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Relationship between state-space and input-output models via observer Markov parameters

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Abstract

This paper describes the relationship between two types of commonly used models in control and identification theory: state-space and input-output models. The relationship between the two model structures can be explained in terms of a newly formulated set of parameters called the observer Markov parameters. This is different from the usual connection between the two model structures via well-known canonical realizations. The newly defined observer Markov parameters generalize the standard system Markov parameters by incorporating information of an associated observer. In the deterministic case, the observer Markov parameters subsume a state-space model and a deadbeat observer gain. In the stochastic case, the observer Markov parameters contain information of an optimal observer such as a Kalman filter. The observer Markov parameters can be identified directly from experimental input-output data. Making use of the new connection, one can extract from the identified observer Markov parameters not only a state-space model of the system, but also an associated observer gain for use in modern state-space based feedback controller design.

1 Introduction

Those who design attitude (and shape) control systems for flexible spacecraft need a structural dynamics model of the vehicle as the basis for developing a design. Certain parts of control theory, such as classical digital control and adaptive control, often use higher order difference equation models, or input-output models. Other control approaches make use of discrete-time state-space models, and the identification methods for these usually start from pulse response histories, i.e. the Markov parameters, and these must be found from the input-output data. Certain new identification methods have been developed by the authors and co-workers, which have proved very successful in structural dynamics testing at NASA. These methods find the Markov parameters of an observer (in certain cases, a Kalman filter) from input-output data rather than finding the Markov parameters of the system, and then from these they obtain a state-space model of the system. This paper develops a bridge between the model types, presenting certain intimate relationships between input-output models, state-space models, system Markov parameters, and observer Markov parameters.

Discrete-time state-space models and input-output models both describe the sampled-data input-output relationship of a dynamic system. The state-space representation describes the relationship in terms of an intermediate variable, the system state. The state-variable approach allows one to elegantly address issues such as controllability, observability, and optimal control, etc. In contrast with state-space based control is adaptive control, which is based on input-output models. In an input-output model, the output variable is expressed directly in terms of the input variable, bypassing the intermediate state variable. A commonly used input-output model is the auto-regressive model with exogenous input (ARX), where the system output at the current time step is expressed as a linear combination of a certain number of past input and output values. Since adaptive control does not assume a priori knowledge of the system, on-line identification is made part of the control approach. As a result, input-output models are preferred over state-space models in adaptive control because the relationship between input-output data and the coefficients of an input-output model is linear. On the other hand, the relationship between input-output data and the state-space model parameters is non-linear. Due to the different types of models used, state-space and adaptive control methods tend to have different flavors, although in principle one would expect significant connections between the two control approaches.

To relate control approaches based on the two types of models, one needs to understand how an input-output model is related to a state-space model and vice versa. This connection is important not only in the area of control but also in the area of system identification. It would allow, for example, the extraction of a state-space model from the coefficients of an identified input-output model. Canonical realization allows one to express an input-output model in the form of an equivalent state-space model and vice versa. This kind of relationship between the two models is significant for theoretical reasons, but it is rather abstract, and not often used in practice. In this paper, we will describe another way to relate the two model structures via a new set of parameters, which are called observer Markov parameters. Among other things, it will be shown that the coefficients of an ARX model are related to those of a state-space model through an observer gain. Since the coefficients of the ARX model can be identified directly from input-output data, this result

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implies that not only a state-space model of the system, but also an associated observer gain can be identified simultaneously from input-output data. The identified model and observer can then be used directly in an observer-based controller design. This is in contrast to the traditional approach where system identification and observer design are two separate processes. In the presence of process and measurement noise, the identified observer, in a certain sense, characterizes the noise inputs as they are actually embedded in the input-output data. This is in contrast to a designed observer, which is based on an assumed model and assumed characteristics of the noise present in the system.

The paper starts out with various ways to describe the dynamics of a linear system using input-output or state-space models. The observer Markov parameters will be used to relate the two model structures. In the deterministic case, the relationship between the coefficients of the commonly used ARX model and the state-space model via a deadbeat observer gain will be explained. The key relationship between the observer Markov parameters and the system Markov parameters will be derived. This relationship allows one to extract from the observer Markov parameters the system Markov parameters and an associated observer gain. In the stochastic case, conditions for which the observer Markov parameters subsume an optimal Kalman filter will be described. A connection between the coefficients of another common input-output model known as the auto-regressive moving average model with exogenous input (ARMAX) and the state-space model will be made in terms of a deadbeat observer gain, and an optimal Kalman filter gain. Knowledge of the system Markov parameters is sufficient to deduce a state-space model of the system using realization theory. Such an algorithm will be briefly described. Throughout the paper, the role of the observer Markov parameters in connecting various types of models, both in the deterministic and stochastic cases, will be clearly seen.

2 Deterministic models

In this section we will examine various deterministic input-output models and their connections to state-space models. It will be clear that the system Markov parameters are the coefficients of the pulse response model and the observer Markov parameters are the coefficients of the ARX model.

2.1 Pulse response model and state-space model

For a continuous-time system governed by a linear differential equation, if the response to a Dirac delta impulse is known, then its response to any other general input can be determined by the convolution integral. Similarly, for a discrete-time system governed by a linear difference equation, we have the Kronecker unit pulse sequence defined as

$$\delta(k) = \begin{cases} 1 & , \ k = 0 \\ 0 & , \ k \neq 0 \end{cases} \tag{1}$$

The unit pulse reponse sequence, denoted by $h_0, h_1, h_2, \ldots$, is obtained by seperately applying this unit pulse sequence to each input, holding the others zero. The system input-output relationship, assuming zero initial conditions, is given by

$$y(k) = h_0 u(k) + h_1 u(k-1) + h_2 u(k-2) + \cdots + h_k u(0) \tag{2}$$

For a system with q outputs and m inputs, each h_i is a $q \times m$ matrix. The contribution to the current output $y(k)$ by the current input $u(k)$ and past input $u(k-1)$, $u(k-2)$, ... is weighted by the pulse response sequence. For this reason this input-output description is also known as the weighting sequence description.

One basic drawback of the pulse response model is that the number of parameters in the model, h_i, $i = 0,1,\ldots,k$, increases with k. Another drawback is that it ignores the effects of the initial conditions. To see why this is the case, we examine a state-space representation of the same system, which takes the general form of a first-order matrix difference equation

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{3}$$

where the dimensions of A, B, C, and D are $n \times n$, $n \times m$, $q \times n$, and $q \times m$, respectively. The $n \times 1$ vector $x(k)$ denotes the state of the system. Let $x(0)$ denote the initial state, solving for the output $y(k)$ in terms of the previous input values yields

$$y(k) = CA^k x(0) + Du(k) + CBu(k-1) + CABu(k-2) + \cdots + CA^{k-1}Bu(0) \tag{4}$$

The coefficient D is sometimes referred to as the direct transmission term, and the products $CA^{k-1}B$ are known as the system Markov parameters

$$Y(0) = D \ , \quad Y(k) = CA^{k-1}B \tag{5}$$

Comparing Eq. (4) with Eq. (2) yields

$$h_0 = Y(0) \ , \quad h_k = Y(k) \tag{6}$$

if $x(0) = 0$. Thus the system Markov parameters are precisely the coefficients of the pulse response model. They are unique for a given system. For an asymptotically stable system, the effect of the initial condition becomes negligible after some sufficiently large number of time steps, say p_s. The system input-output map can then be approximated by a finite number of Markov parameters,

$$y(k) \approx h_0 u(k) + h_1 u(k-1) + h_2 u(k-2) + \cdots + h_{p_s} u(k-p_s) \quad , \; k \geq p_s \tag{7}$$

where $CA^k B \approx 0,\ k \geq p_s$. For a lightly damped system, however, this model is still cumbersome because of the necessarily large number of coefficients needed to make the approximation valid.

2.2 ARX model and observer model

A commonly used input-output model is known as the auto-regressive model with exogenous input or ARX model for short. This model assumes the general form

$$y(k) = \alpha_1 y(k-1) + \cdots + \alpha_p y(k-p) + \beta_0 u(k) + \beta_1 u(k-1) + \cdots + \beta_p u(k-p) \tag{8}$$

which states that the current input value $y(k)$ is a linear combination of the current input value $u(k)$, past p input values $u(k-1)$, ..., $u(k-p)$, and past p output values $y(k-1)$, ..., $y(k-p)$. This model is considerably more convenient than the pulse response model because it has a fixed number of parameters. It is often used in adaptive control and estimation theory.[1,2]

In the following we will describe how this model is related to the original state-space model given in Eq. (3). To this end, adding and subtracting the term $My(k)$ to the right hand side of the state equation in (3) yields

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + My(k) - My(k) \\ &= (A+MC)x(k) + (B+MD)u(k) - My(k) \end{aligned} \tag{9}$$

If a matrix M exists such that $(A+MC)^p$ vanishes identically, i.e.,

$$(A+MC)^k \equiv 0 \;, \quad k \geq p \tag{10}$$

then for $k \geq p$, the output $y(k)$ can be expressed in the form of an ARX model, where the coefficients α_k, β_k are related to the state-space model as

$$\begin{aligned} \alpha_k &= -C(A+MC)^{k-1} M \;, \quad k = 1,2,\ldots,p \\ \beta_k &= C(A+MC)^{k-1}(B+MD) \;, \quad \beta_0 = D \end{aligned} \tag{11}$$

The matrix M in the above development can be interpreted as an observer gain for the following reason. The original state-space model in Eqs. (3) has an observer of the form

$$\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) - M[y(k) - \hat{y}(k)] \\ \hat{y}(k) &= C\hat{x}(k) + Du(k)\end{aligned} \tag{12}$$

where $\hat{x}(k)$ is the estimated state. Even in the absence of noise, $\hat{y}(k)$ differs from $y(k)$ if the actual initial condition $x(0)$ is not known, and some different initial condition is assumed for $\hat{x}(0)$. Defining the state estimation error $e(k) = x(k) - \hat{x}(k)$, the equation that governs $e(k)$ is

$$e(k+1) = (A + MC)e(k) \tag{13}$$

Thus the estimated state $\hat{x}(k)$ will converge to the true state $x(k)$ as k tends to infinity provided the matrix $A + MC$ is asymptotically stable. Furthermore, if M is such that $(A+MC)^k \equiv 0,\ k \geq p$, then the state estimation error will vanish identically after a finite number of time steps (deadbeat observer gain). The observer equation can be re-written in the form

$$\begin{aligned}\hat{x}(k+1) &= (A+MC)\hat{x}(k) + [B+MD,\ -M]\begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \\ \hat{y}(k) &= C\hat{x}(k) + Du(k)\end{aligned} \tag{14}$$

The combinations $\overline{Y}(k) = C(A+MC)^{k-1}[B+MD\ ,\ -M] = [\beta_k \ \ \alpha_k]$ are the Markov parameters of the observer. Hence they are referred to as observer Markov parameters.[3] To simplify the notation, we define

$$\overline{A} = A + MC, \quad \overline{B} = [B + MD\ ,\ -M] \tag{15}$$

so that the observer Markov parameters can be expressed as

$$\overline{Y}(k) = C\overline{A}^{k-1}\overline{B}\ , \qquad k = 1, 2, \ldots, p \tag{16}$$

and $\overline{Y}(0) = D$.

For system identification, the significance of the above result is as follows. The observer mechanism compresses an infinite number of system Markov parameters into exactly p observer Markov parameters. The observer Markov parameters can be identified from input-output data because they are precisely the coefficients of an ARX model of order p. If the identified observer Markov parameters can be unscrambled, then a state-space model of the system and an associated observer gain can be recovered. The unscrambling process is indeed possible and will be described in Sections 3 and 4 later.

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Consider the case where M is a deadbeat observer gain and the system is deterministic (no noise is present). The observer Markov parameters become identically zero after a finite number of terms since $(A+MC)^p$ vanishes identically. The input-output relationship given by the model in Eq. (8) becomes exact for $k \geq p$ regardless of any non-zero initial state $x(0)$. This is an advantage over the pulse response model using system Markov parameters. From linear system theory, the existence of such an observer gain is assured as long as the system is observable. From the point of view of system identification, the observability requirement is automatically satisfied because only the observable (and controllable) part of the system can be identified from input-output measurements. The number of observer Markov parameters is the same as the order of the ARX model. For a multivariable system with q outputs, p must be chosen such that it is at least $p_{\min}$ (Section 3). $p_{\min}$ is the minimum value of p such that $qp_{\min} \geq n$, where n is the order of the state-space model. This result implies that an n-th order (observable) state-space model with multiple outputs can be represented by an ARX model with p less than n. For example, a 2-output system with three lightly damped flexible modes, $(n=6,\ q=2)$, can be described exactly by 3 observer Markov parameters, $p=3$. If the system Markov parameters are used instead, one would need, say 100-1000 of them, and yet the description is only approximate (the actual number depends on the sampling rate and the amount of damping in the system). Of course, it is also possible to represent the same system by more than the minimum number of observer Markov parameters corresponding to an over-parameterized ARX model. In fact, over-parameterization is desirable in the stochastic case. It will be shown later that if p is taken to be sufficiently large relative to n then the observer subsumed in the identified ARX model approximates an optimal Kalman filter. Otherwise, for p to remain "small", the same Kalman filter will be subsumed in a different model instead. This model turns out to be an ARMAX model. This issue will be addressed in Section 5. The extra freedom inherent in the observer Markov parameters (due to the observer gain M) can be used to develop various identification algorithms, including the Observer/Kalman filter Identification method (OKID) for open-loop identification [4-7], and the Observer/Controller Identification (OCID) for closed-loop identification.[8]

3 Recovery of system Markov parameters from observer Markov parameters

The coefficients of an ARX model can be identified from input-output data. These coefficients are the observer Markov parameters, which contain information about the system and an observer. As mentioned previously, it is possible to extract the system state-space model and an associated observer gain from the observer Markov parameters. In this section we describe how the system Markov parameters can be recovered uniquely from a set of observer Markov parameters. The system Markov parameters will allow us to realize a state-space model of the system eventually. First, the direct transmission term D is simply $Y(0)=\beta_0$. The Markov parameter $Y(1)=CB$ is simply

$$Y(1)=CB=\beta_1+\alpha_1 Y(0) \tag{17}$$

The next Markov parameter $Y(2)=CAB$ is obtained from

$$Y(2)=CAB=\beta_2+\alpha_1 Y(1)+\alpha_2 Y(0) \tag{18}$$

Similarly,

$$\begin{aligned} Y(3) &= CA^2B \\ &= \beta_3 - CMCAB - C(A+MC)MCB - C(A+MC)^2 MD \\ &= \beta_3+\alpha_1 Y(2)+\alpha_2 Y(1)+\alpha_3 Y(0) \end{aligned} \tag{19}$$

In general, the system Markov parameters can be recovered from the observer Markov parameters according to the following relationship

$$Y(k)=\beta_k+\sum_{i=1}^{k}\alpha_k Y(k-i) \tag{20}$$

where $\alpha_k=0,\ \beta_k=0$ for $k>p$. The above recursive equation can be written in matrix form as

$$\begin{bmatrix} I & & & & \\ -\alpha_1 & I & & & \\ -\alpha_2 & -\alpha_1 & \ddots & & \\ \vdots & \vdots & \ddots & I & \\ -\alpha_{k-1} & -\alpha_{k-2} & \cdots & -\alpha_1 & I \end{bmatrix} \begin{bmatrix} Y(1) \\ Y(2) \\ Y(3) \\ \vdots \\ Y(k) \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} \alpha_1\beta_0 \\ \alpha_2\beta_0 \\ \alpha_3\beta_0 \\ \vdots \\ \alpha_k\beta_0 \end{bmatrix} \tag{21}$$

The left most matrix in the above equation is square and full rank, which implies that from a given set of observer Markov parameters, the system Markov parameters can be uniquely recovered.

Furthermore, let the matrices $\overline{\mathbf{Y}}_\alpha$ and $\mathbf{Y}$ be defined as follows

$$\begin{aligned} \overline{\mathbf{Y}}_\alpha &= \begin{bmatrix} \alpha_p & \alpha_{p-1} & \alpha_{p-2} & \cdots & \alpha_1 \end{bmatrix} \\ \mathbf{Y} &= \begin{bmatrix} Y(p+2) & Y(p+3) & Y(p+4) & \cdots & Y(p+N+1) \end{bmatrix} \end{aligned} \tag{22}$$

where N is an arbitrary integer. Then from Eq. (20)

$$\overline{\mathbf{Y}}_\alpha \mathbf{H}(1) = \mathbf{Y} \tag{23}$$

where

$$\mathbf{H}(1) = \begin{bmatrix} Y(2) & Y(3) & Y(4) & \cdots & Y(N+1) \\ Y(3) & Y(4) & Y(5) & \cdots & Y(N+2) \\ Y(4) & Y(5) & Y(6) & \cdots & Y(N+3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y(p+1) & Y(p+2) & Y(p+3) & \cdots & Y(N+p) \end{bmatrix} \tag{24}$$

Using the definitions of the system Markov parameters, the $qp \times Nm$ matrix $\mathbf{H}(1)$ can be expressed as

$$\mathbf{H}(1) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{p-1} \end{bmatrix} A \begin{bmatrix} B & AB & A^2B & \cdots & A^{N-1}B \end{bmatrix} = V_{p-1} A W_{N-1} \tag{25}$$

where A is the system matrix for a state-space model, V_{p-1} is an observability matrix, and W_{N-1} is a controllability matrix. The rank of a sufficiently large (size) matrix $\mathbf{H}(1)$ is the order n of the (controllable and observable) system. If $Nm \geq qp$, the maximum rank of $\mathbf{H}(1)$ is qp. Thus, for a system of order n, the number of observer Markov parameters p must be such that $qp \geq n$ where q is the number of outputs. Furthermore, the maximum order of a system that can be described with p observer Markov parameters is qp. For a multiple-output system, therefore, the number of observer Markov parameters required to be identified can be less than the true order of the system. The minimum number of observer Markov parameters that can describe the system is $p_{\min}$, which is the smallest value of p such that $qp_{\min} \geq n$.

4 Recovery of observer gain from observer Markov parameters

Since the observer Markov parameters contain information of both the system and an observer gain, it is possible to recover the observer gain M from the observer Markov parameters as well. First, the sequence

$$Y_M(k) = CA^{k-1}M \ , \quad k = 1, 2, \ldots \tag{26}$$

can be computed from the observer Markov parameters as follows

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$$Y_M(k) = -\alpha_k + \sum_{i=1}^{k-1} \alpha_i Y_M(k-i) \tag{27}$$

The parameters $Y_M(k)$ contain information about the observer gain, therefore they may be referred to as observer gain Markov parameters. Given A, C, and a sufficient number of $Y_M(k)$, the observer gain M can be obtained from

$$M = \left(V_k^T V_k\right)^{-1} V_k^T Y_M \tag{28}$$

where

$$V_k = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^k \end{bmatrix} \qquad Y_M = \begin{bmatrix} CM \\ CAM \\ \vdots \\ CA^k M \end{bmatrix}$$

provided V_k is of rank n. Alternatively, the observer gain matrix M can be realized simultaneously with the system matrices A, B, C from the combined Markov parameters

$$\begin{aligned} P(k) &= \left[CA^{k-1}B \quad CA^{k-1}M\right] \\ &= CA^{k-1}\left[B \quad M\right], \qquad k = 1,\ 2,\ \ldots \end{aligned} \tag{29}$$

using realization theory. This problem is discussed in Section 6.

5 Stochastic models

This section examines stochastic input-output models as they are related to state-space models in the presence of noise. In particular, we consider the following two input-output models which can be connected to a Kalman filter.

5.1 Stochastic ARX model

Consider the case where the state-space equations given in Eq. (3) are extended to include process and measurement noise

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w_1(k) \\ y(k) &= Cx(k) + w_2(k) \end{aligned} \tag{30}$$

The process noise $w_1(k)$ and measurement noise $w_2(k)$ are two statistically independent, zero-mean, stationary white noise processes with covariances Q

and R, respectively. If A is stable and (A,C) is an observable pair, the same system can also be expressed in the form of a Kalman filter [9]

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) - K\varepsilon(k) \\ y(k) &= C\hat{x}(k) + Du(k) + \varepsilon(k) \end{aligned} \tag{31}$$

where the optimal residual $\varepsilon(k)$ is white with covariance $\Sigma = CPC^T + R$, where P is the unique positive definite symmetric solution of the algebraic Riccati equation

$$P = APA^T - APC^T\left(CPC^T + R\right)^{-1}CPA^T + Q \tag{32}$$

The Kalman filter gain K is given by $K = -APC^T\Sigma^{-1}$. The Kalman filter in Eq. (31) can also be expressed as

$$\begin{aligned} \hat{x}(k+1) &= (A + KC)\hat{x}(k) + (B + KD)u(k) - Ky(k) \\ y(k) &= C\hat{x}(k) + Du(k) + \varepsilon(k) \end{aligned} \tag{33}$$

If $\bar{p}$ is sufficiently large such that

$$\bar{A}^k = (A + KC)^k \approx 0 \qquad k \geq \bar{p} \tag{34}$$

then the input-output description can be approximated by

$$\begin{aligned} y(k) &= \tilde{\alpha}_1 y(k-1) + \cdots + \tilde{\alpha}_p y(k-p) \\ &+ \tilde{\beta}_0 u(k) + \tilde{\beta}_1 u(k-1) + \cdots + \tilde{\beta}_p u(k-p) + \varepsilon(k) \end{aligned} \tag{35}$$

where

$$\begin{aligned} \tilde{\alpha}_k &= -C(A + KC)^{k-1}K \ , \quad k = 1,2,\ldots,\bar{p} \\ \tilde{\beta}_k &= C(A + KC)^{k-1}(B + KD) \ , \quad \tilde{\beta}_0 = D \end{aligned} \tag{36}$$

Observe that for a sufficiently large $\bar{p}$ (such that the condition given in Eq. (34) holds) this model has the same internal structure as the ARX model considered previously in Eq. (1) where the Kalman filter gain K now plays the role of the observer gain M. It is therefore possible to recover the Kalman filter gain from the coefficients of an identified ARX model if p is chosen to be sufficiently large. An important advantage of the structure in Eq. (35) is that the residual $\varepsilon(k)$ appears as an additive term. In the identification of the coefficients $\tilde{\alpha}_k, \tilde{\beta}_k$, $k = 1,2,\ldots,p$, this residual can be minimized by a simple least squares solution. On the other hand, the major disadvantage of this model

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is that a large value of p may be needed. Theoretically, p is required to be several times larger than the order n of the system, $p >> n$, so that Eq. (34) holds. In practice, however, one may obtain a fairly accurate result with a much smaller value of p if the noise level is low. The relationship between the ARX model and a state-space model developed here has been used in the Observer/Kalman filter identification algorithm (OKID) which identifies from experimental input-output data a state-space model of the system together with an associated observer/Kalman filter gain. This technique bypasses the need to obtain estimates of the process and measurement noise covariances as normally required if one has to design a Kalman filter instead of identifying it.

5.2 ARMAX model

It has been shown in the previous section that for an ARX model to subsume an optimal Kalman filter, p must be sufficiently large. In this section, a more general stochastic input-output model is considered where p need not be as large yet the model still subsumes a Kalman filter. To derive such a model, add and subtract the term $My(k)$ to the right hand side of Eq. (31),

$$\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + My(k) - My(k) - K\varepsilon(k) \\ &= A\hat{x}(k) + Bu(k) + M\left[C\hat{x}(k) + Du(k) + \varepsilon(k)\right] - My(k) - K\varepsilon(k) \\ &= (A+MC)\hat{x}(k) + (B+MD)u(k) - My(k) + (M-K)\varepsilon(k)\end{aligned} \tag{37}$$

If M exists such that $(A+MC)^p$ vanishes identically, then for $k \geq p$, the output $y(k)$ can be expressed as

$$\begin{aligned}y(k) = \alpha_1 y(k-1) + \cdots + \alpha_p y(k-p) + \beta_0 u(k) + \beta_1 u(k-1) + \cdots + \beta_p u(k-p) \\ + \gamma_1 \varepsilon(k-1) + \cdots + \gamma_p \varepsilon(k-p) + \varepsilon(k)\end{aligned} \tag{38}$$

where the coefficients α_k, β_k of this input-output model are related to the state-space model as

$$\begin{aligned}\alpha_k &= -C(A+MC)^{k-1}M \;, \quad k = 1,2,\ldots,p \\ \beta_k &= C(A+MC)^{k-1}(B+MD) \;, \quad \beta_0 = D \\ \gamma_k &= C(A+MC)^{k-1}(M-K)\end{aligned} \tag{39}$$

This stochastic model turns out to be an auto-regressive moving average with exogenous input model (ARMAX). It is widely used in adaptive estimation, filtering, and control literature. Equations (39) provide an interpretation of the coefficients of the ARMAX model in terms of a state-space model $A,B,C,$ a

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deadbeat observer gain M, and an optimal Kalman filter gain K. This model has the advantage that $p << \tilde{p}$. As discussed previously, for an observable system, the minimum value of p is p_{min}, which is the smallest value of p such that $qp_{min} \geq n$ where q is the number of outputs and n is the order of the state-space model. However, this advantage is off-set by the introduction of the noise dynamics which causes the identification problem to become non-linear because both γ_k and $\varepsilon(k)$ are not known. The identification of such a model using a generalized least-squares approach (OKID with residual whitening) is formulated in Ref. 6. The method identifies from experimental input-output data a state-space model of the system, a deadbeat observer gain, and an optimal Kalman filter gain without requiring a large value of p as in the original OKID algorithm.

The system Markov parameters, $Y(k) = CA^{k-1}B$, and the observer gain Markov parameters, $Y_M(k) = CA^{k-1}M$, can be recovered from the coefficients α_k, β_k as in Eq. (20), and Eq. (27), respectively. Furthermore, it is possible to recover the Kalman filter gain Markov parameters $Y_K(k) = CA^{k-1}K$ from α_k, γ_k

$$Y_K(k) = -\alpha_k + \sum_{i=1}^{k-1} \alpha_i Y_K(k-i) - \gamma_k \tag{40}$$

where $\alpha_k = 0$ for $k > p$. The Kalman filter gain K can be estimated from A, C, and a sufficient number of recovered Kalman filter gain Markov parameters. Alternately, application of a realization procedure to the sequence of combined Markov parameters $P(k)$ defined as

$$\begin{aligned} P(k) &= \begin{bmatrix} Y(k) & Y_M(k) & Y_K(k) \end{bmatrix} \\ &= CA^{k-1}\begin{bmatrix} B & M & K \end{bmatrix} \end{aligned} \tag{41}$$

will realize simultaneously the matrices A, B, C, M, and K. As a final note, compare the ARMAX model in Eq. (38) with the ARX model considered in Eq. (8). In the stochastic case, if the model in Eq. (8) is used, but p is not sufficiently large, then the residual in the model will be colored. The dynamics of this colored residual is modeled by the coefficients γ_k.

6 Realization from Markov parameters

If a sufficiently long sequence of Markov parameters $Y(k) = CA^{k-1}B$ is known, realization theory can be applied to obtain the individual matrices A, B, C from the sequence. One such method is the Eigensystem Realization Algorithm (ERA), which is often used in structural identification.[8,10,11] The same theory can be applied to factorize the sequence $P(k) = CA^{k-1}[B,\ M]$ to obtain A, B, C, M, or the sequence $CA^{k-1}[B,\ M,\ K]$ to obtain A, B, C, M, K simultaneously. In the following, we describe the realization of the sequence

$CA^{k-1}[B,\ M]$. Realization of the sequence $CA^{k-1}[B,\ M,\ K]$ can be carried out similarly.

Define the Hankel matrix of $P(k)$ as

$$\mathbf{H}(k) = \begin{bmatrix} P(k+1) & P(k+2) & \cdots & P(k+s+1) \\ P(k+2) & P(k+3) & \cdots & P(k+s+2) \\ \vdots & \vdots & \vdots & \vdots \\ P(k+r+1) & P(k+r+2) & \cdots & P(k+r+s+1) \end{bmatrix} \tag{42}$$

A minimal order realization of the n-dimensional state-space model $A,B,C,$ and the observer gain M is given as

$$A' = \Sigma_n^{-1/2} P_n^T \mathbf{H}(1) Q_n \Sigma_n^{-1/2}$$
$$[B' \quad M'] = \Sigma_n^{1/2} Q_n^T E_{m+q} \qquad C' = E_q^T P_n \Sigma_n^{1/2} \tag{43}$$

where matrices P_n, Q_n are formed by the n left and right singular vectors in P and Q, corresponding to the n (non-zero) singular values of $\mathbf{H}(0)$

$$\mathbf{H}(0) = P \Sigma Q^T = P_n \Sigma_n Q_n^T \tag{44}$$

and E_α are selection matrices, $E_\alpha^T = [I_{\alpha\times\alpha} \ \ \underline{0} \ \ \cdots \ \ \underline{0}]$, $\alpha = m+q,\ q$. The realized model A', B', C', and realized observer gain M' are equivalent to the original state-space model $A,B,C,$ and the original observer gain $M,$ in the sense that they are related to each other by a similarity transformation. This is due to the non-uniqueness of the state-space representation. A coordinate change in the state vector does not affect the input-output map, nor does it alter the Markov parameters.

An important aspect in connection with system realization from the Markov parameters is the order of the minimal realization. In the noise-free case, given a sufficient number of Markov parameters, the minimal order n is equal to the rank of the Hankel matrix, which is equal to the number of positive singular values in Σ_n. In practice, because all structural systems have noise and non-linearities, the problem of rank determination is not trivial. The singular value decomposition in the above realization procedure is a critical step to determine this order. Certain smaller singular values are attributed to noise, and truncated. The number of retained singular values then determines an "effective" system order. The corresponding retained columns of P and Q in the singular value decomposition of $\mathbf{H}(0)$ are used to obtain a realization.

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7 Concluding remarks

This paper presents a new connection between various standard input-output models and state-space models in both the deterministic and stochastic cases. The coefficients of the ARX and ARMAX models are explained in terms of the state-space model and the associated observer or Kalman filter gains. The newly defined observer Markov parameters are useful in that they provide a new way to view how state-space models and input-output models are related to each other. This is in contrast to the known connection between the two model types via canonical realizations. We have found a mechanism that compresses an infinite number of system Markov parameters into a finite number of observer Markov parameters, which can be identified from input-output data. The identified observer Markov parameters can then be uncompressed to recover the original system Markov parameters, which completely characterize the dynamics of a linear system because they are its pulse response samples. From the Markov parameters, a state-space model of the system can be realized as well.

This connection is also significant because it shows that not only a state-space model can be extracted from the observer Markov parameters, but an observer or Kalman filter gain as well. This additional capability helps link traditional input-output model identification to modern state-space control, because the identified state-space model and its associated observer gain can be used directly in a state feedback control design. State-space models and observers are two extremely fundamental elements of modern control theory. Furthermore, within the domain of modern control itself, the ability to identify an observer from input-output data is also significant. Traditionally, observer design and system identification are two separate processes. One typically designs an observer rather than identifying it. To design an optimal observer one not only needs accurate knowledge of the system model, but accurate statistics of the process and measurement noise as well. In the proposed approach, both the system state-space model and an associated observer gain are identified from input-output data. There is no need to estimate the statistics of the noises, which is a very difficult task in practice. Intuitively, the fact that this is possible should not be too surprising, because a sufficiently large amount of input-output data does in fact contain information about the system as well as the characteristics of the process and measurement noise. It is expected that several new identification techniques for complex flexible structures can be developed by exploiting this fundamental relationship.

References

1. Ljung, L. & Södertröm, T. *Theory and Practice of Recursive Identification*, The MIT Press, Cambridge, Massachusetts, 1983.

2. Goodwin, G.C. & Sin, K.S. *Adaptive Filtering, Prediction, and Control*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1984.

3. Phan, M., Horta, L.G., Juang, J.-N., & Longman, R.W. Linear system identification via an asymptotically stable observer, *Journal of Optimization Theory and Applications*, 1993, **79**, 59-86.

4. Chen, C.-W., Huang, J.-K., Phan, M., & Juang, J.-N. Integrated system identification and modal state estimation for control of large flexible space structures, *Journal of Guidance, Control, and Dynamics*, 1992, **15**, 88-95.

5. Juang, J.-N., Phan, M., Horta, L.G., & Longman, R.W. Identification of observer/Kalman filter Markov parameters: theory and experiments, *Journal of Guidance, Control, and Dynamics*, 1993, **16**, 320-329.

6. Phan, M., Horta, L.G., Juang, J.-N., & Longman, R.W. Improvement of observer/Kalman filter identification (OKID) by residual whitening, *Journal of Vibrations and Acoustics*, 1995, Vol. 117, 232-239.

7. Juang, J.-N. *Applied System Identification*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1994.

8. Juang, J.-N. & Phan, M. Identification of system, observer, and controller from closed-loop experimental data, *Journal of Guidance, Control, and Dynamics*, 1994, **17**, 91-96.

9. Goodwin, G.G. & Payne, R.L. *Dynamic System Identification: Experimental Design and Data Analysis*, Academic Press, New York, 1977.

10. Ho, B.L. & Kalman, R.E. Effective construction of linear state variable models from input/output data, *Proceedings of the 3rd Annual Allerton Conference on Circuits and Systems Theory*, Monticello, Illinois, 1965.

11. Juang, J.-N. & Pappa, R.S. An eigensystem realization algorithm for modal parameter identification and model reduction, *Journal of Guidance, Control, and Dynamics*, Sept.-Oct. 1985, **8**.