# Relationship Between Hyper $M V$-algebras and Hyperlattices 

R. A. Borzooei, Akefe Radfar, and Sogol Niazian


#### Abstract

Sh. Ghorbani, et al. [9], generalized the concept of $M V$-algebras and defined the notion of hyper $M V$-algebras. Now, in this paper, we try to prove that any hyper $M V$-algebra is a hyperlattice. First we prove that any hyper $M V$-algebra that satisfies the semi negation property is a hyperlattice. Then with a computer program, we show that any hyper $M V$-algebra of order less than 6 , is a hyperlattice. Finally, we claim that this result is correct for any hyper $M V$-algebra.


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## 1 Introduction

The first studies regarding multiple-valued logics were conducted by J. Lukasiewicz and E. Post when they introduced a three-valued logical system in 1920 [14]. The latter built a different n-valued logical system in 1921 [17]. Then Lukasiewicz and Tarski developed in 1930 [15] a logic for which the truth values are the rationales in [0, 1]. In 1940, Gr.C. Moisil introduced the three-valued Lukasiewicz algebras as algebraic models for the corresponding logic of Lukasiewicz. In 1941, Moisil also defined n-valued Lukasiewicz algebras. Then, in 1956, A. Rose showed that for a number of truth values greater than 5 the Lukasiewicz algebras are no longer the algebras of Lukasiewicz
logic. In fact, by defining the n-valued Lukasiewicz algebras, Moisil invented a distinct logical system. In 1958, C.C. Chang defined $M V$-algebras as models for the infinitely valued Lukasiewicz-Tarski logic [5]. In 1977, R. Grigolia introduced $M V_{n}$-algebras to model the n-valued Lukasiewicz logic [10].

The study of hyperstructures, started in 1934 by Marty's paper at the 8th Congress of Scandinavian Mathematicians [16] where hypergroups were introduced. Sh. Ghorbani et al. [9] applied the hyperstructure to $M V$-algebras and introduce the concept of hyper $M V$-algebras which is a generalization of $M V$-algebras and investigated some results. They also discussed quotient structure and category of hyper $M V$-algebras ([8], [7]). Specially, they clarified the relation between the class of hyper $M V$-algebras and hyper Kalgebras [2]. R. A. Borzooei et al. [1] proved that these relations are not true, which unfortunately is used to prove some important results of several hyper $M V$-algebras paper. L. Torkzadeh et al [18] discussed hyper $M V$-ideals and define some hyperoperations on it. Then they get some results and give a problem which want to prove or disprove the hyperoperations $\vee$ and $\wedge$ are associative. As another hyper algebraic structures the notions of (weak) hyper $M V$-deductive systems and (weak) implicative hyper $M V$-deductive systems are introduced in [12]. Then the relation among them are discussed. Also, as a continue, new types of hyper $M V$-deductive systems are introduced in [?newded]. Now, in this paper, we try to find a relationship between hyper $M V$-algebras and hyperlattices.

## 2 Preliminary

In this section we give some definitions and properties of $M V$-algebras and hyper $M V$-algebras which we need in the next section.

Definition 2.1. [5] An $M V$-algebra is an algebra $(A, \oplus, *, 0)$ of type $(2,2,0)$ that satisfying the following axioms:
$(M V 1) x \oplus(y \oplus z)=(x \oplus y) \oplus z$,
(MV2) $x \oplus y=y \oplus x$,
(MV3) $x \oplus 0=x$,
(MV4) $x^{* *}=x$,
(MV5) $x \oplus 0^{*}=0^{*}$,
(MV6) $\left(x^{*} \oplus y\right)^{*} \oplus y=\left(y^{*} \oplus x\right)^{*} \oplus x$.
Let $A$ be an $M V$-algebra. We define the operations $\odot$ and $\ominus$ on $A$ by, $x \odot y=:\left(x^{*} \oplus y^{*}\right)^{*}$ and $x \ominus y=: x \odot y^{*}$, for any $x, y \in A$ and we consider $1=: 0^{*}$. Moreover, the relation $x \leq y$ on $A$ is defined by $x \leq y$ if and
only if $x^{*} \oplus y=1$, for any $x, y \in A$. The relation $\leq$ is a partial order on $A$ which is called the natural order of $A$. This natural order determines a lattice structure $(A, \vee, \wedge)$, where $x \vee y=:\left(x \odot y^{*}\right) \oplus y$ and $x \wedge y=:\left(x^{*} \vee y^{*}\right)^{*}$, for any $x, y \in A$. As a first example of nontrivial $M V$-algebra, consider the real unit interval $[0,1]$ with $x \oplus y=\min \{x+y, 1\}$ and $x^{*}=1-x$. It is easy to see that $([0,1], \oplus, *, 0)$ is an $M V$-algebra.

Proposition 2.1. [5] Let $A$ be an $M V$-algebra and $x, y \in A$. Then the following is hold:
(i) $1^{*}=0$,
(ii) $x \oplus y=\left(x^{*} \odot y^{*}\right)^{*}$,
(iii) $x \oplus 1=1$,
(iv) $(x \ominus y) \oplus y=(y \ominus x) \oplus x$,
(v) $x \oplus x^{*}=1$,
(vi) $x \leq y$ if and only if $y^{*} \leq x^{*}$,
(vii) if $x \leq y$, then for each $z \in A, x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z$,
(viii) $x \odot y \leq z$ if and only if $x \leq y^{*} \oplus z$,
$(i x) x \odot(y \vee z)=(x \odot y) \vee(x \odot z)$,
(x) $x \oplus(y \wedge z)=(x \oplus y) \wedge(x \oplus z)$.

Definition 2.2. [6] A hyperoperation on a nonempty set $H$ is a map ○: $H \times H \rightarrow P^{\star}(H)=P(H)-\{\emptyset\}$. In this case, $(H, \circ)$ is called a hypergroupoid. Let $(H, \circ)$ be a hypergroupoid. Then an element $a \in H$ is called scalar if $|a \odot x|=1$, for any $x \in H$. Moreover, if $A$ and $B$ are two non-empty subsets of $H$, then we define $A \circ B, a \circ B$ and $A \circ b$ as follows, for any $a \in A$ and $b \in B$ :

$$
A \circ B=\bigcup_{a \in A, b \in B}(a \circ b), \quad a \circ B=\{a\} \circ B, \quad A \circ b=A \circ\{b\} .
$$

Definition 2.3. [9] A hyper MV-algebra is a nonempty set $M$ endowed with a hyperoperation " $\oplus$ ", a unary operation " $*$ " and a constant " 0 " satisfying the following axioms, for all $x, y, z \in M$,

$$
(H M V 1) x \oplus(y \oplus z)=(x \oplus y) \oplus z,
$$

$$
\text { (HMV2) } x \oplus y=y \oplus x,
$$

(HMV3) $\left(x^{*}\right)^{*}=x$,
(HMV4) $\left(x^{*} \oplus y\right)^{*} \oplus y=\left(y^{*} \oplus x\right)^{*} \oplus x$,
(HMV5) $0^{*} \in x \oplus 0^{*}$,
(HMV6) $0^{*} \in x \oplus x^{\star}$,
(HMV7) $x \ll y, y \ll x \Rightarrow x=y$
where $x \ll y$ is defined by $0^{*} \in x^{\star} \oplus y$. For any $A, B \subseteq M$, we define $A \ll B$ if and only if there exist $a \in A$ and $b \in B$ such that $a \ll b$. We define $0^{*}:=1$ and $A^{*}=\left\{a^{*}: a \in A\right\}$.

Proposition 2.2. [9] Let $(M, \oplus, *, 0)$ be a hyper $M V$-algebra. Then for all $x, y, z \in M$ and for all nonempty subsets $A, B$ and $C$ of $M$ the following hold:
(i) $A \oplus(B \oplus C)=(A \oplus B) \oplus C$,
(ii) $0 \ll x, x \ll 1, x \ll x$ and $A \ll A$,
(iii) If $x \ll y$, then $y^{*} \ll x^{*}$ and $A \ll B$ implies $B^{*} \ll A^{*}$,
(iv) $\left(A^{*}\right)^{*}=A$,
(v) $0 \oplus 0=\{0\}$ and $x \in x \oplus 0$,
(vi) If $y \in x \oplus 0$, then $y \ll x$.

Theorem 2.3. [1] Let $M$ be a finite hyper $M V$-algebra such that $0 \oplus x=\{x\}$, for all $x \in M$. Then $M$ is an $M V$-algebra.

Proposition 2.4. [18] Let $(M, \oplus, *, 0)$ be a hyper $M V$-algebra. Define the following hyperopoerations on $M$ as follows:

$$
x \vee y=\left(x^{*} \oplus y\right)^{*} \oplus y, \quad x \wedge y=\left(x^{*} \vee y^{*}\right)^{*}
$$

Then for all $x, y, z \in M$ :
(i) $x \in(x \wedge x) \cap(x \vee x)$,
(ii) $x \vee y=y \vee x$ and $x \wedge y=y \wedge x$,
(iii) $x \in(x \wedge(x \vee y)) \cap(x \vee(x \wedge y))$,
(iv) if $x \ll y$, then $y \in x \vee y$ and $x \in x \wedge y$,
(vi) $x, y \ll x \vee y$ and $x \wedge y \ll x, y$.

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## 3 Relationship between hyper $M V$-algebras and hyperlattices

In this section, we try to show that any finite hyper $M V$-algebra is a hyperlattice.

Definition 3.1. If $x^{*}=x$, for any $x \in M-\{0,1\}$, then we say that $M$ satisfied the Semi Negation Property (or (SNP), for short).

Example 3.1. Let $M=\{0, a, b, 1\}$ and hyperoperation $\oplus$ and unary operation $*$ on $M$ are defined as follows;

| $\oplus$ | 0 | a | b | 1 |
| :---: | :---: | :---: | :---: | :---: |
| 0 | $\{0\}$ | $\{0, \mathrm{a}\}$ | $\{0, \mathrm{~b}\}$ | A |
| a | $\{0, \mathrm{a}\}$ | A | $\{0, \mathrm{a}, \mathrm{b}\}$ | A |
| b | $\{0, \mathrm{~b}\}$ | $\{0, \mathrm{a}, \mathrm{b}\}$ | A | A |
| 1 | A | A | A | A |


| $*$ | 0 | a | b | 1 |
| :--- | :--- | :--- | :--- | :--- |
| 1 | a | b | 0 |  |

Then it is easy to see that $(M, \oplus, *, 0)$ is a hyper $M V$-algebra that satisfying the (SNP).

Note: Throughout this section, we let $M$ be a hyper $M V$-algebra and satisfies the (SNP), unless otherwise stated.

Lemma 3.1. For all $x, y \in M-\{0,1\}$ :
(i) $x \ll y$, implies $x=y$,
(ii) if $0 \oplus x=\{x\}$, then $y \notin 1 \oplus x$.

Proof. (i) If $x \ll y$, then by Proposition 2.2(iii), $y^{*} \ll x^{*}$ and so $y \ll x$. Hence, by (HMV7), $x=y$.
(ii) On the contrary, let $y \in 1 \oplus x$, for $y \in M-\{0,1\}$. By (HMV4), we get

$$
y \oplus x=y^{*} \oplus x \subseteq(1 \oplus x)^{*} \oplus x=(x \oplus 0)^{*} \oplus 0=x^{*} \oplus 0=x \oplus 0=\{x\} .
$$

Thus $y \oplus x=\{x\}=y^{*} \oplus x$. Now, by (HMV4),

$$
\begin{aligned}
x \oplus x & =x^{*} \oplus x=\left(y^{*} \oplus x\right)^{*} \oplus x=\left(x^{*} \oplus y\right)^{*} \oplus y \\
& =(x \oplus y)^{*} \oplus y=x^{*} \oplus y=x \oplus y=\{x\} .
\end{aligned}
$$

Hence, $x \oplus x=\{x\}$. Also, by (HMV6), $1=0^{*} \in x^{*} \oplus x=x \oplus x$ which is a contradiction. Therefore, $y \notin 1 \oplus x$.

Lemma 3.2. For any $x, y \in M$ and $A \subseteq M$,
(i) if $1 \in x \oplus y$, then $x=y$ or $x=1$ or $y=1$;
(ii) if $x \notin 0 \oplus 1$, then $x \in 0 \oplus A$ implies $x \in A$;
(iii) if $\{0,1\} \subseteq A$ or $0,1 \notin A$, then $A^{*}=A$.

Proof. (i) Let $1=0^{*} \in x \oplus y$. If $x, y \in M-\{0,1\}$, then $x \ll y$ and so by Lemma 3.1(i), $x=y$. If $x \neq 0,1$ and $y=0$, then $0^{*}=1 \in x \oplus 0=x^{*} \oplus 0$ and implies that $x \ll 0$. Hence $x=0$, which is a contradiction. Therefore, if $x \neq 0,1$, then $y=1$ and similarly, $y \neq 0,1$ implies that $x=1$. If $x, y \in\{0,1\}$ and $x=y=0$, then $1 \in x \oplus y=0 \oplus 0=\{0\}$, which is a contradiction. So, $x=1$ or $y=1$.
(ii) If $x \in 0 \oplus A$, then there is $a \in A$ such that $x \in 0 \oplus a$. By Proposition $2.2(\mathrm{vi}), x \ll a$. By $(i), a=1, x=1$ or $x=a$. Since $1 \in 0 \oplus 1$, by Proposition $2.2(\mathrm{v})$, we get $x \neq 1$. Also, $a=1$ means that $x \in 0 \oplus 1$ which against the assumption. Thus $x=a \in A$.
(iii) We know $A^{*}=\left\{x^{*}: x \in A\right\}$. If $\{0,1\} \subseteq A$, then for any $x \in A, x=0$ or $x=1$ or $x \in M-\{0,1\}$ and so $x^{*}=1$ or $x^{*}=0$ or $x^{*}=x$. Hence, $x^{*} \in A$ i.e. $A^{*}=A$. Now, let $0,1 \notin A$. Then $A \subseteq M-\{0,1\}$ and since $M$ satisfies the (SNP), we get $A^{*}=A$.

Theorem 3.3. [1] Let $M$ be a hyper $M V$-algebra and $x$ be an element of $M$ such that $0 \oplus x=\{x\}$ and $x^{*}=x$. Then $0, x \notin 1 \oplus x$.

Lemma 3.4. Let $x$ be an element of $M-\{0,1\}$ such that $0 \oplus x=\{x\}$. Then we get
(i) $1 \oplus x=\{1\}, 0 \oplus 1=\{1\}$;
(ii) $x \oplus x=\{1\}$;
(iii) $0 \oplus y=\{y\}$, for all $y \in M-\{0,1\}$.

Proof. (i) By Theorem 3.3, $0, x \notin 1 \oplus x$ and by Lemma 3.1(ii), $y \notin 1 \oplus x$, for all $y \in M-\{0,1\}$. Thus $1 \oplus x=\{1\}$. Also, we get $0 \oplus 1=0 \oplus(1 \oplus x)=$ $1 \oplus(0 \oplus x)=1 \oplus x=\{1\}$.
(ii) By part (i) and (HMV4), we get

$$
\begin{aligned}
x \oplus x & =x^{*} \oplus x=(0 \oplus x)^{*} \oplus x=\left(x^{*} \oplus 1\right)^{*} \oplus 1 \\
& =(x \oplus 1)^{*} \oplus 1=1^{*} \oplus 1=0 \oplus 1=\{1\} .
\end{aligned}
$$

(iii) Let $y \in M-\{0,1\}$ and $y \neq x$. By Proposition 2.2(v), $y \in 0 \oplus y$. Now, by the contrary, let $0 \oplus y \neq\{y\}$. Then there exists $z \in M$ such that $z \neq y$ and $z \in 0 \oplus y$.

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If $z \neq 0,1$, then by Proposition $2.2(\mathrm{vi}), z \ll y$ and so by Lemma $3.1(\mathrm{i})$ we get $z=y$, which is a contradiction.
If $z=1$, then $1 \in 0 \oplus y$ and so by Proposition $2.2(v i), 1 \ll y$. Hence $y=1$ which is a contradiction by $y \neq 0,1$.
If $z=0$, since $x \in 0 \oplus x \subseteq(0 \oplus y) \oplus x=(0 \oplus x) \oplus y=x \oplus y$, we get $x \in x \oplus y$. So, by (HMV4),

$$
1 \in x^{*} \oplus x \subseteq(x \oplus y)^{*} \oplus x=\left(y^{*} \oplus x\right)^{*} \oplus x=\left(x^{*} \oplus y\right)^{*} \oplus y=(x \oplus y)^{*} \oplus y
$$

Hence there is $t \in x \oplus y$ such that $1 \in t^{*} \oplus y$. By Lemma $3.2(\mathrm{i}), t^{*}=1$ or $t^{*}=y$ and so $t=0$ or $t=y$. It means that $0 \in x \oplus y$ or $y \in x \oplus y$. If $0 \in x \oplus y$, then by (HMV1) and (ii),

$$
x \in 0 \oplus x \subseteq(x \oplus y) \oplus x=(x \oplus x) \oplus y=1 \oplus y
$$

Hence, by (HMV4), we get $x \in x \oplus y=x^{*} \oplus y \subseteq(1 \oplus y)^{*} \oplus y=(y \oplus 0)^{*} \oplus 0$. Since $x \notin 0 \oplus 1=\{1\}$, by Lemma 3.2(ii), we get $x \in(y \oplus 0)^{*}$. So, $x=x^{*} \in$ $\left((y \oplus 0)^{*}\right)^{*}=y \oplus 0$. Thus $x \ll y$ and so by Lemma 3.1(i), $x=y$ which is a contradiction. Similarly, for the case $y \in x \oplus y$, we get a contradiction. Therefore, $0 \oplus y=\{y\}$, for all $y \neq 0,1$.

Theorem 3.5. If $M$ is finite and $x$ be an element of $M-\{0,1\}$ such that $0 \oplus x=\{x\}$, then $M$ is an $M V$-algebra.

Proof. Let $M$ be finite and $x \in M-\{0,1\}$ such that $0 \oplus x=\{x\}$. Then by Lemma 3.4(iii), $0 \oplus y=\{y\}$, for all $y \in M-\{0,1\}$. Moreover, by Lemma $3.4(\mathrm{i}), 0 \oplus 1=\{1\}$ and by Proposition $2.2(\mathrm{v}), 0 \oplus 0=\{0\}$. Hence $0 \oplus y=\{y\}$ for all $y \in M$ and so by Theorem $2.3, M$ is an $M V$-algebra.

Proposition 3.6. Let $M$ be finite and proper. Then for all distinct elements $x, y, z \in M-\{0,1\}$,
$\left(P_{1}\right) \quad 0 \oplus x=\{0, x\}$,
$\left(P_{2}\right) \quad$ if $x \in 0 \oplus 1$, then $0 \in 0 \oplus 1$,
$\left(P_{3}\right) \quad x, y \in x \oplus y$,
$\left(P_{4}\right) \quad x \in 1 \oplus x$ or $0 \in 1 \oplus x$. Indeed, $1 \oplus x \neq\{1\}$,
$\left(P_{5}\right) \quad$ if $y \notin 0 \oplus 1$, then $y \notin 1 \oplus x$,
$\left(P_{6}\right) \quad$ if $x \notin 0 \oplus 1$, then $x \in 1 \oplus x$,
$\left(P_{7}\right) \quad 1 \oplus x \subseteq(0 \oplus 1) \cup\{0, x\}$,
$\left(P_{8}\right) \quad 1 \oplus 1 \subseteq 0 \oplus 1$,
$\left(P_{9}\right) \quad$ if $y \in x \oplus x$, then $y \in 0 \oplus 1$. Indeed, $x \oplus x \subseteq(0 \oplus 1) \cup\{0, x\}$,
$\left(P_{10}\right) \quad 0 \oplus 1 \backslash\{0, x\} \subseteq x \oplus x$,
$\left(P_{11}\right) \quad 0 \oplus 1 \subseteq(x \oplus x) \cup(1 \oplus x)$ and $\{0, x\} \subseteq(x \oplus x) \cup(1 \oplus x)$,
$\left(P_{12}\right) \quad(x \oplus x) \cup(1 \oplus x)=(0 \oplus 1) \cup\{0, x\}$,
$\left(P_{13}\right) \quad z \in x \oplus y$ implies $x \in y \oplus z$,
$\left(P_{14}\right) \quad$ if $z \in x \oplus y$, then $x, y, z \in 0 \oplus 1$ or $x, y, z \notin 0 \oplus 1$,
$\left(P_{15}\right) \quad(0 \oplus 1) \cup(x \oplus y) \cup\{0\} \subseteq x \vee y$,
$\left(P_{16}\right) \quad$ if $x, y \notin 0 \oplus 1$, then $x \oplus(x \oplus y) \cup\{0\}=x \oplus y \cup\{0\}$.
Proof. $\left(P_{1}\right)$ : Let there exists $x \in M-\{0,1\}$ such that $0 \oplus x=\{x\}$, by the contrary. Then by Theorem 3.5, $M$ is an $M V$-algebra and so it is not proper which is a contradiction. Hence $0 \oplus x \neq\{x\}$, for all $x \in M-\{0,1\}$. Thus there is $y \in 0 \oplus x$ and $y \neq x$. By Lemma 3.1 (i), we imply that $y \in\{0,1\}$. Thus $y=0$ or $y=1$. If $y=1$, then $1 \in 0 \oplus x$ and so $1 \ll x$. Hence, by Proposition 2.2 (vi), $x=1$ which is a contradiction with $x \in M-\{0,1\}$. Thus, $y=0 \in 0 \oplus x$, for all $x \in M-\{0,1\}$. Therefore, $0 \oplus x=\{0, x\}$, for all $x \in M-\{0,1\}$.
$\left(P_{2}\right)$ : Let $x \in 0 \oplus 1$. Then by $\left(P_{1}\right), 0 \in 0 \oplus x \subseteq 0 \oplus(0 \oplus 1)=(0 \oplus 0) \oplus 1=$ $0 \oplus 1$. Hence, $0 \in 0 \oplus 1$.
$\left(P_{3}\right)$ : Let $x, y \in M-\{0,1\}$ be two distinct elements. By $\left(P_{1}\right)$ and (HMV1),

$$
x \in 0 \oplus x \subseteq(0 \oplus y) \oplus x=0 \oplus(x \oplus y)
$$

Then there exists $t \in x \oplus y$ such that $x \in 0 \oplus t$. Thus $x \ll t$ and so $1 \in x \oplus t$. Now, by Lemma 3.2(i), we get $x \in x \oplus y$ or $1 \in x \oplus y$. But $1 \in x \oplus y=x^{*} \oplus y$ implies $x=y$, which is a contradiction. Therefore, $x \in x \oplus y$. By the similar way, $y \in x \oplus y$.
$\left(P_{4}\right):$ Since $0 \ll 0,1 \in 0^{*} \oplus 0$. By $\left(H M V_{4}\right)$ and $\left(P_{1}\right), 1 \in 0^{*} \oplus 0 \subseteq$ $(0 \oplus x)^{*} \oplus 0=(1 \oplus x)^{*} \oplus x$. By Lemma 3.2(i), $1 \in(1 \oplus x)^{*}$ or $x \in(1 \oplus x)^{*}$. Therefore, $0 \in 1 \oplus x$ or $x \in 1 \oplus x$. So $1 \oplus x \neq\{1\}$.
$\left(P_{5}\right)$ : Let $y \notin 0 \oplus 1$. On the contrary, if $y \in 1 \oplus x$, then by $\left(P_{1}\right),\left(P_{3}\right)$ and (HMV4), $y \in y \oplus x=y^{*} \oplus x \subseteq(1 \oplus x)^{*} \oplus x=(x \oplus 0)^{*} \oplus 0=\{0, x\}^{*} \oplus 0=$
$(1 \oplus 0) \cup(x \oplus 0)=(0 \oplus 1) \cup\{0, x\}$. Thus $y \in 0 \oplus 1$, which is a contradiction. Hence $y \notin 1 \oplus x$.
$\left(P_{6}\right)$ : Let $x \notin 0 \oplus 1$. If $0 \oplus 1 \neq\{1\}$, then by $\left(P_{2}\right), 0 \in 0 \oplus 1$ and so by (HMV1) and ( $P_{1}$ ),
$\{0, x\}=0 \oplus x \subseteq(1 \oplus 0) \oplus x=1 \oplus(0 \oplus x)=1 \oplus\{0, x\}=(1 \oplus 0) \cup(1 \oplus x)$.
Now, since $x \notin 0 \oplus 1$, hence $x \in 1 \oplus x$. If $0 \oplus 1=\{1\}$, then by routine calculations, we get $1 \oplus x=\{0, x, 1\} \ni x$.
$\left(P_{7}\right)$ : Let $y \in 1 \oplus x$ and $y \neq\{0, x\}$. Then $y=1$ or $y \neq 0,1$ and $y \neq x$. If $y=1$, then by Proposition 2.2(v), $y=1 \in 0 \oplus 1$. Now, let $x, y \in M-\{0,1\}$ be distinct and $y \notin 0 \oplus 1$, by the contrary. Then by $\left(P_{5}\right), y \notin 1 \oplus x$, which is a contradiction. Hence $y \in 0 \oplus 1$ i.e. $(1 \oplus x) \backslash\{0, x\} \subseteq 0 \oplus 1$. So $1 \oplus x \subseteq(0 \oplus 1) \cup\{0, x\}$.
$\left(P_{8}\right):$ Let $x \in 1 \oplus 1$. Then by $\left(P_{1}\right)$ and (HMV1), $x \in 0 \oplus x \subseteq 0 \oplus(1 \oplus 1)=$ $(0 \oplus 1) \oplus 1$. Thus there is $t \in 0 \oplus 1$ such that $x \in t \oplus 1$. By $\left(P_{7}\right)$, $x \in 1 \oplus t \subseteq(0 \oplus 1) \cup\{0, t\}$. Thus $x \in 0 \oplus 1$ or $x=t$. We note that $x=t$ means $x=t \in 0 \oplus 1$. Hence $x \in 0 \oplus 1$ and so $1 \oplus 1 \subseteq 0 \oplus 1$.
$\left(P_{9}\right)$ : Let $y \in x \oplus x$ and $y \neq x$. Then by (HMV4) and $\left(P_{1}\right)$,
$y \in(x \oplus x) \cup(1 \oplus x)=\{x, 1\} \oplus x=\{0, x\}^{*} \oplus x=(0 \oplus x)^{*} \oplus x=(x \oplus 1)^{*} \oplus 1$.
Hence there is $t \in 1 \oplus x$ such that $y \in 1 \oplus t^{*}$. We note that $t=1$ or $t=0$ or $t \in M-\{0,1\}$.

If $t=1$, then $y \in 1 \oplus 1^{*}=1 \oplus 0$. If $t=0$, then $y \in 1 \oplus 1$ and so by $\left(P_{8}\right)$, $y \in 1 \oplus 1 \subseteq 0 \oplus 1$.

If $t \in M-\{0,1\}$, then by $\left(P_{7}\right), y \in 1 \oplus t \subseteq(0 \oplus 1) \cup\{0, t\}$. Thus $y \in 0 \oplus 1$ or $y=t$. If $y=t$, then $y=t \in 1 \oplus x$. Again by $\left(P_{7}\right), y \in 1 \oplus x \subseteq(0 \oplus 1) \cup\{0, x\}$. Since $x$ and $y$ are distinct and $y \neq 0$, we get $y \in 0 \oplus 1$ in all cases. Therefore, $x \oplus x \backslash\{0, x, 1\} \subseteq 0 \oplus 1$ and so, by $1 \in 0 \oplus 1, x \oplus x \subseteq(0 \oplus 1) \cup\{0, x\}$.
$\left(P_{10}\right)$ : Since $x \ll x$, we get $1 \in x \oplus x$. By (HMV1), we get

$$
0 \oplus 1 \subseteq 0 \oplus(x \oplus x)=(0 \oplus x) \oplus x=\{0, x\} \cup(x \oplus x)
$$

Hence $0 \oplus 1 \backslash\{0, x\} \subseteq x \oplus x$.
$\left(P_{11}\right)$ : Since $0 \ll x$, we conclude that $0^{*} \in 1 \oplus x$. By (HMV4),
$0 \oplus 1 \subseteq(x \oplus 1)^{*} \oplus 1=(0 \oplus x)^{*} \oplus x=\{0, x\}^{*} \oplus x=(x \oplus x) \cup(1 \oplus x)$

Also, if $x \in 0 \oplus 1$, then by $\left(P_{2}\right)$ we get $0 \in 0 \oplus 1$. Thus $\{0, x\} \subseteq 0 \oplus 1 \subseteq$ $(x \oplus x) \cup(1 \oplus x)$. If $x \notin 0 \oplus 1$, then by $\left(P_{6}\right), x \in 1 \oplus x$. Hence, by $\left(P_{1}\right)$, (HMV1) and (1), $\{0, x\}=0 \oplus x \subseteq 0 \oplus(1 \oplus x)=(0 \oplus x) \oplus 1=\{0, x\} \oplus 1=$ $(0 \oplus 1) \cup(x \oplus 1) \subseteq(x \oplus x) \cup(x \oplus 1)$.
$\left(P_{12}\right)$ : By $\left(P_{7}\right),\left(P_{9}\right)$ and $\left(P_{11}\right)$, the proof is clear.
$\left(P_{13}\right):$ Let $z \in x \oplus y$. Then $1 \in z \oplus z^{*}=z \oplus z \subseteq(x \oplus y) \oplus z=x \oplus(y \oplus z)$. By Lemma 3.2(i), $1 \in y \oplus z$ or $x \in y \oplus z$. If $1 \in y \oplus z$, then by Lemma 3.2(i), $y=z$, that is a contradiction. Therefore, $x \in y \oplus z$.
$\left(P_{14}\right):$ Let $z \in x \oplus y, z \in 0 \oplus 1$ and $x \notin 0 \oplus 1$, by the contrary. By $\left(P_{13}\right)$, $z \in x \oplus y$, implies $x \in y \oplus z$. Since $z \in 0 \oplus 1$, we get $x \in z \oplus y \subseteq(0 \oplus 1) \oplus y=$ $0 \oplus(1 \oplus y)$. Since $x \notin 0 \oplus 1$, by Lemma 3.2(ii), $x \in 1 \oplus y$. Also, by $\left(P_{5}\right)$, $x \notin 1 \oplus y$ which is a contradiction. So, $x \in 0 \oplus 1$. Similarly, $y \in 0 \oplus 1$. Therefore, $x, y, z \in 0 \oplus 1$.

Now, let $z \notin 0 \oplus 1$ and $x \in 0 \oplus 1$, by the contrary. Then $z \in x \oplus y \subseteq$ $(0 \oplus 1) \oplus y=0 \oplus(1 \oplus y)$ and so by Lemma 3.2(ii), $z \in 1 \oplus y$. Also, by $\left(P_{5}\right), z \notin 1 \oplus y$, which is a contradiction. So $x \notin 0 \oplus 1$. Similarly, $y \notin 0 \oplus 1$. Therefore, $x, y, z \notin 0 \oplus 1$.
$\left(P_{15}\right): \operatorname{By}\left(P_{3}\right), x, y \in x \oplus y$ and so $x \oplus y=x^{*} \oplus y \subseteq(x \oplus y)^{*} \oplus y=$ $\left(x^{*} \oplus y\right)^{*} \oplus y=x \vee y$. Also by $\left(P_{14}\right)$

$$
(0 \oplus 1) \backslash\{0, y\} \subseteq y \oplus y=y^{*} \oplus y \subseteq(x \oplus y)^{*} \oplus y=\left(x^{*} \oplus y\right)^{*} \oplus y=x \vee y
$$

Since $y \in x \oplus y \subseteq x \vee y$, it is enough to show that $0 \in x \vee y$ :
By $\left(P_{12}\right),(x \oplus x) \cup(1 \oplus x)=(0 \oplus 1) \cup\{0, x\}$. Thus $0 \in x \oplus x$ or $0 \in 1 \oplus x$. If $0 \in x \oplus x$, then $0 \in x \oplus x=x^{*} \oplus x \subseteq(y \oplus x)^{*} \oplus x=\left(y^{*} \oplus x\right)^{*} \oplus x=x \vee y$. So $0 \in x \vee y$ and the proof is complete.
If $0 \in 1 \oplus x$, then by $1 \in y \oplus y, 0 \in x \oplus 1 \subseteq x \oplus(y \oplus y)=(x \oplus y) \oplus y$. It means that there is $t \in x \oplus y$ such that $0 \in t \oplus y$. We note that by Lemma 3.1(i), $1 \notin x \oplus y$ and so $t \neq 1$. If $t=0$, then $0 \in x \oplus y \subseteq x \vee y$ and the proof is complete. Otherwise, $t \in M-\{0,1\}$ and $0 \in t \oplus y=t^{*} \oplus y \subseteq(x \oplus y)^{*} \oplus y=$ $x \vee y$. So, $0 \in x \vee y$ in all cases. Now, we get

$$
(x \oplus y) \cup(0 \oplus 1) \cup\{0\} \subseteq x \vee y
$$

$\left(P_{16}\right)$ : Let $x, y \notin 0 \oplus 1$. Since by $\left(P_{3}\right), y \in x \oplus y$, we conclude that $x \oplus y \subseteq x \oplus(x \oplus y)$. Now, let $t \in x \oplus(x \oplus y)$ be arbitrary. Then there is $u \in x \oplus y$ such that $t \in x \oplus u$. Since $x, y \notin 0 \oplus 1$ by $\left(P_{14}\right)$, we get $u \notin 0 \oplus 1$.

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Again since $x, u \notin 0 \oplus 1$, we conclude that $t \notin 0 \oplus 1$. Also, we have

$$
\begin{aligned}
x \oplus(x \oplus y) & =(x \oplus x) \oplus y \subseteq((0 \oplus 1) \cup\{0, x\}) \oplus y \quad \text { by }\left(P_{9}\right) \\
& =((0 \oplus 1) \oplus y) \cup\{0, y\} \cup(x \oplus y) \quad \text { by }\left(P_{1}\right) \\
& =(0 \oplus(1 \oplus y)) \cup\{0, y\} \cup(x \oplus y) \quad \text { by }(H M V 1) \\
& \subseteq(0 \oplus((0 \oplus 1) \cup\{0, y\})) \cup\{0, y\} \cup(x \oplus y) \quad \text { by }\left(P_{7}\right) \\
& =(0 \oplus 0 \oplus 1) \cup(0 \oplus 0) \cup(0 \oplus y) \cup\{0, y\} \cup(x \oplus y) \\
& =(0 \oplus 1) \cup\{0, y\} \cup(x \oplus y) \quad \text { by Proposition } 2.2(\mathrm{v}) \\
& =(0 \oplus 1) \cup\{0\} \cup(x \oplus y) \quad \text { by }\left(P_{3}\right)
\end{aligned}
$$

Since $t \notin 0 \oplus 1$, we get $t \in(x \oplus y) \cup\{0\}$ and so $x \oplus(x \oplus y) \subseteq(x \oplus y) \cup\{0\}$. Therefore, $x \oplus(x \oplus y) \cup\{0\}=(x \oplus y) \cup\{0\}$.

Lemma 3.7. For all distinct elements $x, y \in M-\{0,1\}$,
(i) $x \vee y=(x \oplus y) \cup(0 \oplus 1) \cup\{0\}$,
(ii) $0 \vee x=1 \vee x=(0 \oplus 1) \cup\{0, x\}$,
(iii) $1 \vee 1=0 \oplus 1$ and $0 \vee 0=\left\{\begin{array}{cl}\{0\} & \text { if } 0 \oplus 1=\{1\}, \\ 0 \oplus 1 & \text { if } 0 \oplus 1 \neq\{1\},\end{array}\right.$
(iv) $x \vee x=\left\{\begin{array}{cc}(0 \oplus 1 \backslash\{1\}) \cup\{0, x\} & \text { if } 0, x \notin x \oplus x, \\ (0 \oplus 1) \cup\{0, x\} & \text { otherwise },\end{array}\right.$
(v) $0 \vee 1=0 \oplus 1$.

Proof. (i) By $\left(P_{15}\right),(0 \oplus 1) \cup(x \oplus y) \cup\{0\} \subseteq x \vee y$. We note that by Lemma 3.1(i), $1 \notin x \oplus y$ and so $0 \in x \oplus y$ or $x \oplus y \subseteq M-\{0,1\}$. Thus $(x \oplus y)^{*} \subseteq(x \oplus y \backslash\{0\}) \cup\{1\}$. Now, we get

$$
\begin{aligned}
& x \vee y \\
= & (x \oplus y)^{*} \oplus y \subseteq((x \oplus y \backslash\{0\}) \cup\{1\}) \oplus y \\
\subseteq & ((x \oplus y) \oplus y) \cup(1 \oplus y)=(x \oplus(y \oplus y)) \cup(1 \oplus y), \text { by }(H M V 1) \\
\subseteq & x \oplus((0 \oplus 1) \cup\{0, y\}) \cup(1 \oplus y), \text { by }\left(P_{9}\right) \\
= & (x \oplus(0 \oplus 1)) \cup(x \oplus 0) \cup(x \oplus y) \cup(1 \oplus y) \\
\subseteq & ((x \oplus 0) \oplus 1) \cup\{0, x\} \cup(x \oplus y) \cup(0 \oplus 1) \cup\{0, y\}, \text { by }\left(P_{1}\right) \text { and }\left(P_{7}\right) \\
= & (0 \oplus 1) \cup(x \oplus 1) \cup\{0, x\} \cup(x \oplus y) \cup\{0, y\} \\
\subseteq & (0 \oplus 1) \cup\{0, x, y\} \cup(x \oplus y) \quad \text { by }\left(P_{7}\right) \\
= & (0 \oplus 1) \cup\{0\} \cup(x \oplus y) \quad \text { by }\left(P_{3}\right) .
\end{aligned}
$$

Therefore, $x \vee y=(0 \oplus 1) \cup(x \oplus y) \cup\{0\}$.
(ii) $\mathrm{By}\left(P_{1}\right), 0 \vee x=(x \oplus 0)^{*} \oplus 0=\{0, x\}^{*} \oplus 0=\{1, x\} \oplus 0=(1 \oplus 0) \cup$ $(x \oplus 0)=(0 \oplus 1) \cup\{0, x\}$. Also, by $\left(P_{12}\right), 1 \vee x=(0 \oplus x)^{*} \oplus x=\{1, x\} \oplus x=$ $(1 \oplus x) \cup(x \oplus x)=(0 \oplus 1) \cup\{0, x\}=0 \vee x$.
(iii) If $0 \oplus 1=\{1\}$, then $0 \vee 0=(1 \oplus 0)^{*} \oplus 0=1^{*} \oplus 0=0 \oplus 0=\{0\}$ and $1 \vee 1=(0 \oplus 1)^{*} \oplus 1=0 \oplus 1$. If $0 \oplus 1 \neq\{1\}$, then by $\left(P_{2}\right), 0 \in 0 \oplus 1$ and so by Lemma 3.2(iii), $(0 \oplus 1)^{*}=0 \oplus 1$. Thus

$$
0 \vee 0=(1 \oplus 0)^{*} \oplus 0=(1 \oplus 0) \oplus 0=(0 \oplus 0) \oplus 1=0 \oplus 1 .
$$

Also, by $\left(P_{8}\right)$, we get

$$
1 \vee 1=(0 \oplus 1)^{*} \oplus 1=(0 \oplus 1) \oplus 1=0 \oplus(1 \oplus 1) \subseteq 0 \oplus(0 \oplus 1)=(0 \oplus 0) \oplus 1=0 \oplus 1 .
$$

Since $0^{*}=1 \in 0 \oplus 1$, we have $0 \in(0 \oplus 1)^{*}$ and so $0 \oplus 1 \subseteq(0 \oplus 1)^{*} \oplus 1=1 \vee 1$. Therefore, in the two cases, $1 \vee 1=0 \oplus 1$.
(iv) At the first, we prove that $1 \in x \vee x$ if and only if $0 \in x \oplus x$ or $x \in x \oplus x$. If $1 \in x \vee x=(x \oplus x)^{*} \oplus x$, then there is $z \in x \oplus x$, such that $1 \in z^{*} \oplus x$. By Lemma 3.2(i), $z^{*}=1$ or $z^{*}=x$. Thus $z=0 \in x \oplus x$ or $z=x \in x \oplus x$. Conversely, if $0 \in x \oplus x$ or $x \in x \oplus x$, then since $x \ll 1$ and $x \ll x$, we conclude that $1 \in x \oplus 1$ and $1 \in x \oplus x$. So

$$
\begin{aligned}
& 1 \in x \oplus 1=x \oplus 0^{*} \subseteq x \oplus(x \oplus x)^{*}=x \vee x \\
& 1 \in x \oplus x=x^{*} \oplus x \subseteq(x \oplus x)^{*} \oplus x=x \vee x
\end{aligned}
$$

Thus $1 \in x \vee x$, for two cases.
Now, let $t \in x \vee x=(x \oplus x)^{*} \oplus x$. Then there is $u \in x \oplus x$ such that $t \in u^{*} \oplus x$. If $u=0$, then by $\left(P_{7}\right), t \in 1 \oplus x \subseteq(0 \oplus 1) \cup\{0, x\}$. If $u=1$, then by $\left(P_{1}\right)$, $t \in 0 \oplus x=\{0, x\} \subseteq(0 \oplus 1) \cup\{0, x\}$. If $u \in M-\{0,1\}$, then we get

$$
\begin{aligned}
t \in u \oplus x & \subseteq(x \oplus x) \oplus x \subseteq((0 \oplus 1) \cup\{0, x\}) \oplus x \quad \text { by }\left(P_{9}\right) \\
& =((0 \oplus 1) \oplus x) \cup(0 \oplus x) \cup(x \oplus x) \\
& =((0 \oplus x) \oplus 1) \cup\{0, x\} \cup(x \oplus x) \quad \text { by }(H M V 1) \text { and }\left(P_{1}\right) \\
& \subseteq(0 \oplus 1) \cup(x \oplus 1) \cup\{0, x\} \quad \text { by }\left(P_{9}\right) \\
& \subseteq(0 \oplus 1) \cup\{0, x\} \quad \text { by }\left(P_{7}\right) .
\end{aligned}
$$

Hence $t \in(0 \oplus 1) \cup\{0, x\}$ in all cases. Therefore, $x \vee x \subseteq(0 \oplus 1) \cup\{0, x\}$. Conversely, let $t \in(0 \oplus 1) \backslash\{0, x, 1\}$. Then by $\left(P_{10}\right), t \in x \oplus x$. By $\left(P_{3}\right)$, $t \in t \oplus x=t^{*} \oplus x \subseteq(x \oplus x)^{*} \oplus x=x \vee x$. Thus $(0 \oplus 1) \backslash\{0, x, 1\} \subseteq x \vee x$. On the other hand, $\{0, x\}=0 \oplus x=1^{*} \oplus x \subseteq(x \oplus x)^{*} \oplus x=x \vee x$. Hence,

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$(0 \oplus 1 \backslash\{1\}) \cup\{0, x\} \subseteq x \vee x$.
(v) $0 \vee 1=\left(1^{*} \oplus 0\right)^{*} \oplus 0=(0 \oplus 0)^{*} \oplus 0=0^{*} \oplus 0=1 \oplus 0$.

Lemma 3.8. For all distinct element $x, y, z \in M-\{0,1\}$,
(i) if $x, y \in 0 \oplus 1$, then $0 \vee x=1 \vee x=x \vee y=0 \oplus 1$ and $x \vee x=0 \oplus 1 \backslash\{1\}$ or $0 \oplus 1$,
(ii) $\quad$ if $x \in 0 \oplus 1$ and $y \notin 0 \oplus 1$, then $x \vee y=(0 \oplus 1) \cup\{y\}$ and $y \vee y=$ $(0 \oplus 1) \cup\{0, y\}$,
(iii) $\quad(x \oplus y) \vee z=x \vee(y \oplus z)=(0 \oplus 1) \cup\{0\} \cup((x \oplus y) \oplus z)$.

Proof. (i) Let $x, y \in 0 \oplus 1$. Since, by $\left(P_{2}\right), 0 \in 0 \oplus 1$, by Lemma 3.7, $0 \vee x=1 \vee x=0 \oplus 1$ and $x \vee x=0 \oplus 1 \backslash\{1\}$ or $0 \oplus 1$. Moreover, since, by $\left(P_{14}\right), x \oplus y \subseteq 0 \oplus 1$, by Lemma 3.7, $x \vee y=(x \oplus y) \cup(0 \oplus 1) \cup\{0\}=0 \oplus 1$.
(ii) Let $x \in 0 \oplus 1$ and $y \notin 0 \oplus 1$. Then by $\left(P_{2}\right), 0 \in 0 \oplus 1$. Also, by Lemma 3.2 (i) and assumption we get $1 \notin x \oplus y$. By $\left(P_{14}\right)$, we imply that $z \notin x \oplus y$ for all distinct elements $x, y, z \in M-\{0,1\}$. Hence, by $\left(P_{3}\right), x \oplus y=\{x, y\}$ or $x \oplus y=\{0, x, y\}$. So, by Lemma 3.7(i), $x \vee y=(x \oplus y) \cup(0 \oplus 1) \cup\{0\}=$ $(0 \oplus 1) \cup\{y\}$.
(iii) By Lemma 3.7(i) and (HMV1) we get

$$
\begin{aligned}
(x \oplus y) \vee z & =\bigcup_{t \in x \oplus y} t \vee z=\bigcup_{t \in x \oplus y}(0 \oplus 1) \cup\{0\} \cup(t \oplus z) \\
& =(0 \oplus 1) \cup\{0\} \cup((x \oplus y) \oplus z) \\
& =(0 \oplus 1) \cup\{0\} \cup(x \oplus(y \oplus z)) \\
& =\bigcup_{u \in y \oplus z}(0 \oplus 1) \cup\{0\} \cup(x \oplus u) \\
& =\bigcup_{u \in y \oplus z} x \vee u=x \vee(y \oplus z) .
\end{aligned}
$$

We note that, since by $\left(P_{3}\right), x, y \in x \oplus y$ and by Lemma 3.7, $z \vee z \subseteq 0 \vee z=$ $1 \vee z \subseteq x \vee z$, we can suppose that $z \neq t \in M-\{0,1\}$, without loss of generality (similarly, $x \neq u \in M-\{0,1\}$ ).

Lemma 3.9. For all distinct elements $x, y, z \in M-\{0,1\}$ we have
(i) $0 \vee(0 \oplus 1)=0 \oplus 1=1 \vee(0 \oplus 1)$;
(ii) $(0 \oplus 1) \vee x=0 \vee x=1 \vee x=\left\{\begin{array}{cc}(0 \oplus 1), & \text { if } x \in 0 \oplus 1, \\ (0 \oplus 1) \cup\{0, x\}, & \text { otherwise. }\end{array}\right.$
(iii) $x \vee(x \vee z)=(0 \oplus 1) \cup\{0\} \cup(x \oplus(x \oplus z))$;
(iv) $0 \vee(y \oplus z)=(0 \oplus 1) \cup\{0\} \cup(y \oplus z)$.

Proof. (i) If $0 \oplus 1=\{1\}$, then by Lemma 3.7(v), $0 \vee(0 \oplus 1)=0 \vee 1=0 \oplus 1$. If $0 \oplus 1=\{0,1\}$, then by Lemma 3.7(iii), $0 \vee(0 \oplus 1)=(0 \vee 0) \cup(0 \vee 1)=$ $(0 \oplus 1) \cup(0 \oplus 1)=0 \oplus 1$. Otherwise, there exists $x \in M-\{0,1\}$ such that $x \in 0 \oplus 1$ and so by $\left(P_{2}\right), 0 \in 0 \oplus 1$. Thus

$$
\begin{aligned}
0 \vee(0 \oplus 1) & =\bigcup_{t \in 0 \oplus 1}(0 \vee t)=(0 \vee 0) \cup(0 \vee 1) \cup \bigcup_{t \in(0 \oplus 1)-\{0,1\}}(0 \vee \dot{t}) \\
& =(0 \oplus 1) \cup(0 \oplus 1) \cup(0 \oplus 1) \cup\{0, \hat{t}\}, \text { by Lemma 3.7(ii),(iii) } \\
& =0 \oplus 1 \quad \text { Since } 0, \hat{t} \in 0 \oplus 1 .
\end{aligned}
$$

Therefore, $0 \vee(0 \oplus 1)=0 \oplus 1$, for all cases. Similarly, we can prove $1 \vee(0 \oplus 1)=0 \oplus 1$.
(ii) If $0 \oplus 1=\{1\}$ or $0 \oplus 1=\{0,1\}$, then by the similar way of (i) and using Lemma $3.7(\mathrm{ii})$, we get $(0 \oplus 1) \vee x=(0 \oplus 1) \cup\{0, x\}$. Let there exists $s \in M-\{0,1\}$ such that $s \in 0 \oplus 1$. Then by $\left(P_{2}\right), 0 \in 0 \oplus 1$. Now, if $x \notin 0 \oplus 1$, then we get

$$
\begin{aligned}
(0 \oplus 1) \vee x & =\bigcup_{t \in 0 \oplus 1}(t \vee x)=(0 \vee x) \cup(1 \vee x) \cup \bigcup_{x \neq t^{\prime} \in(0 \oplus 1)-\{0,1\}}\left(t^{\prime} \vee x\right) \\
& =(0 \oplus 1) \cup\{0, x\} \cup(0 \oplus 1) \cup\{x\}, \text { by Lemmas 3.7(ii), 3.8(ii) } \\
& =(0 \oplus 1) \cup\{0, x\}=0 \vee x=1 \vee x .
\end{aligned}
$$

For the case $x \in 0 \oplus 1$, the proof is similar. Therefore, $(0 \oplus 1) \vee x=$ $(0 \oplus 1) \cup\{0, x\}$, for all cases.
The proof of (iii) and (iv) is routine.
Theorem 3.10. Let $M$ be finite. Then for all $x, y, z \in M$,

$$
(x \vee y) \vee z=x \vee(y \vee z) \quad \text { and } \quad(x \wedge y) \wedge z=x \wedge(y \wedge z)
$$

Proof. Case 1: Let $x, y, z \in M$ such that $x=z$. Then by commutativity of " $\vee$ ", we get

$$
x \vee(y \vee x)=(y \vee x) \vee x=(x \vee y) \vee x .
$$

Case 2: Let $x, y, z \in M-\{0,1\}$ be distinct elements and $x \notin 0 \oplus 1$ (for $x \in 0 \oplus 1$ the proof is similar). Then

$$
\begin{aligned}
x \vee(y \vee z) & =x \vee((0 \oplus 1) \cup\{0\} \cup(y \oplus z) \\
& =(\underbrace{(x \vee(0 \oplus 1)) \cup(x \vee 0) \cup(x \vee(y \oplus z)), \text { by Lemma 3.7(i) }} \\
& =(0 \oplus 1) \cup\{0, x\} \cup(x \vee(y \oplus z)), \text { by Lemmas3.9(ii), 3.7(ii) } \\
& =(0 \oplus 1) \cup\{0\} \cup(x \oplus y \oplus z), \text { by Lemma 3.8(iii) and }\left(P_{3}\right) .
\end{aligned}
$$

By the similar way, we get $(x \vee y) \vee z=(0 \oplus 1) \cup\{0\} \cup((x \oplus y) \oplus z)$. Therefore, $x \vee(y \vee z)=(x \vee y) \vee z$.

Case 3: Let $x, y, z \in M-\{0,1\}$ such that $x=y$. Then
(i) if $x, z \in 0 \oplus 1$, then by Lemma 3.8(i) and Lemma 3.9(ii), $x \vee(x \vee z)=$ $x \vee(0 \oplus 1)=0 \oplus 1$. Also, it is routine to see that $(x \vee x) \vee z=0 \oplus 1$. Therefore, $x \vee(x \vee z)=0 \oplus 1=(x \vee x) \vee z$.
(ii) If $x \in 0 \oplus 1$ and $z \notin 0 \oplus 1$, then by Lemma 3.8(ii) and Lemma 3.9(ii), $x \vee(x \vee z)=x \vee((0 \oplus 1) \cup\{z\})=(x \vee(0 \oplus 1)) \cup(x \vee z)=(0 \oplus 1) \cup(0 \oplus 1) \cup\{z\}=$ $(0 \oplus 1) \cup\{z\}$. Since by Lemma 3.8(i), $x \vee x=(0 \oplus 1) \backslash\{1\}$ or $0 \oplus 1$, we get $(x \vee x) \vee z=(0 \oplus 1) \cup\{z\}=x \vee(x \vee z)$ in both cases. Thus $x \vee(x \vee z)=(x \vee x) \vee z$.
(iii) If $x, z \notin 0 \oplus 1$, then

$$
\begin{aligned}
& x \vee(x \vee z) \\
&= x \vee((0 \oplus 1) \cup\{0\} \cup(x \oplus z)) \\
&= \underbrace{(x \vee(0 \oplus 1)) \cup(x \vee 0) \cup(x \vee(x \oplus z)), \text { by Lemma 3.7(i) }}= \\
&=(0 \oplus 1) \cup\{0, x\} \cup(x \vee(x \oplus z)), \text { by Lemma 3.9(ii), 3.7(ii) } \\
&=(0 \oplus 1) \cup\{0, x\} \cup(0 \oplus 1) \cup\{0\} \cup(x \oplus(x \oplus z), \text { by Lemma 3.9(iii) } \\
&=(0 \oplus 1) \cup\{0\} \cup(x \oplus(x \oplus z) \\
&=(0 \oplus 1) \cup(x \oplus z) \cup\{0\}, \text { by }\left(P_{3}\right) \text { and }\left(P_{16}\right) .
\end{aligned}
$$

Also, we have,

$$
\begin{aligned}
& (x \vee x) \vee z \\
= & ((0 \oplus 1) \cup\{0, x\}) \vee z \\
= & \underbrace{(0}_{((0 \oplus 1) \vee z) \cup(0 \vee z)} \cup(x \vee z), \text { by Lemma 3.8(ii) } \\
= & (0 \oplus 1) \cup\{0, z\} \cup(0 \oplus 1) \cup\{0\} \cup(x \oplus z), \text { by Lemmas 3.9(ii), 3.7(ii) } \\
= & (0 \oplus 1) \cup\{0\} \cup(x \oplus z), \text { by } P_{3} .
\end{aligned}
$$

Thus $x \vee(x \vee z)=(x \vee x) \vee z$. For the case $x, y, z \in M-\{0,1\}$ such that $y=z$, the proof is similar.

Case 4: Let $x \in\{0,1\}$ and $y, z \in M-\{0,1\}$ be distinct elements. We suppose $x=0$ (for $x=1$ the proof is similar)

$$
\begin{aligned}
0 \vee(y \vee z) & =0 \vee((0 \oplus 1) \cup\{0\} \cup(y \oplus z)), \text { Lemma 3.7(i) } \\
& =(0 \vee(0 \oplus 1)) \cup(0 \vee 0) \cup(0 \vee(y \oplus z)) \\
& =(0 \oplus 1) \cup\{0\} \cup(0 \vee(y \oplus z)), \text { by Lemma 3.9(i) } \\
& =(0 \oplus 1) \cup\{0\} \cup(y \oplus z), \text { by Lemma 3.9(iv). }
\end{aligned}
$$

On the other hand,
(i) If $z \notin 0 \oplus 1$, then

$$
\begin{aligned}
& (0 \vee y) \vee z \\
= & ((0 \oplus 1) \cup\{0, y\}) \vee z \\
= & \underbrace{((0 \oplus 1) \vee z) \cup(0 \vee z)} \cup(y \vee z), \text { by Lemma 3.7(ii) } \\
= & (0 \oplus 1) \cup\{0, z\} \cup(0 \oplus 1) \cup\{0\} \cup(y \oplus z), \text { by Lemmas 3.9(ii), 3.7(ii),(i) } \\
= & (0 \oplus 1) \cup\{0\} \cup(y \oplus z), \text { by Lemma 3.7(i), (ii) and }\left(P_{3}\right) .
\end{aligned}
$$

(ii) If $z \in 0 \oplus 1$, then by Lemma 3.9(ii) and Lemma 3.8(i), this is routine to see that $(0 \vee y) \vee z=(0 \oplus 1) \cup\{0\} \cup(y \oplus z)$. Hence, $(0 \vee y) \vee z=0 \vee(y \vee z)$ for any cases. If $y \in\{0,1\}$ and $x, z \in M-\{0,1\}$ or $z \in\{0,1\}$ and $x, y \in M-\{0,1\}$, then we can prove by the similar way.

Case 5: Let $x \in\{0,1\}$ and $y, z \in M-\{0,1\}$ such that $y=z$. Suppose that $x=0$ (for $x=1$ the proof is similar).
(i) If $y \notin 0 \oplus 1$, then

$$
\begin{aligned}
(0 \vee y) \vee y & =((0 \oplus 1) \cup\{0, y\}) \vee y \\
& =\underbrace{((0 \oplus 1) \vee y) \cup(0 \vee y)} \cup(y \vee y), \text { by Lemma 3.7(ii) } \\
& =(0 \oplus 1) \cup\{0, y\} \cup(y \vee y) \quad \text { by Lemmas 3.9(ii), 3.7(ii) } \\
& =(0 \oplus 1) \cup\{0, y\}, \text { by Lemma 3.8(ii). }
\end{aligned}
$$

Moreover, by Lemma 3.9(i) and Lemma 3.8(ii),

$$
\begin{aligned}
0 \vee(y \vee y) & =0 \vee((0 \oplus 1) \cup\{0, y\}=(0 \vee(0 \oplus 1)) \cup(0 \vee 0) \cup(0 \vee y) \\
& =(0 \oplus 1) \cup\{0, y\}=(0 \vee y) \vee y .
\end{aligned}
$$

(ii) If $y \in 0 \oplus 1$, then it is routine to see that $0 \vee(y \vee y)=0 \oplus 1=(0 \vee y) \vee y$. Similarly, $x=y \in M-\{0,1\}, z \in\{0,1\}$ or $x=z \in M-\{0,1\}, y \in\{0,1\}$

Vol. LIV (2016) Relationship Between Hyper $M V$-algebras and Hyperlattices91 can be proved.

Case 6: Let $x, y \in\{0,1\}$ and $z \in M-\{0,1\}$. Suppose $x=y=0$ (for $x=1$ or $y=1$ the proof is similar).

$$
\begin{aligned}
0 \vee(0 \vee z) & =0 \vee((0 \oplus 1) \cup\{0, z\}) \\
& =(0 \vee(0 \oplus 1)) \cup(0 \vee 0) \cup(0 \vee z), \text { by Lemma 3.7(ii) } \\
& =(0 \oplus 1) \cup\{0\} \cup(0 \oplus 1) \cup\{0, z\} \\
& =(0 \oplus 1) \cup\{0, z\}, \text { by Lemma } 3.9(\mathrm{i}) \\
& =0 \vee x=(0 \vee 0) \vee z
\end{aligned}
$$

Similarly, $x, z \in\{0,1\}, y \in M-\{0,1\}$ or $y, z \in\{0,1\}, x \in M-\{0,1\}$ can be proved.

Case 7: Let $x, y, z \in\{0,1\}$. Suppose $x=y=0, z=1$ and $0 \oplus 1=\{1\}$. Then by Lemas 3.7 (v) and 3.9(i), we get $0 \vee(0 \vee 1)=0 \vee(0 \oplus 1)=0 \oplus 1=$ $0 \vee 1=(0 \vee 0) \vee 1$. For other cases the proof is similar.

Finally, by definition of $\wedge$, it can easily prove that $x \wedge(y \wedge z)=(x \wedge y) \wedge$ $z$.

Definition 3.2. [11] Let $L$ be a nonempty set endowed with hyperoperations $\wedge$ and $\vee$. Then $(L, \wedge, \vee)$ is called a hyperlattice if for any $a, b, c \in L$, the following conditions are satisfied:
(i) $a \in a \wedge a, a \in a \vee a$;
(ii) $a \wedge b=b \wedge a, a \vee b=b \vee a$;
(iii) $(a \wedge b) \wedge c=a \wedge(b \wedge c),(a \vee b) \vee c=a \vee(b \vee c)$;
(iv) $a \in a \wedge(a \vee b), a \in a \vee(a \wedge b)$.

Corollary 3.11. If $M$ is a finite hyper $M V$-algebra that satisfies the (SNP), then $M$ is a hyperlattice.

Proof. By Theorem 3.10 and Proposition 2.4, the proof is clear.

Corollary 3.12. Any finite hyper MV-algebra of the orders 2 and 3, satisfies the (SNP), and so is a hyperlattice.

Computer Check: All hyper $M V$-algebras of orders 4, 5 and 6, are hyperlattices.

Example 3.2. (i) Let $M=\{0, a, b, 1\}$ and hyperoperation $\oplus$ and unary operation $*$ on $M$ are defined as follows;

| $\oplus$ | 0 | a | b | 1 |
| :---: | :---: | :---: | :---: | ---: |
| 0 | $\{0\}$ | $\{0, \mathrm{a}\}$ | $\{\mathrm{b}\}$ | $\{\mathrm{b}, 1\}$ |
| a | $\{0, \mathrm{a}\}$ | $\{0, \mathrm{a}\}$ | $\{\mathrm{b}, 1\}$ | $\{\mathrm{b}, 1\}$ |
| b | $\{\mathrm{b}\}$ | $\{\mathrm{b}, 1\}$ | $\{\mathrm{b}, 1\}$ | $\{\mathrm{b}, 1\}$ |
| 1 | $\{\mathrm{~b}, 1\}$ | $\{\mathrm{b}, 1\}$ | $\{\mathrm{b}, 1\}$ | $\{\mathrm{b}, 1\}$ |


| $*$ | 0 | a | b | 1 |
| :--- | :--- | :--- | :--- | :--- |
|  | 1 | b | a | 0 |

Then by routine calculations $(M, \oplus, *, 0)$ is a hyper $M V$-algebra, which dose not satisfies the (SNP.) But $(M, \vee, \wedge)$ is a hyperlattice.
(ii) Let $M=[0,1]$. We define unary operetion " *" and hyperoperation " $\oplus$ " on $M$ by $x^{*}=1-x$ and $x \oplus y=[0, \min \{1, x+y\}]$. Then $(M, \oplus, *, 0)$ is a hyper $M V$-algebra. It is easy to see that $x^{*} \neq x$ for any $x \neq \frac{1}{2}$ i.e. $M$ is not satisfied (SNP). But by routine calculation we get $x \vee y=[0,1]$ and so $x \vee(y \vee z)=[0,1]=(x \vee y) \vee z$ for all $x, y, z \in M$. By the similar way, $x \wedge(y \wedge z)=(x \wedge y) \wedge z$, for all $x, y, z \in M$. Hence $(M, \vee, \wedge)$ is a hyperlattice.
Note: In Corollary 3.11, the condition "finite with (SNP)" is sufficient but it is not necessary. Indeed, we have not found any finite or infinite hyper $M V$-algebra, which is not a hyperlattice.

Open problem: Any hyper $M V$-algebra is a hyperlattice.

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R. A. Borzooei

Department of Mathematics, Shahid Beheshti University Tehran, Iran
E-mail: borzooei@sbu.ac.ir

Akefe Radfar
Department of Mathematics, Payame Noor University
p.o.box 19395-3697

Tehran, Iran
E-mail: Radfar@pnu.ac.ir

Sogol Niazian
Tehran Medical Sciences Branch, Islamic Azad University
Tehran, Iran
E-mail: s.niazian@iautmu.ac.ir
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