the inclusion $i : c_2 \longrightarrow c_1$ is a morphism of $\mathcal{C}$.

(ii) If $c_1, c_2 \in \operatorname{Ob} \mathcal{C}$, there is at most one morphism $f \in \operatorname{Mor} \mathcal{C}$ such that $f(c_1) \subset c_2$.

The *faces* of $\mathcal{C}$ are the elements of $\operatorname{Ob} \mathcal{C}$. A face c of $\mathcal{C}$ is a *facet* if $f(c) = c'$ for all $\{f : c \longrightarrow c'\} \in \operatorname{Mor} \mathcal{C}$. The complex $\mathcal{C}$ is *pure of dimension* n if all its facets have dimension n. The *underlying space* of $\mathcal{C}$ is the topological space $|\mathcal{C}| = \coprod_{c \in \operatorname{Ob} \mathcal{C}} c/\sim$ where $c \sim f(c)$ for all $f \in \operatorname{Mor} \mathcal{C}$ and $c \in \operatorname{Ob} \mathcal{C}$.

Definition 2.2. Let $\mathcal{C}$ be a pure polyhedral complex of dimension n. The *Nerve of* $\mathcal{C}$ is the simplicial complex $\operatorname{Ner}(\mathcal{C})$ defined by the following conditions.

(i) Vertices of $\operatorname{Ner} \mathcal{C}$ are dual to facets of $\mathcal{C}$; if c is a facet of $\mathcal{C}$, c^* denotes the corresponding vertex of $\operatorname{Ner} \mathcal{C}$.

(ii) Vertices $c_0^*, \dots, c_k^*$ span a k-simplex if there is a face c of $\mathcal{C}$ of dimension $n-k$ and a collection of morphisms $\{f_i : c \longrightarrow c_i\} \in \operatorname{Mor} \mathcal{C}$. The k-simplex $\sigma = [c_0^*, \dots, c_k^*]$ is *dual* to c.

Remark 2.3. For $k < n$, a k-dimensional face c of $\mathcal{C}$ needs not have a dual k-simplex; when it does, there may be several dual simplices. When $n = 1$, $\mathcal{C}$ is simplicial and dual to $\operatorname{Ner} \mathcal{C}$.

Definition 2.4. Let K be a simplicial complex and $K^{(m)}$ its m-skeleton.

An *edge path in* K is a finite sequence $(v_0, v_1, \dots, v_n)$ of vertices of K such that $\{v_i, v_{i+1}\}$ spans a simplex for each i. An *edge loop* is an edge path with $v_0 = v_n$; the product of $(v_0, \dots, v_n)$ and $(v_n, \dots, v_m)$ is $(v_0, \dots, v_m)$.

Edge paths are *equivalent* if they are related by a finite sequence of *elementary equivalences* given by $(\dots, v_i, v_{i+1}, v_{i+2}, \dots) \sim (\dots, v_i, v_{i+2}, \dots)$ when $\{v_i, v_{i+1}, v_{i+2}\}$ spans a simplex of K.

The *edge path group* $E(K, v)$ is the group of equivalence classes of edge loops based at v; if $|K|$ is a geometric realisation of K, $E(K, v) \simeq \pi_1(|K|, v)$.

Lemma 2.5. *If $\mathcal{C}$ is a finite pure polyhedral complex of dimension n, $|\mathcal{C}|$ and $|\operatorname{Ner} \mathcal{C}|$ are homotopy equivalent.*

Proof. This is [KK11, Lemma 10]. □

2.2 Models of effective divisors and the classical MMP

Let Z be a normal projective variety and $\mathbf{R} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. The group of $\mathbf{R}$-Cartier $\mathbf{R}$-divisors is denoted by $\operatorname{Div}_{\mathbf{R}}(Z)$, and $\sim_{\mathbf{R}}$ and $\equiv$ denote $\mathbf{R}$-linear and numerical equivalence of $\mathbf{R}$-divisors. If $Z \longrightarrow X$ is a morphism to a normal projective variety, numerical equivalence over X is denoted by $\equiv_X$. Let $\operatorname{Pic}(Z)_{\mathbf{R}} = \operatorname{Div}_{\mathbf{R}}(Z)/\sim_{\mathbf{R}}$ and $N^1(Z)_{\mathbf{R}} = \operatorname{Div}_{\mathbf{R}}(Z)/\equiv$. Denote by $\rho(Z) = \dim N^1(Z)_{\mathbb{Q}}$

the *Picard rank* of Z. If $D \in \mathrm{Div}_{\mathbf{R}}(Z)$, then $[D] \in N^1(Z)_{\mathbf{R}}$ is the image of D under the natural map $\mathrm{Div}_{\mathbf{R}}(Z) \to N^1(Z)_{\mathbf{R}}$.

The ample, big, nef, effective, and pseudoeffective cones in $N^1(Z)_{\mathbb{R}}$ are denoted by $\mathrm{Amp}(X)$, $\mathrm{Big}(Z)$, $\mathrm{Nef}(Z)$, $\mathrm{Eff}(Z)$, and $\overline{\mathrm{Eff}}(Z)$. If $D \in \mathrm{Eff}(Z)$, $\mathbf{B}(D)$ is the stable base locus of D (see [Laz04] for definitions).

A *pair* (Z, Δ) consists of a normal projective variety Z and an effective $\mathbb{R}$-divisor Δ on Z such that $K_Z + \Delta$ is $\mathbb{R}$-Cartier; $K_Z + \Delta$ is an *adjoint divisor*. I say that (Z, Δ) is $\mathbb{Q}$-factorial if Z is; I often assume that $K_Z + \Delta$ is $\mathbb{Q}$-Cartier. The pair (Z, Δ) has klt (respectively *log canonical*) singularities if, for every log resolution $f : W \longrightarrow Z$, the coefficients of the divisor $K_W - f^*(K_Z + \Delta)$ are all greater than -1 (respectively greater than or equal to -1). A klt pair (Z, Δ) is *terminal* if, in addition, all f-exceptional divisors appear in $K_W - f^*(K_Z + \Delta)$ with positive coefficients.

I recall the definitions of models of divisors introduced in [BCHM10], and show that the results of the classical (log)-MMP are models of suitable effective divisors.

Definition 2.6. Let Z be a normal projective variety and $D \in \mathrm{Div}_{\mathbb{Q}}(Z)$. Let $f : Z \dashrightarrow X$ be a birational contraction such that $D' = f_*D$ is $\mathbb{Q}$-Cartier.

(i) The map f is *D-nonpositive* if, for a resolution $(p, q) : W \to Z \times X$,
$$p^*D = q^*D' + E,$$
where $E \geqslant 0$ is q-exceptional. When $\mathrm{Supp}\, E$ contains the strict transform of all f-exceptional divisors, f is *D-negative*.

(ii) When D is effective, f is a *semiample model* of D if f is D-nonpositive, X is normal and projective and D' is semiample. If $\varphi : X \to S$ is the semiample fibration defined by D', the *ample model* of D is $\varphi \circ f : Z \dashrightarrow X \to S$.

Lemma 2.7. *Let Z be a $\mathbb{Q}$-factorial projective variety and D_1 and D_2 be numerically equivalent big $\mathbb{Q}$-divisors. Assume that $R(Z, D_1)$ and $R(Z, D_2)$ are finitely generated and denote by φ_i the map $Z \dashrightarrow \mathrm{Proj}\, R(Z, D_i)$.*

There is an isomorphism $\eta : \mathrm{Proj}\, R(Z, D_1) \to \mathrm{Proj}\, R(Z, D_2)$ such that $\varphi_2 = \eta \circ \varphi_1$ and the stable base loci of D_1, D_2 satisfy $\mathbf{B}(D_1) = \mathbf{B}(D_2)$.

Proof. This is [KKL13, Lemma 3.11]. □

Definition 2.8. Let (Z, Δ) be a $\mathbb{Q}$-factorial klt pair with $K_Z + \Delta \in \mathrm{Div}_{\mathbb{Q}}(Z)$. A birational contraction $\varphi : Z \dashrightarrow X$ is *the result of a $(K_Z + \Delta)$-MMP* if X is projective, $\mathbb{Q}$-factorial and if the following hold.

(i) If $K_Z + \Delta$ is pseudoeffective, φ is a semiample model of $K_Z + \Delta$.

(ii) Otherwise, φ is a semiample model of $K_Z + \Theta$ for some klt pair (Z, Θ) such that $\Theta - \Delta$ is nef and $[K_Z + \Theta] \in \partial\, \overline{\mathrm{Eff}}_{\mathbb{Q}}(Z)$. The contraction $\varphi : Z \dashrightarrow X$ is called *an MMP with scaling by* $\mathbb{R}_+(\Theta - \Delta)$.

The MMP terminates for (Z, Δ) if a contraction $\varphi : Z \dashrightarrow X$ that is the result of a $(K_Z + \Delta)$-MMP exists. The *classical MMP* refers to the K_Z-MMP, where Z is terminal.

Remark 2.9. [BCHM10] shows that the MMP terminates for (Z, Δ) klt when Δ and $K_Z + \Delta$ are not both strictly pseudoeffective (i.e. not big).

The goal of the following lemma is to verify that Definition 2.8(ii) is consistent with the classical formulation of the MMP.

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LEMMA 2.10. *Let (Z, Δ) be a $\mathbb{Q}$-factorial klt pair and assume that $K_Z + \Delta$ is not pseudoeffective. Then, any result $\varphi : Z \dashrightarrow X$ of a $(K_Z + \Delta)$-MMP is $(K_Z + \Delta)$-nonpositive.*

Let Θ be a $\mathbb{Q}$-divisor with $\Theta - \Delta$ nef, $[K_Z + \Theta] \in \partial\, \overline{\mathrm{Eff}}_{\mathbb{Q}}(Z)$, and such that φ is a semiample model of $K_Z + \Theta$. If $f : X \to S$ is the Iitaka fibration associated to $\varphi_*(K_Z + \Theta)$, then $-\varphi_*(K_Z + \Delta)$ is f-nef.

Remark 2.11. If Z has terminal singularities, and if $\varphi : Z \dashrightarrow X$ is the result of a K_Z-MMP, then X is terminal because φ is K_Z-nonpositive.

Proof. Let $\varphi : Z \dashrightarrow X$ be the result of a $(K_Z + \Delta)$-MMP that is a semiample model for $K_Z + \Theta$, where Θ is as in Definition 2.8. Let $(p, q) : W \longrightarrow Z \times X$ be a resolution of φ. Then, $\varphi_*\Theta$ and $\varphi_*\Delta$ are $\mathbb{R}$-Cartier because X is $\mathbb{Q}$-factorial, and, by definition of φ,

$$p^*(K_Z + \Theta) = q^*(K_X + \varphi_*\Theta) + E,$$

where $E \geqslant 0$ is q-exceptional. Since φ is a contraction, p-exceptional divisors are q-exceptional, and there is a q-exceptional divisor E' such that

$$p^*(\Theta - \Delta) - q^*(\varphi_*(\Theta - \Delta)) = E'.$$

Since $\Theta - \Delta$ is nef, the negativity lemma [Kol92, Lemma 2.19] shows that $E' \leqslant 0$, and

$$p^*(K_Z + \Delta) = q^*(K_X + \varphi_*\Delta) + E - E',$$

so that φ is also $(K_Z + \Delta)$-nonpositive.

Let $C \subset X$ be an irreducible effective curve contracted by f, i.e. with $(K_X + \varphi_*\Theta) \cdot C = 0$. Such curves cover X because $K_X + \varphi_*\Theta$ is not big, and we may assume that $C \not\subseteq q(\mathrm{Exc}\, q)$. Then, $(K_X + \varphi_*\Delta) \cdot C = \varphi_*(\Delta - \Theta) \cdot C$. Let $\widetilde{C} \subset W$ be an effective curve that maps one-to-one to C. By the projection formula, $\varphi_*(\Delta - \Theta) \cdot C = q^*(\varphi_*(\Delta - \Theta)) \cdot \widetilde{C}$. Since $q^*(\varphi_*(\Delta - \Theta)) = -(p^*(\Theta - \Delta) + (E_q - E_p))$ and $(E_q - E_p) \cdot \widetilde{C} \geqslant 0$ by construction of C, we have $(K_X + \varphi_*\Delta) \cdot \widetilde{C} \leqslant 0$. This shows that $-\varphi_*(K_Z + \Delta)$ is f-nef. □

The following structures arise in the MMP of uniruled varieties.

DEFINITION 2.12. A *log-Mori fibre space* is a projective $\mathbb{Q}$-factorial klt pair (X, Δ) equipped with a morphism $f : X \to S$ such that $\dim S < \dim X$, $\rho(X) = \rho(S) + 1$ and $-(K_X + \Delta)$ is f-ample.

When $\Delta = 0$ and X is terminal, X/S is a *Mori fibre space* (Mfs).

By [BCHM10], if (Z, Δ) is a klt pair with $[K_Z + \Delta] \notin \overline{\mathrm{Eff}}(Z)$, there is a contraction $\varphi : Z \dashrightarrow X$ that is the result of a $(K_Z + \Delta)$-MMP such that the fibration $f : X \to S$ has $\rho(X/S) = 1$. Then, $(X, \varphi_*\Delta)$ is a log-Mori fibre space, because by Lemma 2.10, $-(K_X + \varphi_*\Delta)$ is f-nef, and $-(K_X + \varphi_*\Delta) \not\equiv_f 0$, so that $-(K_X + \varphi_*\Delta)$ is f-ample.

The *Sarkisov program* studies birational maps between (log-)Mori fibre spaces $(X, \Delta_X)/S$ and $(Y, \Delta_Y)/T$. There is a nonsingular uniruled variety Z (respectively a klt pair (Z, Δ) with $K_Z + \Delta$ nonpseudoeffective) such that $Z \dashrightarrow X/S$ and $Z \dashrightarrow Y/T$ are results of K_Z-MMPs (respectively $(K_Z + \Delta)$-MMPs). The simplest examples of birational maps between Mori fibre spaces arise in the *two-ray game*.

Example 2.13. Let X/S be a Mori fibre space and $Z \to X$ a divisorial contraction such that $-K_Z$ is nef and big over S. By the cone theorem, there is another Mori fibre space Y/T

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that is the result of a K_Z-MMP over S. Explicitly, $Z \dashrightarrow Y$ is either an isomorphism in codimension 1 followed by a divisorial contraction and $S \simeq T$, or $Z \dashrightarrow Y$ is an isomorphism in codimension 1 and $T \to S$ is a morphism with $\rho(T) = \rho(S) + 1$.

Definition 2.14 [Cor95]. An *elementary Sarkisov link* between log-Mori fibre spaces $(X, \Delta_X)/S$ and $(Y, \Delta_Y)/T$ is a diagram

$$\begin{array}{ccccc} \widetilde{X} & & \dashrightarrow & & \widetilde{Y} \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle g} \\ X & & & & Y \\ {\scriptstyle \varphi}\downarrow & & & & \downarrow{\scriptstyle \psi} \\ S & & & & T \\ & {\scriptstyle p}\searrow & & \swarrow{\scriptstyle q} & \\ & & R & & \end{array} \tag{2}$$

where $\widetilde{X} \overset{\Phi}{\dashrightarrow} \widetilde{Y}$ is an isomorphism in codimension 1, $(g \circ \Phi \circ f^{-1})_* \Delta_X = \Delta_Y$, $\rho(\widetilde{X}/R) = \rho(\widetilde{Y}/R) = 2$, and f, g, p and q are either isomorphisms or $(K + \Delta)$-nonpositive morphisms of relative Picard rank 1. One of f and p and one of g and q are isomorphisms. The link is of type I (respectively III) when p and g (respectively f and q) are isomorphisms and of type II (respectively IV) when p and q (respectively f and g) are isomorphisms.

A Sarkisov link is thus always induced by a two-ray configuration as in Example 2.13: it is the birational map between two Mori fibre spaces that are end products of the $K_{\widetilde{X}}$-MMP over $W = S, T$ or R with $\rho(\widetilde{X}/W) = 2$ and $\dim W < \dim \widetilde{X}$.

2.3 Geographies of models on good regions

By the preceding section, the Sarkisov program is the study of birational maps between end products of the MMP on smooth uniruled varieties, or, equivalently, between models of strictly pseudoeffective adjoint divisors $K_Z + \Delta$, where Z is smooth and Δ nef. In this subsection, I recall the construction of some good regions of $\mathrm{Div}_{\mathbb{R}}(Z)$, for Z nonsingular, where every effective divisor D admits a $\mathbb{Q}$-factorial semiample model $\varphi_D : Z \dashrightarrow X_D$, and φ_D can be decomposed into elementary maps analogous to Mori's contractions of extremal rays. The birational maps between models X_D as D varies in the region can then be studied easily.

Set up 2.15. Let $\Delta_1, \dots, \Delta_r$ be big $\mathbb{Q}$-divisors on a projective $\mathbb{Q}$-factorial variety Z, and assume that (Z, Δ_i) is klt for all i. The *adjoint ring* $\mathfrak{R} = R(Z; K_Z + \Delta_1, \dots, K_Z + \Delta_r)$ is

$$\mathfrak{R} = \bigoplus_{(n_1, \dots, n_r) \in \mathbb{N}^r} H^0(Z, n_1(K_Z + \Delta_1) + \cdots + n_r(K_Z + \Delta_r)),$$

where, for each $D \in \mathrm{Div}_{\mathbb{R}}(Z)$, $H^0(Z, D) = \{f \in k(Z) \mid \operatorname{div} f + D \geqslant 0\}$. By [BCHM10, Corollary 1.1.5], the *support of* $\mathfrak{R}$

$$\mathcal{C}_{\mathfrak{R}} = \{D \in \sum \mathbb{R}_+(K_Z + \Delta_i) \mid H^0(Z, D) \neq \{0\}\} \subseteq \mathrm{Div}_{\mathbb{R}}(Z)$$

is a rational polyhedral cone (see [KKL13, Theorem 3.2]).

As is shown in [KKL13], the convex geometry of $\mathcal{C}_{\mathfrak{R}}$ determines a *geography* of ample models of divisors $K + \Delta$ as they vary in $\mathcal{C}_{\mathfrak{R}}$. Since $\mathcal{C}_{\mathfrak{R}}$ is spanned by klt adjoint divisors, an additional

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condition on the dimension of $\mathcal{C}_{\mathfrak{R}}$ ensures that these ample models are $\mathbb{Q}$-factorial, so that they are the results of $(K_Z+\Delta)$-MMPs for $K_Z+\Delta\in\mathcal{C}_{\mathfrak{R}}$. The results relevant to this paper are recalled in the following theorem and proposition.

THEOREM 2.16. *Let Z, $\Delta_1,\dots,\Delta_r$ and $\mathfrak{R}$ be as in Setup 2.15. Assume that there is a big divisor $D\in\mathcal{C}_{\mathfrak{R}}$. Then, there is a finite decomposition*

$$\mathcal{C}_{\mathfrak{R}}=\coprod\mathcal{A}_i$$

with the following properties.

(i) *Each $\overline{\mathcal{A}_i}$ is a rational polyhedral cone.*

(ii) *There is a rational map $\varphi_i: X \dashrightarrow X_i$, with X_i is normal and projective, such that φ_i is the ample model of all $D\in\mathcal{A}_i\cap\operatorname{Div}_{\mathbb{Q}}(Z)$.*

(iii) *If $\mathcal{A}_j\subseteq\overline{\mathcal{A}_i}$, there is a morphism $\varphi_{ij}: X_i\to X_j$ with $\varphi_i\simeq\varphi_{ij}\circ\varphi_j$.*

(iv) *If $\mathcal{A}_i$ contains a big divisor, φ_i is a semiample model of all $D\in\overline{\mathcal{A}_i}\cap\operatorname{Div}_{\mathbb{Q}}(Z)$ and X_i has rational singularities.*

Proof. See [KKL13, Theorems 4.2, 4.5]. □

Remark 2.17. I recall the properties of the decomposition of $\mathcal{C}_{\mathfrak{R}}$ used in what follows. Since $\mathfrak{R}$ is finitely generated, [Ein+06, Theorem 4.1] implies the following.

(i) The cone $\mathcal{C}_{\mathfrak{R}}$ is closed and rational polyhedral.

(ii) There is a positive integer d and a resolution $f:\widetilde{Z}\longrightarrow Z$ such that $\operatorname{Mob} f^*(dD)$ is basepoint free for every $D\in\mathcal{C}_{\mathfrak{R}}\cap\operatorname{Div}_{\mathbb{Z}}(Z)$, and $\operatorname{Mob} f^*(kdD)=k\operatorname{Mob} f^*(dD)$ for every $k\in\mathbb{N}$.

(iii) There is a finite rational polyhedral subdivision $\mathcal{C}_{\mathfrak{R}}=\bigcup\mathcal{C}_i$ into cones with disjoint interiors such that the restriction of $D\mapsto\operatorname{Fix}|f^*dD|$ to each $\mathcal{C}_i$ is linear.

The cones $\mathcal{A}_i$ in Theorem 2.16 are the relative interiors of all cones that are intersections of the $\mathcal{C}_i$ and their proper faces. For each i, φ_i is induced by the Iitaka fibration $\psi_i:\widetilde{Z}\to\operatorname{Proj} R(\widetilde{Z},\operatorname{Mob} f^*dD)$ for any $D\in\mathcal{A}_i\cap\operatorname{Div}_{\mathbb{Z}}(Z)$.

PROPOSITION 2.18. *Let Z and $\mathfrak{R}$ be as in Theorem 2.16, and denote by $\pi:\operatorname{Div}_{\mathbb{R}}(Z)\longrightarrow N^1(Z)_{\mathbb{R}}$ the natural map.*

(i) *Let $L\subseteq\operatorname{Div}_{\mathbb{R}}(Z)$ be the maximal subspace with $A+L\subseteq\mathcal{C}_{\mathfrak{R}}$ for some $\mathbb{Q}$-divisor A, and let $L^*\subseteq\operatorname{Div}_{\mathbb{R}}(Z)$ be a complementary subspace. Denote by $p:\operatorname{Div}_{\mathbb{R}}(Z)\to L^*$ the projection. There is a finitely generated ring $\mathfrak{R}'$ with support $\mathcal{C}_{\mathfrak{R}'}=p(\mathcal{C}_{\mathfrak{R}})$. The cone $\mathcal{C}_{\mathfrak{R}'}$ contains no line and π restricts to an isomorphism on $\mathcal{C}_{\mathfrak{R}'}$. If $D\in\mathcal{C}_{\mathfrak{R}}$ is a big $\mathbb{Q}$-divisor, D and $p(D)$ have the same ample model.*

(ii) *Let $\mathcal{C}_{\mathfrak{R}}=\coprod\mathcal{A}_i$ be the coarsest decomposition of Theorem 2.16. Then, $\mathcal{A}_i=L+\mathcal{A}'_i$, where $\mathcal{A}'_i=p(\mathcal{A}_i)$. The decomposition $\mathcal{C}_{\mathfrak{R}'}=\coprod\mathcal{A}'_i$ satisfies Theorem 2.16(i)–(iv).*

Assume that $\dim\mathcal{C}_{\mathfrak{R}'}=\rho(Z)$. Up to refining $\{\mathcal{A}_i; i\in I\}$, we may assume that for all $i,j\in I$, every $\pi(\mathcal{A}_i)$ is contained in one of the subspaces bounded by H, where $H\subset N^1(Z)_{\mathbb{R}}$ is any supporting hyperplane of $\pi(\mathcal{A}_j)$.

(iii) *If $\dim\mathcal{A}'_i=\rho(Z)$, φ_i is the result of a $(K_Z+\Delta)$-MMP for all (Z,Δ) klt with $K_Z+\Delta\in\mathcal{A}_i$.*

(iv) *If $\mathcal{C}_{\mathfrak{R}'}$ contains an ample divisor, φ_i is the composition of a finite number of* elementary contractions, *where an elementary contraction is an isomorphism in codimension* 1 *or a morphism that contracts a single divisor.*

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Proof. Let $L \subset \mathrm{Div}_{\mathbb{R}}(Z)$ be such that $A + L \subseteq \mathcal{C}_{\mathfrak{R}}$ for some $\mathbb{Q}$-divisor A. If $D \in L \cap \mathrm{Div}_{\mathbb{Q}}(Z)$, then $D \equiv 0$, because both $A + nD$ and $A - nD$ are effective for all $n \gg 0$. By the results on convex sets recalled above, $L^* \cap \mathcal{C}_{\mathfrak{R}} = p(\mathcal{C}_{\mathfrak{R}})$ is a cone that contains no line, and hence the cone over a rational polytope $\mathcal{P}$. The vertices of $\mathcal{P}$ are of the form $\varepsilon_i(K_Z + \Delta_i') \in \mathcal{C}_{\mathfrak{R}}$ where $\varepsilon_i \in \mathbb{Q}_+$ and Δ_i' is a big $\mathbb{Q}$-divisor. Then, $p(\mathcal{C}_{\mathfrak{R}}) = \mathcal{C}_{\mathfrak{R}'}$, where $\mathfrak{R}' = R(Z; K_Z + \Delta_1', \dots, K_Z + \Delta_m')$ is finitely generated.

If $D \in \mathcal{C}_{\mathfrak{R}} \cap \mathrm{Div}_{\mathbb{Q}}(Z)$, since $p(D) \in \mathcal{C}_{\mathfrak{R}} \cap \mathrm{Div}_{\mathbb{Q}}(Z)$ as well, both $R(Z, D)$ and $R(Z, p(D))$ are finitely generated. When D is big, by Lemma 2.7, the ample models of D and $p(D)$ coincide up to isomorphism because $D \equiv p(D)$.

Let $f : \widetilde{Z} \to Z$ be a resolution as in Remark 2.17. Again by Lemma 2.7, if $D \in \mathcal{C}_{\mathfrak{R}}$ is a big $\mathbb{Q}$-divisor, $\mathbf{B}(D) = \mathbf{B}(p(D))$, and $\mathrm{Fix}\,|f^*dD| = \mathrm{Fix}\,|f^*(dpD)|$. The coarsest decomposition of $\mathcal{C}_{\mathfrak{R}}$ of [Ein+06, Theorem 4.1] depends only on regions of linearity of $D \mapsto \mathrm{Fix}\,|f^*dD|$ on $\{D \in \mathcal{C}_{\mathfrak{R}} \mid D \text{ is big}\}$, so that $\mathcal{C}_i = L + p(\mathcal{C}_i)$, and $\mathcal{A}_i = L + \mathcal{A}_i'$. The last two assertions are part of [KKL13, Theorems 4.4 and 5.4]. □

Notation 2.19. Assume that $\mathcal{C}_{\mathfrak{R}}$ is a rational polyhedral cone that contains no line. There is a hyperplane H such that $\mathcal{C}_{\mathfrak{R}}$ is the cone over $\mathcal{P}_{\mathfrak{R}} = H \cap \mathcal{C}_{\mathfrak{R}}$. The rational polytope $\mathcal{P}_{\mathfrak{R}}$ is a *base of* $\mathcal{C}_{\mathfrak{R}}$; any two bases of $\mathcal{C}_{\mathfrak{R}}$ are related by a smooth affine transformation. When $\mathcal{C}_{\mathfrak{R}}$ is as in Theorem 2.16, any $D \in \mathcal{P}_{\mathfrak{R}}$ is a positive rational multiple of a klt adjoint divisor and

$$\mathcal{P}_{\mathfrak{R}} = \coprod (\mathcal{A}_i \cap H) = \coprod \mathcal{Q}_i$$

where $\mathcal{Q}_i$ is the relative interior of a rational polytope. This decomposition has the properties listed in Theorem 2.16 and Proposition 2.18.

Definition–Lemma 2.20. Let $Z, \mathfrak{R}$ and $\{\mathcal{A}_i; i \in I\}$ of $\mathcal{C}_{\mathfrak{R}}$ be as in Proposition 2.18. Assume that $\mathcal{C}_{\mathfrak{R}} = \mathcal{C}_{\mathfrak{R}'}$ contains no line.

Denote by $\mathcal{C}_{\mathfrak{R}}$ the polyhedral complex with $\mathrm{Ob}\,\mathcal{C}_{\mathfrak{R}} = \{\overline{\mathcal{A}_i}; i \in I\}$ and morphisms the inclusions of faces. The complex $\mathrm{Ner}(\mathcal{C}_{\mathfrak{R}})$ is connected and has the following properties.

(i) There is a one-to-one correspondence between vertices $\overline{\mathcal{A}_i}^*$ of $\mathrm{Ner}\,\mathcal{C}_{\mathfrak{R}}$ and the results $\varphi_i : Z \dashrightarrow X_i$ of $(K_Z + \Delta)$-MMPs that are ample models of $K_Z + \Delta \in \mathcal{C}_{\mathfrak{R}}$.

(ii) Vertices $\overline{\mathcal{A}_i}^*, \overline{\mathcal{A}_j}^*$ of $\mathrm{Ner}\,\mathcal{C}_{\mathfrak{R}}$ form an edge precisely when $\varphi_j \circ \varphi_i^{-1}$ or $\varphi_i \circ \varphi_j^{-1}$ is an elementary contraction.

(iii) There is a one-to-one correspondence between edge-paths on $\mathcal{N}_{\mathfrak{R}}$ from v_i to v_j and decompositions of $X_i \dashrightarrow X_j$ into elementary contractions and inverses of elementary contractions dominated by Z.

Proof. This is a straightforward reformulation of Proposition 2.18. □

Remark 2.21. Let $\mathcal{C}_{\mathfrak{R}} = \bigcup_{1 \leqslant j \leqslant N} \mathcal{C}_j$ be the decomposition into cones of dimension $\rho(Z)$ associated to $\{\mathcal{A}_i; i \in I\}$. Edge paths on $\mathrm{Ner}\,\mathcal{C}_{\mathfrak{R}}$ are dual to piecewise linear paths between points in $\mathrm{int}(\mathcal{C}_j)$ that cross all $\partial\mathcal{C}_j$ transversally along codimension 1 faces.

3. Complexes and the Sarkisov program

Let $\{X_j/S_j, \Phi_{j,j'}\}$ be a collection of Mori fibre spaces and birational maps between them. As mentioned in § 2.2, there are divisors Θ_j on a nonsingular variety Z such that each X_j/S_j is the result of a K_Z-MMP and of a $(K_Z + \Theta_j)$-MMP, where $K_Z + \Theta_j$ is strictly pseudoeffective. In § 3.1, I show that $\{\Theta_j\}_j$ can be chosen so that all $K_Z + \Theta_j$ lie in a good region $\mathcal{C}_{\mathfrak{R}} \subset \mathrm{Div}_{\mathbb{R}}(Z)$

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in the sense of § 2.3. As above, I denote by $\mathcal{C}_{\mathfrak{R}}$ both the cone and the polyhedral complex supported on this cone and defined by a decomposition as in Proposition 2.18. Any $\Phi_{j,j'}$ can then be decomposed into a finite number of steps that are elementary contractions or inverses of elementary contractions dominated by Z. By Definition–Lemma 2.20, a decomposition is determined by considering suitable paths in $\mathcal{C}_{\mathfrak{R}}$. In general, some of the models (of divisors $D \in \mathcal{C}_{\mathfrak{R}}$) that appear in such a decomposition are not Mori fibre spaces. To achieve a decomposition of $\Phi_{j,j'}$ where all elementary steps are birational maps between Mori fibre spaces, I consider paths on $\mathcal{C}_{\mathfrak{R}}$ that lie on the part of $\partial\mathcal{C}_{\mathfrak{R}}$ consisting of strictly pseudoeffective divisors; this is done in § 3.2. Lastly, in § 3.3, given a cone $\overline{\mathcal{A}}$ in the decomposition of $\mathcal{C}_{\mathfrak{R}}$ with $\mathcal{A} \cap \operatorname{Big}(Z) = \emptyset$, I define a *residual ring* $\operatorname{resi}_{\mathcal{A}} \mathfrak{R}$. If $\varphi : Z \longrightarrow W$ is the ample model corresponding to $\mathcal{A}$, the support of $\operatorname{resi}_{\mathcal{A}} \mathfrak{R}$ inherits a geography that is a *geography of models over* W.

3.1 Cones of divisors for the Sarkisov program

Consider a collection $\{X_1/S_1, \dots, X_r/S_r; \Phi_{j,j'} : X_j \dashrightarrow X_{j'}\}$ of Mori fibre spaces and birational maps between them. Denote by $p_j : X_j \to S_j$ the fibration morphisms. The following construction is similar to the one in [HM13, § 4].

Proposition 3.1. *Let $f = (f_1, \dots, f_r) : Z \to X_1 \times \cdots \times X_r$ be a resolution of all maps $\Phi_{j,j'}$. There are ample $\mathbb{Q}$-divisors $\Delta_1, \dots, \Delta_n$ such that the rational polyhedral cone $\mathcal{C}_{\mathfrak{R}}$ associated to $\mathfrak{R} = R(Z; K_Z + \Delta_1, \dots, K_Z + \Delta_n)$ contains an ample divisor and has the following properties.*

Let V be the $\mathbb{R}$-vector space $\sum \mathbb{R}(K_Z + \Delta_i)$, $\partial^+\mathcal{C}_{\mathfrak{R}}$ the cone $\{D \in \mathcal{C}_{\mathfrak{R}} | D \notin \operatorname{Big}(Z)\}$ and let $\mathcal{C}_{\mathfrak{R}} = \coprod_{i \in I} \mathcal{A}_i$ be the coarsest decomposition in Theorem 2.16. Let $\varphi_i : Z \dashrightarrow Z_i$ be the ample model of any $D \in \mathcal{A}_i \cap \operatorname{Div}_{\mathbb{Q}}(Z)$. Then the following hold.

- (i) *The cone $\mathcal{C}_{\mathfrak{R}}$ has dimension $\rho(Z)$ and contains no line.*
- (ii) *There are indices $j_0, j_1 \in I$ such that $f_j \simeq \varphi_{j_0}$ and $p_j \circ f_j \simeq \varphi_{j_1}$. Moreover, $\mathcal{A}_{j_1} \subset \overline{\mathcal{A}_{j_0}}$ and* $\dim \mathcal{A}_{j_0} = \rho(Z) = \dim \mathcal{A}_{j_1} + 1$.
- (iii) *For every $\Phi_{j,j'}$ in the collection, both φ_{j_0} and $\varphi_{j'_0}$ are results of log-MMPs for (Z, Δ) klt with $K_Z + \Delta \in V$. Further, φ_{j_0} and $\varphi_{j'_0}$ are MMPs with scaling by suitable ample divisors $A, A' \in V$.*
- (iv) *$\partial^+\mathcal{C}_{\mathfrak{R}}$ is purely $(\rho(Z) - 1)$-dimensional.*
- (v) *If $\partial^+\mathcal{C}_{\mathfrak{R}} \subseteq \bigcup_{j=1}^r \overline{\mathcal{A}_{j_1}}$, any result of a log-MMP for (Z, Δ) klt with $K_Z + \Delta \in V$ nonpseudoeffective that is an MMP with scaling by an ample divisor $A \in V$ is one of the X_j/S_j in the collection.*

Proof. Observe that it is enough to construct a finitely generated ring $\mathfrak{R} = R(Z; K_Z + \Delta_1, \dots, K_Z + \Delta_n)$ such that assertions (ii)–(v) hold and such that the numerical classes of elements of $\mathcal{C}_{\mathfrak{R}}$ span $N^1(Z)_{\mathbb{R}}$. Indeed, if $L \subset V$ is the maximal subspace with $\mathcal{C}_{\mathfrak{R}} + L \subseteq \mathcal{C}_{\mathfrak{R}}$, then $p(\mathcal{C}_{\mathfrak{R}}) = \mathcal{C}_{\mathfrak{R}'}$ constructed as in Proposition 2.18 satisfies assertions (i)–(v).

Step 1. I first construct a ring $\mathfrak{R}$ satisfying assertion (ii) and such that the classes of divisors in $\mathcal{C}_{\mathfrak{R}}$ span $N^1(Z)_{\mathbb{R}}$.

Let $f = (f_1, \dots, f_r) : Z \to X_1 \times \cdots \times X_r$ be a common resolution that resolves all maps $\Phi_{j,j'}$, and let $\{E_k\}$ be the collection of f-exceptional divisors. Since X_j is terminal,

$$K_Z = f_j^* K_{X_j} + G_j, \tag{3}$$

where $G_j > 0$ is a $\mathbb{Q}$-divisor whose support contains all f_j-exceptional divisors. Fix a rational number $\lambda > 1$. For each j, let $\{A_j^l\}$ be ample $\mathbb{Q}$-divisors on S_j whose numerical classes span $N^1(S_j)_{\mathbb{R}}$ and such that $-(\lambda-1)K_{X_j}+p_j^*A_j^l$, $-\lambda K_{X_j}+p_j^*A_j^l$, and $-K_{X_j}+p_j^*A_j^l$ are ample. Denote the reduced sum of f_j-exceptional divisors by $E^j=\sum E_k$, and by $E_k^j=E^j+E_k$, where E_k ranges over f_j-exceptional divisors. Let $0<\delta\ll 1$ be a rational number such that $G_j-\delta E_j^k>0$ and its support contains all f_j-exceptional divisors. Define a collection of divisors

$$\Theta_j^{k,l}=f_j^*\{-\lambda K_{X_j}+p_j^*A_j^l\}-\delta E_j^k \quad \text{and} \quad D_j^{k,l}=f_j^*\{-K_{X_j}+p_j^*A_j^l\}-\delta E_j^k;$$

we may assume that δ is chosen so that $\Theta_j^{k,l}$, and $D_j^{k,l}$ are ample. Then,

$$K_Z+\Theta_j^{k,l}=f_j^*\{-(\lambda-1)K_{X_j}+p_j^*A_j^l\}+G_j-\delta E_k^j$$

is big, and f_j is $K_Z+\Theta_j^{k,l}$-negative, and is the ample model of $K_Z+\Theta_j^{k,l}$. Similarly, since

$$K_Z+D_j^{k,l}=f_j^*\{p_j^*A_j^l\}+G_j-\delta E_k^j,$$

f_j is a semiample model for $K_Z+D_j^{k,l}$, and $p_j\circ f_j$ is the ample model of $K_Z+D_j^{k,l}$. Since $D_j^{k,l}$ is ample and $K_Z+D_j^{k,l}$ is not big, K_Z is not pseudoeffective, and $Z\dashrightarrow X_j/S_j$ is the result of a K_Z-MMP with scaling by any $D_j^{k,l}$.

Let Δ_i be the divisors in $\{\Theta_j^{k,l},D_j^{k,l}\}_{j,k,l}$. If $\sum\mathbb{R}_+(K_Z+\Delta_i)$ does not contain an ample divisor, add to the collection $\{\Delta_i\}$ a very general ample divisor Δ such that $K_Z+\Delta$ is ample.

Since each Δ_i is ample, up to replacing Δ_i with $\varepsilon_i H_i\sim_{\mathbb{Q}}\Delta_i$, where $\varepsilon_i\in\mathbb{Q}$, $0<\varepsilon_i\ll 1$ and H_i is a very ample $\mathbb{Q}$-divisor, we may assume that the pairs (Z,Δ_i) are klt. The ring $\mathfrak{R}=R(Z;K_Z+\Delta_1,\ldots,K_Z+\Delta_n)$ is finitely generated by [CL12, Theorem 3.2]. Since $\mathcal{C}_{\mathfrak{R}}$ contains an ample divisor, there is a coarsest decomposition $\mathcal{C}_{\mathfrak{R}}=\coprod_{i\in I}\mathcal{A}_i$ as in Theorem 2.16. By uniqueness of ample models, for each j, there are indices $j_0,j_1\in I$ with $f_j=\varphi_{j_0}$ and $K_Z+\Theta_j^{k,l}\in\mathcal{A}_{j_0}$, and $p_j\circ f_j=\varphi_{j_1}$ and $K_Z+D_j^{k,l}\in\mathcal{A}_{j_1}$ for all k,l. The divisors $K_Z+D_j^{k,l}$ are not big, and $\mathcal{A}_{j_1}\subseteq\partial^+\mathcal{C}_{\mathfrak{R}}$, and $\varphi_{j_1}=p_j\circ\varphi_{j_0}$. Also, since λ can be chosen arbitrarily close to 1 in the definition of divisors of the form of $\Theta_j^{k,l}$, $\mathcal{A}_{j_1}\cap\overline{\mathcal{A}_{j_0}}\neq\emptyset$, and by definition of the decomposition (4), $\mathcal{A}_{j_1}\subsetneq\overline{\mathcal{A}_{j_0}}$. The only thing that is left to prove is that the numerical classes of divisors in $\mathcal{A}_{j_0}$ span $N^1(Z)_{\mathbb{R}}$.

The classes $[K_Z+\Theta_j^{k,l}]$ (respectively $[K_Z+D_j^{k,l}]$) span $N^1(Z)_{\mathbb{R}}$ (respectively a subspace of codimension 1) because f_j is a resolution and p_j a Mori fibration, so that

$$N^1(Z)_{\mathbb{R}}=(p_j\circ f_j)^*N^1(S_j)_{\mathbb{R}}\oplus\mathbb{R}[f_j^*(-K_{X_j})]\oplus\bigoplus\mathbb{R}[E_k]$$

where the sum runs over f_j-exceptional divisors, and $\bigoplus\mathbb{R}[E_k]=\bigoplus\mathbb{R}[E_k^j]$.

Step 2. For assertion (iii), let $\Phi_{j,j'}:X_j\dashrightarrow X_{j'}$ be a birational map in the collection. As in Step 1, denote by $j_0,j_1\in I$ (respectively j_0',j_1'), the indices such that f_j and $p_j\circ f_j$ (respectively $f_{j'}$ and $p_{j'}\circ f_{j'}$) are the ample models associated to $\mathcal{A}_{j_0},\mathcal{A}_{j_1}$ (respectively to $\mathcal{A}_{j_0},\mathcal{A}_{j_1'}$). As noted in Step 1, K_Z is not pseudoeffective, and there are ample divisors $D_j,D_{j'}$ with $K_Z+D_j\in\mathcal{A}_{j_1}$ and $K_Z+D_j'\in\mathcal{A}_{j_1'}$. Then, for $0\leqslant\varepsilon\ll 1$, if $\Delta=\varepsilon(D_j+D_{j'})$, (Z,Δ) is klt and $K_Z+\Delta\notin\mathcal{C}_{\mathfrak{R}}$ and f_j (respectively $f_{j'}$) is an MMP with scaling of (Z,Δ) by $\mathbb{R}_+D_j$ (respectively $\mathbb{R}_+D_{j'}$).

Step 3. I now prove the statements on $\partial^+\mathcal{C}_{\mathfrak{R}}$. Using the notation set in Notation 2.19, let $\mathcal{P}_{\mathfrak{R}}$ be a base of $\mathcal{C}_{\mathfrak{R}}$. Since D is big if and only if λD is big for all $\lambda>0$, to prove assertion (iv), it is enough to ensure that $\partial^+\mathcal{P}_{\mathfrak{R}}=\{D\in\mathcal{P}_{\mathfrak{R}}\mid D \text{ is not big}\}$ is purely $(\rho(Z)-2)$-dimensional. Assume that this is not the case, and let $\partial^+\mathcal{P}_{\mathfrak{R}}=T\coprod T'$, where $\dim T=\rho(Z)-2>\dim T'$.

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Note that $T \neq \emptyset$ because for any X_j/S_j in the collection, if $\mathcal{A}_{j_1}$ is defined as above, $\mathcal{A}_{j_1} \cap \mathcal{P}_{\mathfrak{R}} \subseteq \partial^+\mathcal{P}_{\mathfrak{R}}$ is $(\rho(Z)-2)$-dimensional. Let $\mathcal{P}'_{\mathfrak{R}} \subsetneq \mathcal{P}_{\mathfrak{R}}$ be a polytope of the same dimension as $\mathcal{P}_{\mathfrak{R}}$ that contains all the vertices of $\mathcal{P}_{\mathfrak{R}}$ in T but none of those in T'. The vertices of $\mathcal{P}'_{\mathfrak{R}}$ are of the form $\varepsilon_1(K_Z+\Delta'_1), \dots, \varepsilon_m(K_Z+\Delta'_m)$ for $\varepsilon_i \in \mathbb{Q}$, and (Z,Δ'_i) klt with Δ'_i big. Then, $\mathfrak{R}' = R(Z, K_Z+\Delta'_1, \dots, K_Z+\Delta'_m)$ is finitely generated, and $\mathcal{C}'_{\mathfrak{R}}$ is the cone spanned by $\mathcal{P}'_{\mathfrak{R}}$. Further, $\mathfrak{R}'$ satisfies assertions (i)–(iii), because for all X_j/S_j, $\mathcal{A}_{j_0} \cap \mathcal{P}'_{\mathfrak{R}}$ is $(\rho(Z)-1)$-dimensional and $\mathcal{A}_{j_1} \cap \mathcal{P}'_{\mathfrak{R}}$ is $(\rho(Z)-2)$-dimensional. After replacing $\mathfrak{R}$ by $\mathfrak{R}'$, assertion (iv) holds.

For assertion (v), let (Z,Δ) be a klt pair with $K_Z+\Delta \in V$ not pseudoeffective. Let Θ be such that $K_Z+\Theta \in \mathcal{C}_{\mathfrak{R}}$ is not big, (Z,Θ) is klt and $\Theta-\Delta \in V$ is ample. Let $\mathcal{A}_{i_0}$ be the cone of maximal dimension with $K_Z+\Theta \in \overline{\mathcal{A}_{i_0}}$. Since $K_Z+\Theta$ is not big, $K_Z+\Theta$ belongs to a codimension 1 face $\overline{\mathcal{A}_{i_1}}$ of $\overline{\mathcal{A}_{i_0}}$ with $K_Z+\Theta \in \overline{\mathcal{A}_{i_1}}$. In the notation of Theorem 2.16, $\varphi_{i_0,i_1}: Z_{i_0} \to Z_{i_1}$ is a Mori fibre space that is the result of a $(K_Z+\Delta)$-MMP. By Lemma 2.10, $-(K_{Z_{i_0}}+\varphi_{i_0}\Delta)$ is φ_{i_0,i_1}-nef, and since $\rho(Z_{i_0}/Z_{i_1})=1$, it is φ_{i_0,i_1}-ample because $\Theta-\Delta$ is ample and $(\varphi_{i_0})_*(\Theta-\Delta) \not\equiv 0$. If the $\overline{\mathcal{A}_{j_1}}$ cover $\partial^+\mathcal{C}_{\mathfrak{R}}$, $\mathcal{A}_{i_1} = \mathcal{A}_{j_1}$ for some j, and $X_j/S_j \simeq Z_{i_0}/Z_{i_1}$. □

As a by-product, the proof of Proposition 3.1 shows the next lemma.

LEMMA 3.2. *Let $\mathcal{A}_{i_0}, \mathcal{A}_{i_1}$ be cones in the decomposition of $\mathcal{C}_{\mathfrak{R}}$ such that $\mathcal{A}_{i_0}$ has maximal dimension, $\mathcal{A}_{i_1} \subset \overline{\mathcal{A}_{i_0}}$ has codimension 1 in $\mathcal{A}_{i_0}$ and contains no big divisor. Then $(Z_{i_0}, (\varphi_{i_0})_*\Delta)/Z_{i_1}$ is a log Mori fibre space, where (Z,Δ) is any klt pair with $\Delta = \Theta - H$ for H ample and $K_Z+\Theta \in \mathcal{A}_{i_1} \cap \mathrm{Div}_{\mathbb{Q}}(Z)$.*

Remark 3.3. In Proposition 3.1, K_Z is not effective and all Z_{i_0}/Z_{i_1} that are results of K_Z-MMPs with scaling by nef divisors Θ with $K_Z+\Theta \in V$ are Mori fibre spaces.

Remark 3.4. By construction, the movable part of any $D \in \sum \mathbb{R}_+(K_Z+\Delta_i)$ is basepoint free, and hence, in the notation of Theorem 2.16, f is an isomorphism and all φ_i are morphisms.

Remark 3.5. If $\mathcal{A}_{i_1} \cap \mathrm{Big}(Z) = \emptyset$ and $\mathcal{A}_{i_1} \subseteq \overline{\mathcal{A}_{i_0}}$ with $\dim \mathcal{A}_{i_0} = \rho(Z)$, then $\varphi_{i_0,i_1}: Z_{i_0} \to Z_{i_1}$ has $1 \leqslant \rho(Z_{i_0}/Z_{i_1}) \leqslant \dim \mathcal{A}_{i_0} - \dim \mathcal{A}_{i_1}$.

3.2 Polyhedral complexes

In this section, I define a polyhedral complex supported on $\partial^+\mathcal{C}_{\mathfrak{R}}$, where $\mathcal{C}_{\mathfrak{R}}$ is as in Proposition 3.1.

Notation 3.6. Let $\{X_1/S_1, \dots, X_r/S_r; \Phi_{j,j'}\}$ be a collection of Mori fibre spaces and birational maps between them and $\mathfrak{R} = R(Z; K_Z+\Delta_1, \dots, K_Z+\Delta_n)$ the ring constructed in Proposition 3.1. Write $V = \sum \mathbb{R}(K_Z+D_i)$. Fix the coarsest decomposition

$$\mathcal{C}_{\mathfrak{R}} = \coprod_{i\in I} \mathcal{A}_i \tag{4}$$

as in Theorem 2.16. As above, $\varphi_i: Z \dashrightarrow X_i$ is the ample model of all $D \in \mathcal{A}_i$. Let $\mathcal{P}_{\mathfrak{R}}$ be a base of $\mathcal{C}_{\mathfrak{R}}$ and $\mathcal{Q}_i = \mathcal{A}_i \cap \mathcal{P}_{\mathfrak{R}}$ the induced decomposition as in Notation 2.19.

Enlarging the collection if necessary, assume that X_j/S_j exhaust the possible results of $(K_Z+\Delta)$-MMPs with scaling by ample divisors $A \in V$ for noneffective $K_Z+\Delta \in V$. Let $\mathcal{B}^+(I) = \{i \in I | \mathcal{A}_i \cap \mathrm{Big}(Z) = \emptyset\}$, where I is the index set in (4), so that

$$\partial^+\mathcal{C}_{\mathfrak{R}} = \coprod_{i\in\mathcal{B}^+(I)} \mathcal{A}_i. \tag{5}$$

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By construction, $\partial^+\mathcal{C}_{\mathfrak{R}}$ is purely $(\rho(Z)-1)$-dimensional and is the cone over $\partial^+\mathcal{P}_{\mathfrak{R}} = \coprod_{i\in\mathcal{B}^+(I)}\mathcal{Q}_i$, where $\mathcal{Q}_i$ are defined in Notation 2.19.

Notation 3.7. Let Z and $\mathfrak{R}$ be as in Notation 3.6, and let $\mathcal{C}_{\mathfrak{R}}$ be the polyhedral complex of Definition–Lemma 2.20. Let $\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$ be the subcomplex of $\mathcal{C}_{\mathfrak{R}}$ with objects $\overline{\mathcal{A}_i}$ for $i\in\mathcal{B}^+(I)$; $\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$ is a pure polyhedral complex of dimension $\rho(Z)-1$. Similarly, the polyhedral complex $\mathcal{B}^+(\mathcal{Q}_{\mathfrak{R}})$ with objects $\overline{\mathcal{Q}_i}$ for $i\in\mathcal{B}^+(I)$ is pure of dimension $\rho(Z)-2$. Denote by $\mathcal{N}_{\mathfrak{R}}$ the simplicial complex $\operatorname{Ner}\mathcal{B}^+(\mathcal{Q}_{\mathfrak{R}})$.

Remark 3.8. There is a natural identification between $\mathcal{N}_{\mathfrak{R}}$ and the $(\rho(Z)-2)$-skeleton of $\operatorname{Ner}\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$. Any $\rho(Z)$ distinct vertices of $\operatorname{Ner}\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$ span a $(\rho(Z)-1)$-simplex dual to the cone $\{0\}$.

Remark 3.9. By definition, the geometric realisations of the complexes defined above are $|\mathcal{C}_{\mathfrak{R}}| = \mathcal{C}_{\mathfrak{R}}$, $|\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})| = \partial^+\mathcal{C}_{\mathfrak{R}}$ and $|\mathcal{B}^+(\mathcal{Q}_{\mathfrak{R}})| = \partial^+\mathcal{P}_{\mathfrak{R}}$.

In what follows, I always assume that $\rho(Z)\geqslant 3$, as the Sarkisov program reduces to the two-ray game otherwise. In particular, the vertices and edges of $\mathcal{N}_{\mathfrak{R}}$ are precisely the vertices and edges of $\operatorname{Ner}\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$. I use both constructions of $\mathcal{N}_{\mathfrak{R}}$. The next proposition states some easy properties of $\mathcal{C}_{\mathfrak{R}}$, $\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$, and $\mathcal{N}_{\mathfrak{R}}$.

Proposition 3.10. *If $\overline{\mathcal{A}_i}, \overline{\mathcal{A}_j}$ are distinct facets of $\mathcal{C}_{\mathfrak{R}}$, then $\varphi_i \not\simeq \varphi_j$. If $\overline{\mathcal{A}}$ is a facet of $\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$, there is a unique facet $\overline{\mathcal{A}'}$ of $\mathcal{C}_{\mathfrak{R}}$ with $\mathcal{A}\subseteq\overline{\mathcal{A}'}$.*

There is a one-to-one correspondence between the vertices of $\mathcal{N}_{\mathfrak{R}}$ and the Mori fibre spaces produced by MMPs with scaling by nef divisors $H\in V$.

Proof. Let $\overline{\mathcal{A}_i}$ and $\overline{\mathcal{A}_j}$ be distinct facets of $\mathcal{C}_{\mathfrak{R}}$. If φ_i is the ample model for all $D\in\mathcal{A}_i\cup\mathcal{A}_j$, then $D\mapsto \operatorname{Fix}|dD|$ is linear on $\overline{\mathcal{A}_i\cup\mathcal{A}_j}$. This is impossible because the decomposition (4) is the coarsest such.

The second assertion follows immediately from the definition of the decomposition (4). If $\mathcal{A}_i, \mathcal{A}_j$ are facets of $\mathcal{C}_{\mathfrak{R}}$ with $\overline{\mathcal{A}_{i,j}} = \overline{\mathcal{A}_i}\cap\overline{\mathcal{A}_j}$ and $\dim\mathcal{A}_{i,j} = \rho(Z)-1$, any $D\in\mathcal{A}_{i,j}$ can be written $D = D_i + D_j$ for $D_i\in\mathcal{A}_i$ and $D_j\in\mathcal{A}_j$. But then, D is big because D_i and D_j both are, and $\mathcal{A}_{i,j}\notin\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$. In particular, a facet of $\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$ is not the intersection of two facets of $\mathcal{C}_{\mathfrak{R}}$.

Lastly, let $v_i = (\overline{\mathcal{A}_{i_1}})^*$ be a vertex of $\mathcal{N}_{\mathfrak{R}}$, and $\overline{\mathcal{A}_{i_0}}$ the facet of $\mathcal{C}_{\mathfrak{R}}$ that contains $\mathcal{A}_{i_1}$. By Lemma 3.2 and Remark 3.3, Z_{i_0}/Z_{i_1} is a Mori fibre space that is the result of a K_Z-MMP by scaling in V, and all results of K_Z-MMPs with scaling in V are of this form. This finishes the proof. □

Remark 3.11. If $\overline{\mathcal{A}_i}, \overline{\mathcal{A}_j}$ are distinct facets of $\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$, the associated ample models φ_i and φ_j need not be distinct. When $\varphi_i\simeq\varphi_j$, φ_i has distinct factorisations through semiample models that are ample models for the (not necessarily distinct) facets $\mathcal{A}'_i, \mathcal{A}'_j$, compare with Corollary 3.18.

Example 3.12. Let S be the blow up of $\mathbb{P}^2$ in two points $P_1\neq P_2$. Let E_1, E_2 be the (-1)-curves lying over P_1, P_2 and L the proper transform of the line through P_1 and P_2. Figure 1 represents a base $\mathcal{P}_{\mathfrak{R}}$ and the decomposition induced by (4), where $\mathfrak{R} = R(S, K_S+A+E_1, K_S+A+E_2, K_S+A+L)$ for an ample divisor $A\sim_{\mathbb{Q}} -K_S$. Then, $\mathcal{C}_{\mathfrak{R}} = \overline{\operatorname{Eff}}(S) = \mathbb{R}_+[L]+\mathbb{R}_+[E_1]+\mathbb{R}_+[E_2]$. If $\mathcal{A}_i$ denotes the interior of the cones $\mathcal{C}_i$ in Figure 1, the ample models associated to $\mathcal{A}_0,\dots,\mathcal{A}_4$ are $\varphi_0\colon S\xrightarrow{\simeq} S$, $\varphi_1\colon S\to\mathbb{P}^1_a\times\mathbb{P}^1_b$, $\varphi_2\colon S\to\mathbb{F}_{1a}$, $\varphi_3\colon S\longrightarrow\mathbb{F}_{1b}$

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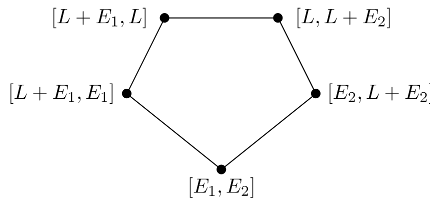
and $\varphi_4 : S \longrightarrow \mathbb{P}^2$. The complex $\mathcal{N}_{\mathfrak{R}}$ is the following graph with five vertices and five edges.



Definition 3.13. A birational map $\Phi : Z_i \dashrightarrow Z_j$ between the ample models associated to $\mathcal{A}_i, \mathcal{A}_j$ in (4) is *dominated by* Z if Z is a resolution of Φ, that is if $\Phi \simeq \varphi_j \circ \varphi_i^{-1}$. Equivalently, Φ is dominated by Z when $\Phi \circ \varphi_i$ is the ample model of some nonzero $\mathbb{Q}$-divisor $D \in \mathcal{A}_j$.

An elementary Sarkisov link $L_{i,j} : X_i/S_i \dashrightarrow X_j/S_j$ as in (2) is *dominated by* Z *in* V if X_i/S_i and X_j/S_j are the results of $(K_Z + \Delta)$-MMPs with scaling in V for (Z, Δ) klt and $K_Z + \Delta \in V$.

Remark 3.14. The map $\Phi : Z_i \dashrightarrow Z_j$ is dominated by Z when Φ is induced by the geography of ample models associated to the decomposition (4) of $\mathcal{C}_{\mathfrak{R}}$. When $\Phi = L_{i,j}$ is a Sarkisov link, this is equivalent to requiring that $X_i \dashrightarrow X_j$ is dominated by Z and that the induced map $Z \dashrightarrow T_{i,j}$ is the ample model for some nonzero $\mathbb{Q}$-divisor $D \in \mathcal{C}_{\mathfrak{R}}$.

3.3 Residual rings and complexes

In this section, I define subrings of $\mathfrak{R}$ associated to affine subspaces $H \subset V$, and show that, for suitable choices of H, the geography of ample model induced by (4) on the support of these subrings corresponds to running suitable relative Minimal Model Programs.

Definition 3.15. Let W be a subspace of $\mathrm{Div}_{\mathbb{R}}(Z)$ and D a nonzero $\mathbb{Q}$-divisor. The affine subspace $H = D + W$ of dimension $d \leqslant \rho(Z) - 1$ is in *general position* with respect to $\mathcal{C}_{\mathfrak{R}}$ if $H \cap \mathcal{C}_{\mathfrak{R}}$ is not contained in any union or intersection of cones $\mathcal{A}_i$ in (4).

Let $\mathcal{A}$ be a cone of dimension d in (5). An affine subspace H of dimension $\rho(Z) - 1 - d$ is in general position with respect to $\mathcal{C}_{\mathfrak{R}}$ and $\mathcal{A}$ if it is in general position with respect to $\mathcal{C}_{\mathfrak{R}}$ and if $H \cap \mathcal{A} \neq \emptyset$.

The following two lemmas are reformulations of results in [HM13].

Lemma 3.16 [HM13, Corollary 3.4]. *Let $D \in V$ be a nonzero adjoint $\mathbb{Q}$-divisor. For any $k \leqslant \rho(Z) - 1$ the subset $U \subset Gr(k, V)$ of k-dimensional affine subspaces $D + W \subset V$ in general position with respect to $\mathcal{C}_{\mathfrak{R}}$ is Zariski dense.*

Proof. This is an immediate consequence of the existence of the finite decomposition (4) into rational polyhedral cones. □

Lemma 3.17. *There is a one-to-one correspondence between edges of $\mathcal{N}_{\mathfrak{R}}$ and elementary Sarkisov links dominated by Z.*

Proof. I use the notation of the proof of Proposition 3.1. First, I show that an edge of $\mathcal{N}_{\mathfrak{R}}$ determines an elementary link dominated by Z. Let v_i, v_j be vertices of $\mathcal{N}_{\mathfrak{R}}$ that span a 1-simplex. Let $i_1, j_1 \in \mathcal{B}^+(I)$ be such that $\overline{\mathcal{A}_{i_1}} = v_i^*$ and $\overline{\mathcal{A}_{j_1}} = v_j^*$, and $i_0, j_0 \in I$ be the indices of the facets of $\mathcal{C}_{\mathfrak{R}}$ with $\mathcal{A}_{i_1} \subseteq \overline{\mathcal{A}_{i_0}}$ and $\mathcal{A}_{j_1} \subseteq \overline{\mathcal{A}_{j_0}}$. Denote by X_i/S_i (respectively X_j/S_j) the Mori fibre spaces Z_{i_0}/Z_{i_1} (respectively Z_{j_0}/Z_{j_1}).

Let $\overline{\mathcal{A}} = \overline{\mathcal{A}_k}$ be the object of $\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$ dual to the 1-simplex $\{v_i, v_j\}$ of $\mathcal{N}_{\mathfrak{R}}$. Then, $\overline{\mathcal{A}} = \overline{\mathcal{A}_{i_1}} \cap \overline{\mathcal{A}_{j_1}}$ and $\mathcal{A}$ has dimension $\rho(Z) - 2$. Let $\varphi : Z \longrightarrow T_{i,j}$ be the ample model associated to the face $\mathcal{A}$.

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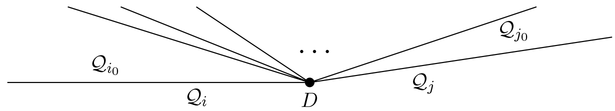


FIGURE 2. Intersection of $\mathcal{P}_{\mathfrak{R}}$ with a suitable affine plane H.

Then, by Theorem 2.16(iii), there are morphisms $\varphi_{i_0,k}$ and $\varphi_{j_0,k}$ that decompose as

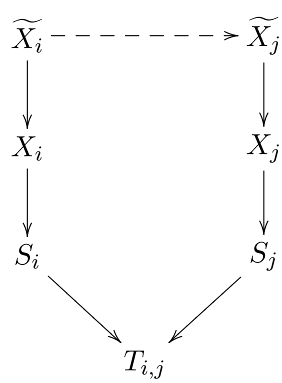
$$X_i \xrightarrow{\varphi_{i_0,i_1}} S_i \xrightarrow{\varphi_{i_1,k}} T_{i,j} \quad \text{and} \quad X_i \xrightarrow{\varphi_{j_0,j_1}} S_j \xrightarrow{\varphi_{j_1,k}} T_{i,j}$$

such that $\varphi \simeq \varphi_{i_0,k} \circ \varphi_{i_0} \simeq \varphi_{j_0,k} \circ \varphi_{j_0}$.

Fix $D \in \mathcal{A} \cap \mathrm{Div}_{\mathbb{Q}}(Z)$, and let $H = D + W$ be a two-dimensional affine subspace in general position with respect to $\mathcal{C}_{\mathfrak{R}}$ and $\mathcal{A}$. Then, $\mathcal{C}_{\mathfrak{R}} \cap H = \mathcal{P}_{\mathfrak{R}} \cap H$ for a suitable base $\mathcal{P}_{\mathfrak{R}}$, and $\mathcal{P}_{\mathfrak{R}} \cap H$ is as in Figure 2, where $\mathcal{Q}_i = \mathcal{A}_i \cap H$.

Since D is an exposed point of $\mathcal{P}_{\mathfrak{R}} \cap H$ and since the decomposition (4) is finite, there is a line L such that $(\mathcal{P}_{\mathfrak{R}} \cap H) \cap L$ contains no other vertices of $\mathcal{P}_{\mathfrak{R}} \cap H$, and the only two-dimensional polytopes $\overline{\mathcal{Q}_i}$ that L intersects contain D (e.g. a small translation of the supporting hyperplane of $\mathcal{P}_{\mathfrak{R}} \cap H$ at D). Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be the facets of $\mathcal{C}_{\mathfrak{R}}$ that have one-dimensional intersection with L, where $\mathcal{A}_i \cap H = \mathcal{Q}_i$. Then, $\mathcal{A} \subset \overline{\mathcal{A}_l}$ for $l = 1, \ldots, n$, and there is a morphism $Z_l \to T_{i,j}$ for all l, with $\rho(Z_l/T_{i,j}) \leqslant 2$ by Remark 3.5 because $\dim \mathcal{A} = \dim \mathcal{A}_l - 2$.

Let l_0 be such that $\rho(Z_{l_0})$ is maximal. Then, again by Proposition 2.18, both $Z_{l_0} \dashrightarrow X_i$ and $Z_{l_0} \dashrightarrow X_j$ are compositions of finitely many elementary contractions and by definition of l_0, these are birational contractions. Since $\rho(X_i)$ and $\rho(X_j)$ are both greater than or equal to $\rho(T_{i,j}) + 1$, there are indices $1 \leqslant i', j' \leqslant n$ and elementary contractions $Z_{i'} = \widetilde{X_i} \to X_i$ and $Z_{j'} = \widetilde{X_j} \to X_j$ with $Z_{l_0}, \widetilde{X_i}, \widetilde{X_j}$ isomorphic in codimension 1, that fit in the following diagram.



This shows that every edge of $\mathcal{N}_{\mathfrak{R}}$ defines a Sarkisov link dominated by Z.

Conversely, let $L_{i,j} : X_i/S_i \dashrightarrow X_j/S_j$ be an elementary Sarkisov link dominated by Z. Let v_i be the vertex of $\mathcal{N}_{\mathfrak{R}}$ associated to X_i/S_i and v_j that associated to X_j/S_j, and, as above, denote by $\overline{\mathcal{A}_{i_1}}, \overline{\mathcal{A}_{j_1}}$ the dual facets of $\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$. By Definition 3.13, $\overline{\mathcal{A}_{i_1}} \cap \overline{\mathcal{A}_{j_1}} \neq \emptyset$. But then, by definition of the decomposition (4), there is an index $k \in \mathcal{B}^+(I)$ with $\overline{\mathcal{A}_k} = \overline{\mathcal{A}_{i_1}} \cap \overline{\mathcal{A}_{j_1}}$, and $\dim \mathcal{A}_k = \rho(Z) - 2$. The vertices $\{v_i, v_j\}$ span a 1-simplex dual to $\overline{\mathcal{A}_k}$. $\square$

COROLLARY 3.18. *Let* $[v_i, v_j]$ *be a* 1*-simplex of* $\mathcal{N}_{\mathfrak{R}}$, $\overline{\mathcal{A}} \in \mathrm{Ob}\, \mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$ *its dual face and let* $\varphi : Z \longrightarrow T_{i,j}$ *be the ample model associated to* $\mathcal{A}$. *The elementary Sarkisov link associated to* $[v_i, v_j]$ *is of*

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type I (respectively type III) if $\varphi_i \simeq \varphi \not\simeq \varphi_j$ *(respectively* $\varphi_i \not\simeq \varphi \simeq \varphi_j$*),*
type II if $\varphi_i \simeq \varphi_j$*, and type IV if* $\varphi_i \not\simeq \varphi \not\simeq \varphi_j$*.*

Proof. This is a reformulation of the definition of the types of links. □

The next proposition is an extension of Lemma 3.17 to affine spaces of arbitrary dimension. Explicitly, if H is in general position with respect to $\mathcal{C}_{\mathfrak{R}}$, there is a subring $\mathfrak{R}_H$ whose support $\mathcal{C}_{\mathfrak{R},H}$ is the cone over $\mathcal{C}_{\mathfrak{R}} \cap H$. There is a natural geography of models associated to the convex geometry of $\mathcal{C}_{\mathfrak{R},H}$, which behaves like the geography of models of a divisorial ring on some Z' of Picard rank $\rho(Z') = \dim H + 1$.

Proposition 3.19. *Let H be an affine space of dimension $d \leqslant \rho(Z) - 1$ in general position with respect to $\mathcal{C}_{\mathfrak{R}}$. There is a finitely generated ring $\mathfrak{R}_H$ whose support $\mathcal{C}_{\mathfrak{R},H}$ has dimension $d+1$ and a decomposition*

$$\mathcal{C}_{\mathfrak{R},H} = \coprod \mathcal{A}'_i \tag{6}$$

into cones $\mathcal{A}'_i$ with base $\mathcal{A}_i \cap H$, where $\mathcal{A}_i$ are the cones in (4); the cones $\mathcal{A}'_i$ in (6) satisfy Theorem 2.16(i)–(iv).

For every $\mathcal{A}_i$ in (4), when $\{0\} \neq \mathcal{A}'_i$, the codimension of $\mathcal{A}'_i$ in $\mathcal{C}_{\mathfrak{R},H}$ is that of $\mathcal{A}_i$ in $\mathcal{C}_{\mathfrak{R}}$. If $\varphi_i : Z \longrightarrow Z_i$ is the ample model associated to $\mathcal{A}'_i$ with $\dim \mathcal{A}'_i = d+1$, Z_i is $\mathbb{Q}$-factorial.

There are polyhedral complexes $\mathcal{C}_{\mathfrak{R},H}$, $\mathcal{B}^+(\mathcal{C}_{\mathfrak{R},H})$ naturally associated to (6); $\mathcal{N}_{\mathfrak{R},H} = \mathrm{Ner}(\mathcal{B}^+(\mathcal{C}_{\mathfrak{R},H}))^{(d)}$ is a subcomplex of $\mathcal{N}_{\mathfrak{R}}$.

Proof. Since H is in general position with respect to $\mathcal{C}_{\mathfrak{R}}$, $H \cap \mathcal{C}_{\mathfrak{R}}$ is a rational polytope of dimension $\rho(Z) - d - 1$ whose vertices are of the form $\varepsilon_1(K_Z + \Delta_1), \dots, \varepsilon_m(K_Z + \Delta_m) \in \mathcal{C}_{\mathfrak{R}}$ for $\varepsilon_i \in \mathbb{Q}$, (Z, Δ_i) klt and Δ_i big $\mathbb{Q}$-divisors. Define $\mathfrak{R}_H = R(Z; K_Z + \Delta_1, \dots, K_Z + \Delta_m)$; then $\mathfrak{R}_H$ is finitely generated and its support is the cone over $H \cap \mathcal{C}_{\mathfrak{R}}$. The properties of the cones in the decomposition of $\mathfrak{R}_H$ then follow from the fact that H is in general position. Also, if $\mathcal{A}_i \subset \overline{\mathcal{A}_j}$ and $\mathcal{A}_i \cap H \neq \emptyset$, then $\mathcal{A}_j \cap H \neq \emptyset$ and $\dim \mathcal{A}_j - \dim \mathcal{A}_i = \dim \mathcal{A}'_j - \dim \mathcal{A}'_i$, so that $\mathcal{N}_{\mathfrak{R},H}$ is a subcomplex of $\mathcal{N}_{\mathfrak{R}}$. □

Definition 3.20. Let $\mathcal{C}$ be a polyhedral complex, and $\overline{\mathcal{A}} \in \mathrm{Ob}(\mathcal{C})$. The *residual complex* $\mathrm{resi}_{\mathcal{A}}(\mathcal{C})$ is the minimal subcomplex of $\mathcal{C}$ containing all objects $c \in \mathrm{Ob}\,\mathcal{C}$ such that there is a map $\{f : \mathcal{A} \longrightarrow c\} \in \mathrm{Mor}\,\mathcal{C}$.

Lemma 3.21. *Let $\mathcal{A} = \mathcal{A}_{i_0}$ be a cone in (5), where $i_0 \in \mathcal{B}^+(I)$.*

(i) *Let $\overline{\mathcal{A}_1}, \overline{\mathcal{A}_2}$ be facets of $\mathrm{resi}_{\mathcal{A}}\,\mathcal{C}_{\mathfrak{R}}$ (and hence of $\mathcal{C}_{\mathfrak{R}}$). Then, $Z_1 \dashrightarrow Z_2$ is the composition of a finite number of maps $Z_j \dashrightarrow Z_{j'}$ that are either isomorphisms in codimension 1, morphisms that contract a single divisor, or inverses of morphisms contracting a single divisor and, for all j, j', the diagram*

$$\begin{array}{ccc} Z_j & \dashrightarrow & Z_{j'} \\ & \searrow \quad \swarrow & \\ & Z_{i_0} & \end{array}$$

commutes.

(ii) *If H is an affine subspace in general position with respect to $\mathcal{C}_{\mathfrak{R}}$ and $\mathcal{A}$, then $\mathrm{resi}_{\mathcal{A}}\,\mathcal{C}_{\mathfrak{R}} \subseteq \mathcal{C}_{\mathfrak{R},H}$.*

(iii) *If $\mathcal{A}_1, \dots, \mathcal{A}_n$ are the cones in (4) such that H is in general position with respect to $\mathcal{A}_1, \dots, \mathcal{A}_n$, then $\mathcal{C}_{\mathfrak{R},H} = \cup\, \mathrm{resi}_{\mathcal{A}_i}(\mathcal{C}_{\mathfrak{R}})$.*

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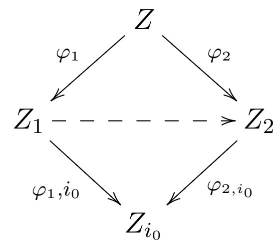
Proof. For assertion (i), recall that by Theorem 2.16(iii), the diagram



commutes; the result follows immediately from the decomposition of the morphisms φ_1, φ_2 into elementary contractions of Proposition 2.18(iv). Assertions (ii) and (iii) follow from the fact that H is in general position and from the definition of the decomposition (4). □

DEFINITION–LEMMA 3.22. Let $\mathcal{A} = \mathcal{A}_{i_0}$ be a cone in (5) of dimension d, where $i_0 \in \mathcal{B}^+(I)$. There is an $(\rho(Z) - d - 1)$-dimensional affine subspace $H_{\mathcal{A}}$ in general position with respect to $\mathcal{C}_{\mathfrak{R}}$ such that the cones $\mathcal{A}'_i$ in the decomposition (6) of $\mathcal{C}_{\mathfrak{R}, H_{\mathcal{A}}}$ are in one-to-one correspondence with the objects of $\operatorname{resi}_{\mathcal{A}} \mathcal{C}_{\mathfrak{R}}$. The ring $\operatorname{resi}_{\mathcal{A}} \mathfrak{R} = \mathfrak{R}_{H_{\mathcal{A}}}$ is *residual to* $\mathcal{A}$.

Proof. Let H be a $(\rho(Z) - d)$-dimensional affine subspace in general position with respect to $\mathcal{C}_{\mathfrak{R}}$ such that $\mathcal{A} \cap H$ is zero-dimensional and $\mathcal{C}_{\mathfrak{R}} \cap H$ is $(\rho(Z) - d)$-dimensional. The intersection $\mathcal{A} \cap H$ is convex, so that $\mathcal{A} \cap H = \{D\}$ is a single point.

Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be the cones distinct from $\mathcal{A}$ in (5) that are of dimension d and that intersect H. Denote by $D_1, \dots, D_n$ the points $\mathcal{A}_1 \cap H, \dots, \mathcal{A}_n \cap H$: D and $D_1, \dots, D_n$ are vertices of the polytope $\mathcal{C}_{\mathfrak{R}} \cap H$. Therefore, we may choose $H_{\mathcal{A}} \subseteq H$ a suitably small translation of the supporting hyperplane of $\mathcal{C}_{\mathfrak{R}} \cap H$ at D that is in general position with respect to $\mathcal{C}_{\mathfrak{R}}$, and such that D belongs to one of the half-space of H defined by $H_{\mathcal{A}}$, the vertices $D_1, \dots, D_n$ to the other half-space, and the only cones $\mathcal{A}'_i$ that have nonempty intersection with $H_{\mathcal{A}}$ contain D (i.e. are of the form $\mathcal{A}'_i = \mathcal{A}_i \cap H$, with $\mathcal{A} \subset \overline{\mathcal{A}_i}$).

Let $\mathcal{C}_{\mathfrak{R}, H_{\mathcal{A}}} = \coprod \mathcal{A}''_i$ be the decomposition induced by (4) on the support of $\mathfrak{R}_{H_{\mathcal{A}}}$. Then, by construction and by Lemma 3.21(ii), there is a one-to-one correspondence between cones $\{0\} \neq \mathcal{A}''_i$ and objects $\overline{\mathcal{A}} \neq \overline{\mathcal{A}_i}$ of $\operatorname{resi}_{\mathcal{A}} \mathcal{C}_{\mathfrak{R}}$; this is what we wanted. □

The next lemma is a reformulation of a key idea in [HM13], and implies Theorem 1.2. It shows that, similarly to Definition–Lemma 2.20, decompositions of birational maps between Mori fibre spaces correspond to edge paths on $\mathcal{N}_{\mathfrak{R}}$, or, equivalently, as in Remark 2.21, to suitable paths on $\partial^+ \mathcal{C}_{\mathfrak{R}}$.

LEMMA 3.23. *Assume the setup of Notation 3.6. Let* $\Phi : X/S \dashrightarrow Y/T$ *be a map in the collection* ($\Phi = \Phi_{j,j'}$ *for some* j, j'). *If* $v_{X/S}$ *and* $v_{Y/T}$ *are the vertices of* $\mathcal{N}_{\mathfrak{R}}$ *associated to* X/S *and* Y/T, *there is an edge path* $(v_0, \dots, v_n)$ *on* $\mathcal{N}_{\mathfrak{R}}$ *with* $v_0 = v_{X/S}$ *and* $v_n = v_{Y/T}$ *on* $\mathcal{N}_{\mathfrak{R}}$. *Further,* $\Phi \simeq L_{n-1,n} \circ \dots \circ L_{0,1}$, *where* $L_{j,j+1}$ *is the elementary Sarkisov link associated to the* 1*-simplex* $\{v_j, v_{j+1}\}$.

There is a one-to-one correspondence between edge paths on $\mathcal{N}_{\mathfrak{R}}$ *and the decompositions of birational maps* $X_j/S_j \dashrightarrow X_{j'}/S_{j'}$ *into elementary Sarkisov links dominated by* Z.

Proof. Let $\overline{\mathcal{A}_S}$ and $\overline{\mathcal{A}_T}$ be the facets of $\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$ dual to $v_{X/S}$ and $v_{Y/T}$, and $\overline{\mathcal{A}_X}, \overline{\mathcal{A}_Y}$ the associated facets of $\mathcal{C}_{\mathfrak{R}}$. Denote by $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ the ample models associated to $\mathcal{A}_X, \mathcal{A}_Y$.

I first show that if Φ is dominated by Z, there is a well-defined edge path on $\mathcal{N}_{\mathfrak{R}}$ from $v_{X/S}$ to $v_{Y/T}$ that decomposes Φ into elementary links. It is enough to prove that there is a two-dimensional affine subspace H of V in general position with respect to $\mathfrak{R}$ such that $v_{X/S}$

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and $v_{Y/T}$ belong to the same connected component of $\mathcal{N}_{\mathfrak{R},H}$. Dually, this is equivalent to showing that $\mathcal{A}_S \cap H$ and $\mathcal{A}_T \cap H$ belong to the same connected component of $\partial^+\mathcal{P}_{\mathfrak{R}} \cap H$.

By Proposition 3.1(iii), there is a klt pair (Z, Δ), with $K_Z + \Delta \in V$ not pseudoeffective, and ample $\mathbb{Q}$-divisors A_X and A_Y such that f and g are results of log-MMPs for (Z, Δ). These results of log-MMPs are MMPs with scaling by A_X and A_Y, i.e. $K_Z + \Delta + A_X \in \mathcal{A}_S$ and $K_Z + \Delta + A_Y \in \mathcal{A}_T$; this forces $K_Z + \Delta + (1+\varepsilon)A_X \in \mathcal{A}_X$ and $K_Z + \Delta + (1+\varepsilon)A_Y \in \mathcal{A}_T$ for $0 < \varepsilon \ll 1$. By Lemma 3.16, up to small perturbation of A_X, A_Y, and Δ, we may further assume that $H = (K_Z + \Delta) + \mathbb{R}_+ A_X + \mathbb{R}_+ A_Y \subseteq V$ is in general position with respect to $\mathcal{C}_{\mathfrak{R}}$. Write $\mathcal{P}_{\mathfrak{R},H} = \mathcal{C}_{\mathfrak{R}} \cap H$ and $\mathcal{Q}_i = \mathcal{A}_i \cap H$. By construction, $\mathcal{Q}_S$ and $\mathcal{Q}_T$ and hence also $\partial^+\mathcal{P}_{\mathfrak{R},H}$ are one-dimensional. Since $K_Z + \Delta \notin \mathcal{P}_{\mathfrak{R},H}$,

$$\mathcal{T} = \{K_Z + \Delta + \lambda(tA_X + (1-t)A_Y) \mid \lambda > 0, t \in [0,1]\} \cap \partial\mathcal{P}_{\mathfrak{R},H}$$

is one-dimensional, and intersects $\mathcal{Q}_S$ and $\mathcal{Q}_T$ in a one-dimensional locus. A divisor $D \in \mathcal{T}$ is not big, as this would imply that there is $t \leqslant 1$ such that $K_Z + \Delta + tA_X$ or $K_Z + \Delta + tA_Y$ is big. Therefore, $\mathcal{T} \subseteq \partial^+\mathcal{P}_{\mathfrak{R},H}$ and $\mathcal{Q}_S$ and $\mathcal{Q}_T$ belong to the same component of $\partial^+\mathcal{P}_{\mathfrak{R},H}$.

Since $v_{X/S}$ and $v_{Y/T}$ belong to the same path-connected component of $\mathcal{N}_{\mathfrak{R}}$, there is an edge path $(v_0, \ldots, v_n)$ on $\mathcal{N}_{\mathfrak{R}}$ with $v_0 = v_{X/S}$ and $v_n = v_{Y/T}$. Since $\Phi \simeq f_n \circ f_0^{-1}$ and since each $L_{j,j-1} \simeq f_j \circ f_{j-1}^{-1}$, then $\Phi \simeq L_{n,n-1} \circ \cdots \circ L_{0,1}$ is a decomposition of Φ into elementary Sarkisov links.

Conversely, an edge path of $\mathcal{N}_{\mathfrak{R}}$ defines a birational map Φ which is the composition of the elementary Sarkisov links associated to its edges, and Φ is dominated by Z by construction. □

4. Relations in the Sarkisov program

Let $\Phi_{j,j'} : X_j/S_j \dashrightarrow X_{j'}/S_{j'}$ be a birational map between Mori fibre spaces; by Theorem 1.2, $\Phi_{j,j'}$ is the composition of a finite number of elementary Sarkisov links. However, in general, this decomposition needs not be unique. In this section, I study *relations in the Sarkisov program*, i.e. distinct factorisations of this form of a given $\Phi_{j,j'}$. More precisely, I show that a factorisation of $\Phi_{j,j'}$ into elementary Sarkisov links corresponds to an *edge path* on a suitable simplicial complex $\mathcal{N}_{\mathfrak{R}}$. As a consequence, *relations* are determined by the fundamental group of $\partial^+\mathcal{C}_{\mathfrak{R}}$, the locus of $\mathcal{C}_{\mathfrak{R}}$ that consists of nonzero divisors that are not big, where $\mathfrak{R}$ is a suitable divisorial ring.

4.1 Sarkisov program and edge paths on $\mathcal{N}_{\mathfrak{R}}$

In this section, I define elementary relations in the Sarkisov program and show that they correspond to generators of the edge path group of $\mathcal{N}_{\mathfrak{R}}$.

Definition 4.1. A *(nontrivial) relation in the Sarkisov program* is a composition of $r > 2$ Sarkisov links

$$\mathrm{Id} \simeq L_{r-1,r} \circ \cdots \circ L_{0,1} \tag{7}$$

which defines an automorphism of $X_0 \simeq X_r$ that commutes with $X_0 \to S_0$.

The relation (7) is *elementary* if, in addition, no proper subchain of links in (7) forms a relation, i.e. $L_{j,j-1} \circ \cdots \circ L_{i+1,i} \not\simeq \mathrm{Id}$ for any $1 < i < j \leqslant r$, and if, for all i, $L_{i+1,i} \circ L_{i,i-1}$ is not an elementary Sarkisov link.

The relation is *dominated by* a normal projective variety Z if Z dominates all the elementary Sarkisov links $L_{j,j-1}$.

Consider a relation in the Sarkisov program, and let Z and $\mathfrak{R}$ be associated to the collection $\{X_1/S_1, \dots, X_{r-1}/S_{r-1}; L_{j+1,j}\}$ as in Proposition 3.1. By Lemma 3.23, we may assume that $\mathcal{N}_{\mathfrak{R}}$ is connected (or equivalently that $\partial^+\mathcal{C}_{\mathfrak{R}}$ is connected and purely $(\rho(Z)-1)$-dimensional) to study birational maps dominated by Z. This is not restrictive: given a map $\Phi : X/S \dashrightarrow Y/T$ dominated by Z, $\mathfrak{R}$ may be replaced by a subring to get rid of components of $\partial^+\mathcal{C}_{\mathfrak{R}}$ that do not contain $v_{X/S}$ and $v_{Y/T}$. This is achieved as in the proof of Proposition 3.1(iv). The next result is a simple consequence of Lemma 3.23.

COROLLARY 4.2. *There is a one-to-one correspondence between elementary relations in the Sarkisov program dominated by Z and equivalence classes of edge paths $(v_0, \dots, v_n)$ on $\mathcal{N}_{\mathfrak{R}}$ with $v_0 = v_n$ and $v_i \neq v_j$ for any $0 < i < j \leqslant n$. Equivalently, there is a one-to-one correspondence between elementary relations in the Sarkisov program dominated by Z and generators of the fundamental group $\pi_1(\partial^+\mathcal{P}_{\mathfrak{R}})$. When $\rho(Z) \geqslant 4$, $\pi_1(\partial^+\mathcal{P}_{\mathfrak{R}}) \simeq \pi_1(\partial^+\mathcal{C}_{\mathfrak{R}})$.*

Proof. Lemma 3.23 associates to any nontrivial relation dominated by Z an edge loop $(v_0, \dots, v_n)$ on $\mathcal{N}_{\mathfrak{R}}$ based at v_0. When the relation is elementary, $(v_0, \dots, v_n)$ is a simple edge loop, because $v_i = v_j$ for $0 \leqslant i < j < N$ would contradict the definition. Also, since $L_{i+1,i} \circ L_{i,i-1}$ is not a Sarkisov link, $\{v_i, v_{i+1}, v_i\}$ never spans a simplex, and $(v_0, \dots, v_n)$ is the 'minimal' edge loop in its equivalence class. This shows that an elementary relation defines the equivalence class of a generator of $E(\mathcal{N}_{\mathfrak{R}})$. The converse is clear by construction of $\mathcal{N}_{\mathfrak{R}}$. Since $\mathcal{N}_{\mathfrak{R}}$ is connected, $E(\mathcal{N}_{\mathfrak{R}}, v)$ is independent of the choice of $v = v_0$.

For the last assertion, recall that $E(\mathcal{N}_{\mathfrak{R}}) \simeq \pi_1(|\mathcal{N}_{\mathfrak{R}}|)$, and $\pi_1(|\mathcal{N}_{\mathfrak{R}}|) \simeq \pi_1(|\mathcal{B}^+(\mathcal{Q}_{\mathfrak{R}})|)$ by Lemma 2.5. However, as noted in Remark 3.9, $\partial^+\mathcal{P}_{\mathfrak{R}}$ is the underlying space of $\mathcal{B}^+(\mathcal{Q}_{\mathfrak{R}})$. Similarly, $E(\operatorname{Ner} \mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})) \simeq \pi_1(\partial^+\mathcal{C}_{\mathfrak{R}})$, and since the edge path group of a simplicial complex only depends on its 2-skeleton, $E(\mathcal{N}_{\mathfrak{R}}) = E(\operatorname{Ner} \mathcal{B}^+(\mathcal{C}_{\mathfrak{R}}))$ when $\rho(Z) \geqslant 4$. □

Standard results in Algebraic Topology give the following description of $E(\mathcal{N}_{\mathcal{R}})$. The 1-skeleton of $\mathcal{N}_{\mathfrak{R}}$ is a connected combinatorial graph. Let $T_{\mathfrak{R}}$ be a *spanning tree*, i.e. an edge path $(v_0, \dots, v_n)$ containing all vertices of $\mathcal{N}_{\mathfrak{R}}$ but no cycle, and let A be the set of edges of $\mathcal{N}_{\mathfrak{R}}$ that are not in $T_{\mathfrak{R}}$. Then, $E(\mathcal{N}_{\mathfrak{R}})$ is the set of equivalence classes of the group freely generated by the *fundamental cycles* $\{\gamma_{i,j} \mid \{v_i, v_j\} \in A\}$, where $\gamma_{i,j}$ is the class of the cycle obtained by adding $\{v_i, v_j\} \in A$ to $T_{\mathfrak{R}}$.

COROLLARY 4.3. *If $\dim \mathcal{C}_{\mathfrak{R}} = 3$, then $E(\mathcal{N}_{\mathfrak{R}})$ is either trivial or generated by a single element γ.*

Proof. When $\dim \mathcal{C}_{\mathfrak{R}} = 3$, $\partial\mathcal{P}_{\mathfrak{R}}$ is homeomorphic to S^1, and $E(\mathcal{N}_{\mathfrak{R}})$ is trivial when $\partial^+\mathcal{P}_{\mathfrak{R}} \subsetneq \partial\mathcal{P}_{\mathfrak{R}}$ and generated by an element γ otherwise. The path γ is naturally dual to the complex $\mathcal{B}^+(\mathcal{Q}_{\mathfrak{R}})$. □

Example 4.4. Recall the setting of Example 3.12. The group $E(\mathcal{N}_{\mathfrak{R}})$ is generated by the unique fundamental cycle through all vertices, and there is a unique elementary relation of the Sarkisov program dominated by Z, which is the relation (1) mentioned in the introduction.

CONVENTION 4.5. So far, I always assumed that $\dim \mathcal{C}_{\mathfrak{R}} \geqslant 2$. When $\mathcal{C}_{\mathfrak{R}}$ is two-dimensional, the situation is that studied in the two-ray game, and there is at most one Sarkisov link: $\mathcal{N}_{\mathfrak{R}}$ consists of at most one vertex. Set $E(\mathcal{N}_{\mathfrak{R}})$ to be the trivial group when $\dim \mathcal{C}_{\mathfrak{R}} = 2$.

4.2 Relations on three-dimensional rings

In this section, I give a geometric description of $E(\mathcal{N}_{\mathfrak{R}})$ when $\rho(Z) = 3$. There is a fairly explicit description of two-ray configurations on varieties Z with $\rho(Z) = 2$ and rational polyhedral pseudoeffective cone. This leads to the definition of four types of elementary Sarkisov links.

When $\rho(Z)=3$, there is no such description: any attempt at a *classification* elementary relations in the Sarkisov program is essentially qualitative.

Definition 4.6. Let $\gamma=(v_0,\dots,v_n)$ be an edge path on $\mathcal{N}_{\mathfrak{R}}$, and denote by $\overline{\mathcal{A}_i}=v_i^*$, and by $Z\longrightarrow S_i$ the ample model associated to $\mathcal{A}_i$. Assume that $\rho(S_0)=\min\{\rho(S_j)\}$, and set $S=S_0$ and $\rho=\rho(S_0)$.

The path γ is *of type A* if, for some $n_1<n_2<n$ or $n_1=n_2=n$,

$\{v_{j-1},v_j\}$ is of type II or IV unless $j=n_1$ or n_2,

$\{v_{n_1-1},v_{n_1}\}$ is of type I, and $\{v_{n_2-1},v_{n_2}\}$ is of type III.

In this case, $\rho(S_j)=\rho+1$ for $n_1\leqslant j\leqslant n_2-1$, and $\rho(S_j)=\rho$ otherwise.

The path γ is *of type B* if there are $n_1<n_2<n$ such that

$\{v_{j-1},v_j\}$ is of type II or IV for $j<n_1$ and $j>n_2$,

$\{v_{n_1-1},v_{n_1}\}$ is of type I and $\{v_{n_2-1},v_{n_2}\}$ of type III, and

$(v_{n_1},\dots,v_{n_2-1})$ is the product of finitely many paths of type A.

In this case, $\rho(S_j)=\rho+1$ or $\rho+2$ for $n_1\leqslant j\leqslant n_2-1$, and $\rho(S_j)=\rho$ otherwise.

Corollary 4.7. *Assume that* $\dim\mathcal{C}_{\mathfrak{R}}=3$. *A nontrivial edge loop* $(v_0,\dots,v_n)\in E(\mathcal{N}_{\mathfrak{R}})$ *is the composition of finitely many paths of type A or B. An elementary relation* $L_{n,n-1}\circ\cdots\circ L_{1,0}$ *of the Sarkisov program dominated by* Z *is the composition of finitely many chains of elementary links of type A or B.*

Proof. Let X_j/S_j be the Mfs associated to each vertex v_j and assume that $\rho(S_0)=\rho=\min\{\rho(S_j)\}$. As noted above, $E(N_{\mathfrak{R}})$ is nontrivial precisely when $\partial^+\mathcal{P}_{\mathfrak{R}}=\partial P_{\mathfrak{R}}$ is homeomorphic to S^1. Consider $\partial\mathcal{P}_{\mathfrak{R}}$ equipped with the triangulation induced by the decomposition (4); then $\partial\mathcal{P}_{\mathfrak{R}}$ is a simplicial complex dual to $\mathcal{N}_{\mathfrak{R}}$. There are two cases. If S_0 is not a point, $\rho\geqslant 1$ and the generator of $E(\mathcal{N}_{\mathfrak{R}})$ (i.e. the dual of $\partial\mathcal{P}_{\mathfrak{R}}$) is the composition of a finite number of edge paths of the form $L_1\cup L_2\cup L_3$, where L_1, L_2 and L_3 are connected, and for each $v_i\in L_1\cup L_3$ (respectively $v_i\in L_2$), $\rho(S_i)=1$ (respectively $\rho(S_i)=2$); $L_1\cup L_2\cup L_3$ is of type A.

If $\rho=0$, the generator of $E(\mathcal{N}_{\mathfrak{R}})$ is the composition of finitely many edge paths of the form $L_1\cup L_2\cup L_3$, where each L_i is connected, L_2 is a path of type A, and S_i is a point (respectively $\rho(S_i)=1$ or 2) for all $v_i\in L_1\cup L_3$ (respectively $v_i\in L_2$); $L_1\cup L_2\cup L_3$ is of type B. □

Corollary 4.8. *Assume that* $\dim\mathcal{C}_{\mathfrak{R}}=\rho\geqslant 3$ *and let* $\mathcal{A}$ *be a cone in (4) of dimension* $\rho-3$ *that contains no big divisor. Then* $E(\operatorname{resi}_{\mathcal{A}}\mathcal{N}_{\mathfrak{R}})$ *is trivial or is the free group generated by an edge-loop* $(v_0,\dots,v_n)$ *which is the composition of finitely many chains of type A or B.*

Proof. Recall from Definition–Lemma 3.22 that $\operatorname{resi}_{\mathcal{A}}\mathfrak{R}$ is a finitely generated ring with three-dimensional support. By Corollary 4.3, $E(\operatorname{resi}_{\mathcal{A}}\mathcal{N}_{\mathfrak{R}})$ is trivial or generated by a single edge loop $(v_0,\dots,v_n)$. Denote by $Z\longrightarrow S$ the ample model associated to the cone $\mathcal{A}$. By Lemma 3.21, the proof of Corollary 4.7 applies to this situation after replacing 'point' by S because all relevant contractions occur over S. □

The goal of Example 4.9 is to show that both types of relation in the Sarkisov program are realised among the end products of the MMP on a terminal $\mathbb{Q}$-factorial Fano 3-fold Z with $\rho(Z)=3$. Let $\{X_i/S_i\}$ be the Mori fibre spaces that are results of the MMP on Z. The 3-fold Z admits no small contraction and is a common resolution of all X_i. If D_1,D_2,D_3 is a basis of $\overline{\mathrm{Eff}}(Z)$ and $A\sim_{\mathbb{Q}}-K_Z$, the ring $\mathfrak{R}(K_Z+A+D_1,K_Z+A+D_2,K_Z+A+D_3)$ is finitely generated and has support $\overline{\mathrm{Eff}}_{\mathbb{R}}(Z)$. Then, Z, $\mathfrak{R}$ are associated to the collection of X_i/S_i and Sarkisov links dominated by Z.

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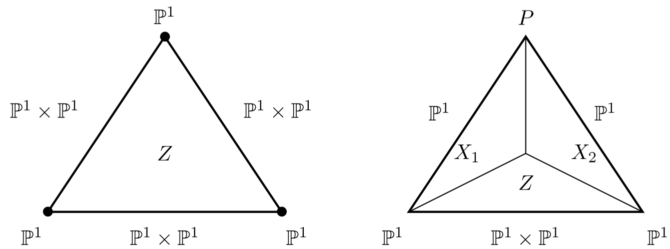

Figure 3. Geography of ample models of a Picard rank 3 Fano 3-fold: Case 1.

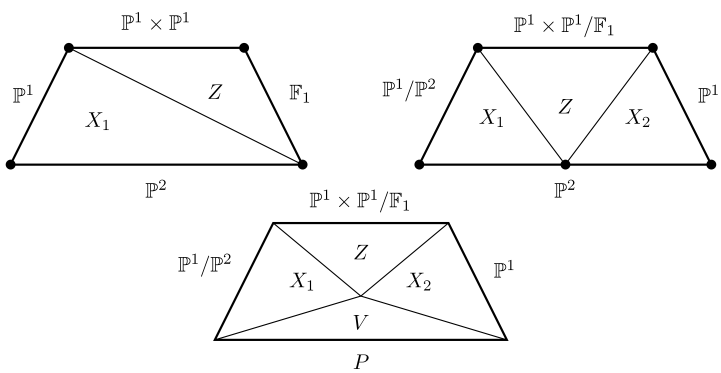

Figure 4. Geography of ample models of a Picard rank 3 Fano 3-fold: Case 2.

Example 4.9. The methods of [MM86] can be used to classify the polytopes $\mathcal{P}_{\mathfrak{R}}$ and complexes $\mathcal{Q}_{\mathfrak{R}}$. The numbers between parentheses are those in the classification of [MM81], and n is the number of chambers of maximal dimension in the coarsest decomposition (4). In the figures, the faces $\mathcal{Q}_i$ of dimension 1 and 2 are labelled by the associated ample models except when this model is $Z \longrightarrow \{P\}$.

Case 1. The 3-fold Z admits no E1-contraction to a Fano 3-fold (Figure 3). Either Z admits no E1 contraction and $n = 1$ (1, 27), or Z admits an E1-contraction to a weak Fano 3-fold and $n = 3$ (2, 31). The relation is of type A.

Case 2. The 3-fold Z is not primitive and admits a structure of Mori fibre space (Figure 4). There are three cases: $n = 2$ (4, 28), $n = 3$ (3, 8, 17, 24) and $n = 4$ (6, 10, 25, 30). Here, $\rho(X_i) = 2$ and $\rho(V) = 1$. The relation is of type A in the first two cases, and of type B in the last one.

Case 3. The 3-fold Z is not primitive and has no Mori fibre space structure (Figure 5). There are two cases with $n = 5$ that correspond to whether the interior chambers yield ample models with $\rho = 2$ only (7,13) or not (11, 12, 15, 16, 20, 26, 22), one with $n = 6$, and one with $n = 8$. Again, $\rho(X_i) = 2$ and $\rho(V_j) = 1$. The relation is always of type A.

Remark 4.10. Example 4.9 shows that both types of elementary relations in the Sarkisov program occur among the end products of the MMP of Fano 3-folds with $\rho = 3$. Note that the end products of the MMP on a Picard rank 3 del Pezzo surface only give an example of a relation of type A (see Example 4.4 and (1)). Since (1) and (25) in the classification of [MM81] are toric, both types of relations actually occur among the end products of the MMP on *toric Fano 3-folds.*

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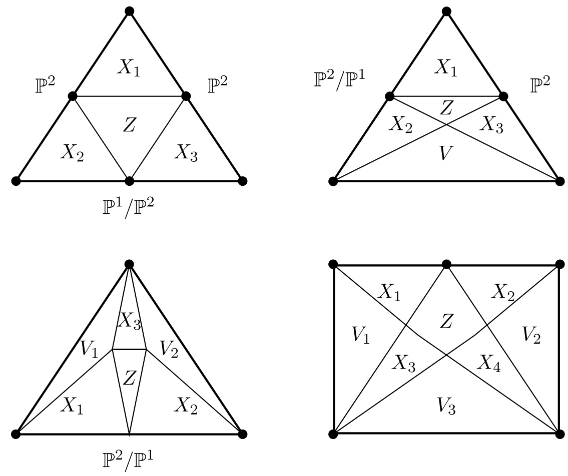

FIGURE 5. Geography of ample models of a Picard rank 3 Fano 3-fold: Case 3.

Lastly, $\mathbb{P}^3/\{P\}$ is among the end products of the MMP on (11) and of (25), and this shows that both types of relations are realised in the Cremona group of $\mathbb{P}^3$.

4.3 General case

I now make precise the statement that relations in the Sarkisov program are *generated* by relations of the type occurring among the end products of the MMP on Picard rank 3 Fano 3-folds.

THEOREM 4.11. *The edge path group $E(\mathcal{N}_{\mathfrak{R}})$ is generated by edge loops $\{\gamma_{\mathcal{A}} \mid E(\operatorname{resi}_{\mathcal{A}} \mathcal{N}_{\mathfrak{R}}) = \mathbb{Z}[\gamma_{\mathcal{A}}]\}$ as $\mathcal{A}$ ranges over $(\rho(Z)-3)$-dimensional cones in (4) that contain no big divisor.*

COROLLARY 4.12. *Relations in the Sarkisov program are generated by elementary relations. These are compositions of finitely many chains of type A or B, and there are examples of relations of type A and B induced by the MMP on nonsingular Fano 3-folds of Picard rank 3.*

Proof. Let $I', I'' \subset \mathcal{B}^+(I)$ be such that $I' = \{i \in \mathcal{B}^+(I) \mid \dim \mathcal{A}_i = \rho(Z) - 3\}$, and $I'' = \{i \in \mathcal{B}^+(I) \mid \dim \mathcal{A}_i = \rho(Z) - 2, \mathcal{A}_j \not\subset \overline{\mathcal{A}_i} \text{ for } j \in I'\}$. For each $i \in \mathcal{B}^+(I)$ and cone $\mathcal{A}_i$ in (6), let $\operatorname{resi}_{\mathcal{A}_i} \mathfrak{R}$ be the $(\rho(Z)-d)$-dimensional residual ring as in Definition–Lemma 3.22. Then, the 1-skeleton of $\mathcal{N}_{\mathfrak{R}}$ is

$$\mathcal{N}_{\mathfrak{R}}^{(1)} = \bigcup_{i \in I'} \operatorname{resi}_{\mathcal{A}_i} \mathcal{N}_{\mathfrak{R}} \cup \bigcup_{i \in I''} \operatorname{resi}_{\mathcal{A}_i} \mathcal{N}_{\mathfrak{R}}.$$

By Corollary 4.3, $E(\operatorname{resi}_{\mathcal{A}_i} \mathcal{N}_{\mathfrak{R}})$ is trivial for all $i \in I$", and by Corollary 4.7, $E(\operatorname{resi}_{\mathcal{A}_i} \mathcal{N}_{\mathfrak{R}})$ is either trivial or of the form $\mathbb{Z}[\gamma_{\mathcal{A}_i}]$ for $i \in I'$. Since $\mathcal{N}_{\mathfrak{R}}$ is connected, by the van Kampen theorem, $E(\mathcal{N}_{\mathfrak{R}}^{(1)})$ is generated by the groups $E(\operatorname{resi}_{\mathcal{A}_i} \mathcal{N}_{\mathfrak{R}})$. Corollary 4.12 then follows immediately from Corollary 4.8, Example 4.9 and Remark 4.10. □

Remark 4.13. Note that $E(\mathcal{N}_{\mathfrak{R}})$ is not in general *freely generated* by the loops $\gamma_{\mathcal{A}_i}$ when $\dim \mathcal{C}_{\mathfrak{R}} > 3$. The loops $\gamma_{\mathcal{A}_i}$ generate the free group $E(\mathcal{N}_{\mathfrak{R}}^{(1)})$. When $\dim \mathcal{C}_{\mathfrak{R}} > 3$, $\mathcal{N}_{\mathfrak{R}} \neq \mathcal{N}_{\mathfrak{R}}^{(1)}$, and the relations in $E(\mathcal{N}_{\mathfrak{R}})$ depend on the 2-skeleton of $\mathcal{N}_{\mathfrak{R}}$, that is by faces $\mathcal{A}$ of dimension $\rho(Z)-4$. For example, there is a relation between $\gamma_{\mathcal{A}_1}$ and $\gamma_{\mathcal{A}_2}$ when $\overline{\mathcal{A}_3} = \overline{\mathcal{A}_1} \cap \overline{\mathcal{A}_2}$ has dimension $\rho(Z)-4$, so that $\operatorname{resi}_{\mathcal{A}_1} \mathcal{N}_{\mathfrak{R}}$ and $\operatorname{resi}_{\mathcal{A}_2} \mathcal{N}_{\mathfrak{R}}$ both are subcompanies of $\operatorname{resi}_{\mathcal{A}_3} \mathcal{N}_{\mathfrak{R}}$. In general, these relations between

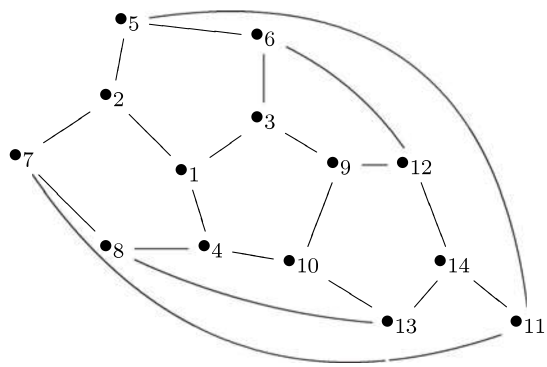

FIGURE 6. The 1-skeleton of $\mathcal{N}_{\overline{\mathrm{Eff}}\,S}$.

relations in the Sarkisov program can be more complicated, but they are accounted for by divisorial rings with four-dimensional support.

Example 4.14. To conclude, I determine the edge loops on $\mathcal{N}_{\overline{\mathrm{Eff}}\,S}$, for S a smooth del Pezzo surface S with $\rho(S)=4$. Let S be the blow up of $\mathbb{P}^2$ in three points $\{P_1, P_2, P_3\}$; then $\mathcal{C}_{\mathfrak{R}} = \overline{\mathrm{Eff}}(S) = \mathrm{Eff}(S)$ is a rational polyhedral cone of dimension 4. Let E_1, E_2, E_3 be the (-1)-curves and L_1, L_2, L_3 the proper transforms of lines through two of the three points P_i on $\mathbb{P}^2$. Let $S \to S_i$ (resp $S \to T_i$) be the contraction of E_i (resp of L_i). The vertices of $\mathcal{N}_{\mathfrak{R}}$ are as follows:

(1) (respectively (14)) corresponds to the ample model $S \longrightarrow \mathbb{P}^2 \to pt$ factoring through the three maps $S \longrightarrow S_i$ (respectively $S \longrightarrow T_i$);

(2, 3, 4) (respectively (11, 12, 13)) correspond to the three ample models $S \longrightarrow \mathbb{F}_1 \to \mathbb{P}^1$ factoring through $S \longrightarrow S_i$ (respectively $S \longrightarrow T_i$);

(5, 6) (respectively (7, 8), respectively (9, 10)) are the ample models $S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ factoring through $S \longrightarrow S_1$ and $S \longrightarrow T_1$ (respectively S_2, T_2 and S_3, T_3).

Figure 6 illustrates the 1-skeleton of $\mathcal{N}_{\mathfrak{R}}$. Here, there are nine faces of $\mathcal{B}^+(\mathcal{C}_{\mathfrak{R}})$ of dimension 1, $\mathcal{N}_{\mathfrak{R}} = \bigcup_{i \in I'} \mathrm{Res}_{\mathcal{A}}\, \mathcal{N}_{\mathfrak{R}}$ and the associated cycles have the following interpretation:

(1, 2, 5, 6, 3) (respectively (1, 3, 9, 10, 4), respectively ((1, 2, 7, 8, 4))) corresponds to relations among the end products of the MMP on S_1 (respectively on S_2, respectively on S_3);

(12, 9, 10, 13, 14) (respectively (11, 5, 6, 12, 14), respectively (11, 7, 8, 13, 14)) corresponds to relations among the end products of the MMP on T_1 (respectively on T_2, respectively on T_3);

(2, 5, 7, 11), (3, 9, 12, 6) and (4, 8, 10, 13) correspond to relations among the end products of the MMP of S over each of the three $\mathbb{P}^1$ that arise as ample models $S \longrightarrow \mathbb{P}^1$ for some $D \in \overline{\mathrm{Eff}}(S)$.

As a by-product, this example shows how to recover all factorisations of the quadratic involution $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$; these correspond to edge-loops through the two vertices of $\mathcal{N}_{\mathfrak{R}}$ associated to the two distinct maps $S \longrightarrow \mathbb{P}^2 \to \{P\}$.

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