Relationship between the p and s Fresnel reflection coefficients of an interface independent of angle of incidence

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The Fresnel reflection coefficients r_p and r_s of p- and s-polarized light at the planar interface between two linear isotropic media are found to be interrelated by $(r_s - r_p)/(1 - r_s r_p) = \cos 2\beta$, independent of the angle of incidence ϕ, where $\tan^2 \beta = \epsilon$ and ϵ is the (generally complex) ratio of dielectric constants of the media of refraction and incidence. This complements another relation (found earlier), $(r_s^2 - r_p)/(r_s - r_s r_p) = \cos 2\phi$, which is valid at a given ϕ independent of ϵ (i.e., for all possible interfaces). Taken together, these two equations specify r_p and r_s completely and can be used to replace the original Fresnel equations.

The complex-amplitude Fresnel reflection coefficients of p- and s-polarized monochromatic light at the planar interface between two linear, homogeneous, isotropic, and nonmagnetic media are given by[1]

$$r_p = \frac{\epsilon \cos \phi - (\epsilon - \sin^2 \phi)^{1/2}}{\epsilon \cos \phi + (\epsilon - \sin^2 \phi)^{1/2}} = f(\epsilon, \phi), \tag{1}$$

$$r_s = \frac{\cos \phi - (\epsilon - \sin^2 \phi)^{1/2}}{\cos \phi + (\epsilon - \sin^2 \phi)^{1/2}} = g(\epsilon, \phi), \tag{2}$$

where ϕ is the angle of incidence and ϵ is the ratio of the dielectric constant of the medium of refraction (which is, in general, complex) to that of the medium of incidence (real).

By eliminating ϵ between Eqs. (1) and (2), we previously obtained[2]

$$r_p = r_s(r_s - \cos 2\phi)/(1 - r_s \cos 2\phi). \tag{3}$$

Equation (3) is a direct relation between r_p and r_s that is valid for all possible interfaces at a given angle of incidence ϕ. Its properties and the insight that one can derive from it are discussed in Refs. 2 and 3.

It is apparent that it should also be possible to eliminate ϕ between Eqs. (1) and (2) and obtain a second direct relation between r_p and r_s that is valid for a given interface (or a given ϵ) at all angles of incidence. After a few algebraic steps,[4] we get

$$(r_s - r_p)/(1 - r_s r_p) = c, \tag{4}$$

where

$$c = (1 - \epsilon)/(1 + \epsilon). \tag{5}$$

Equation (4) is, to our knowledge, new and can be verified by direct substitution from Eqs. (1) and (2). Equation (4) can be rearranged to read as

$$r_p = (r_s - c)/(1 - c r_s) \tag{6}$$

or

$$r_s = (r_p + c)/(1 + c r_p), \tag{7}$$

each of which is in the form of a bilinear transformation. (For a given complex ϵ, as ϕ increases from 0° to 90°, r_p and r_s trace trajectories in the complex plane that are images of each other through such a transformation.)

It is instructive to consider some special cases. At normal incidence ($\phi = 0$), $r_p = -r_s$ (in the Nebraska conventions[5]), and Eq. (4) gives $2r_s = c(1 + r_s^2)$. When this equation is solved for r_s, one retrieves the known result $r_s(0) = (1 - \epsilon^{1/2})/(1 + \epsilon^{1/2})$. At grazing incidence, $r_p = r_s$, and Eq. (4) takes the form $0/(1 - r_s^2) = c \neq 0$, from which one correctly obtains $r_s^2 = 1$.

At $\phi = 45°$, $r_p = r_s^2$; substitution of this result[6] into Eq. (4) gives $r_s/(r_s^2 + r_s + 1) = (1 - \epsilon)/(1 + \epsilon)$. By solving the latter equation for ϵ, one obtains $\epsilon = (1 + r_s^2)/(1 + r_s)^2$, a nice-looking result that has proved to be useful recently.[7]

For a dielectric–dielectric interface, $r_p = 0$ when light is incident at the Brewster angle $\phi = \phi_B$. Setting $r_p = 0$ in Eq. (4) gives

$$r_s(\phi_B) = c = (1 - \epsilon)/(1 + \epsilon). \tag{8}$$

Equation (8) is a simple reduced form of the Fresnel reflection coefficient for the unextinguished s polarization at the Brewster angle. Equation (8) also provides some meaning for the constant c. By combining Eqs. (6) and (8), we obtain

$$r_p(\phi) = \frac{r_s(\phi) - r_s(\phi_B)}{1 - r_s(\phi_B) r_s(\phi)}. \tag{9}$$

Equation (9) is interesting in that it expresses the reflection coefficient of a dielectric–dielectric interface for the p polarization at any angle ϕ in terms of that for the s polarization at the same angle ϕ and at the Brewster angle ϕ_B.

For a dielectric–dielectric interface under conditions of total internal reflection (TIR; $\epsilon < 1$ and $c > 0$), we can write

$$r_p = \exp(j\delta_p), \qquad r_s = \exp(j\delta_s), \tag{10}$$

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where δ_p and δ_s are the phase shifts that the p- and s-polarized components of the electric vector experience on TIR. By substituting Eqs. (10) into Eq. (6) and noting that c is real, one gets after some manipulations

$$\tan \delta_p = \frac{(1 - c^2) \sin \delta_s}{2c + (1 + c^2)\cos \delta_s}. \tag{11}$$

Equation (11) ties directly the TIR phase shifts δ_p and δ_s for a given interface at all angles of incidence, from the critical angle $\phi_c = \sin^{-1} \epsilon^{1/2}$ to grazing incidence $\phi = 90°$.

Before concluding, we note that Eq. (3) can be rewritten as

$$(r_s^2 - r_p)/(r_s - r_s r_p) = \cos 2\phi, \tag{12}$$

which bears resemblence to Eq. (4). To create more symmetry between the two independent relations between r_p and r_s at constant ϵ [Eq. (4)] and at constant ϕ [Eq. (12)], we introduce a (generally complex) angle β such that

$$\epsilon = \tan^2\beta. \tag{13}$$

(It is interesting to note that β reduces to the usual Brewster angle, $\beta = \phi_B$, when ϵ is real, i.e., for an interface between two transparent media.) With Eq. (13), c of Eq. (5) becomes

$$c = \cos 2\beta, \tag{14}$$

and Eq. (4) now reads as

$$(r_s - r_p)/(1 - r_s r_p) = \cos 2\beta. \tag{15}$$

The similarity in structure between Eqs. (12) and (15) is remarkable.[8] These two equations, taken together, specify r_p and r_s completely and can be used to replace the original Fresnel equations.

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1. See, for example, M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1975), Sec. 1.5.2.
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4. The simplest way is to form the ratios $U_\nu = (1 - r_\nu)/(1 + r_\nu)$, where $\nu = p, s$. It follows immediately that $U_s = \epsilon U_p$, from which Eq. (4) is obtained.
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8. With $c = \cos 2\beta$, Eq. (6) also becomes similar to Eq. (3), except for the multiplicative factor r_s that appears in the right-hand side of Eq. (3).