BULL. AUSTRAL. MATH. SOC. 22D05, 22A05
VOL. 13 (1975), 149-151.

Relationships between group topologies

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There are two lines of thought investigated in this thesis, each connecting two group topologies defined on the one group, and each developing one of two central results. For this abstract, let G be a group (with identity 0) on which are defined two group topologies τ_1 and τ_2.

(A) The first result is as follows. Suppose G is divisible, abelian, and has no points (except 0) whose nth-multiple is 0, for some integer n not less than 2. I call a point x of G τ_1-*continuous* if the subsets $\{y : my = x\}$, for positive integers m, eventually intersect each τ_1-neighbourhood of 0. Then let (G, τ_2) be a locally compact and σ-compact group, and denote by ω_2 the outer measure derived from the Haar measure μ_2 on that group. Also suppose that the ratio of the τ_2-measure of $\{nx : x \in A\}$ to the τ_2-measure of A, for any τ_2-Borel-measurable set A (the ratio is the same for any such A with finite measure), does not exceed 1. Then for each τ_2-Borel-measurable set A with nonvoid τ_1-interior, $\mu_2(A) \ge \omega_2(W_1)$, W_1 being the subgroup of all τ_1-continuous points in G.

This result answers the questions posed by Hawley [2] concerning the relationship between a compact group topology and the usual one for the real line (my answers can be found in [6]), which in fact prompted this

Received 7 April 1975. Thesis submitted to La Trobe University, September 1974. Degree approved, March 1975. Supervisor: Dr Graham C. Elton.

line of investigation. The generalised form of the answer ((A) above) and its immediate consequences appear as [8].

An investigation ensues as to the properties of the subgroup of continuous points in a topological group (see [10]), and it turns out that this group bears a strong resemblance to euclidean space. I also find sufficient conditions to make this subgroup negligible in the group. Such an approach can be used to prove known results in topological ring theory ([5], p. 170, Theorem 21), that a locally compact topological division ring is the real line, the complex numbers, or the quaternions (see [10]), and the topological vector space result that the only Hausdorff, locally compact topological vector spaces over the reals are the finite dimensional euclidean spaces. I conclude this first half by demonstrating various things concerning locally compact σ-compact and compact group topologies for the real line, and also for the multiplicative reals under such topologies.

(B) The second central result requires that (G, τ_1) be locally compact, (G, τ_2) be locally countably compact, and there be a Hausdorff topology for G weaker than both τ_1 and τ_2 (that is, that τ_1 and τ_2 be *compatible*). Then if there is a τ_2-open set contained in some τ_1-Lindelöf set, we will have $\tau_1 \subseteq \tau_2$. This result (found in [9]) actually is a generalisation of a result of Kasuga's, which makes τ_1 and τ_2 equal if they are compatible and σ-compact locally compact group topologies ([3], p. 58, 6.19, and [4]).

I use this technique to prove the known results of Bichteler's [1], that, given τ_1 and τ_2 are locally compact group topologies, and the irreducible unitary representations continuous with respect to each of τ_1 and τ_2 coincide, then $\tau_1 = \tau_2$ if

(a) they are comparable, or

(b) there is a normal subgroup of G open and σ-compact in one of the topologies (my proof can be found as [7]).

The question also arises as to what happens if two topologies defined on the one group are not compatible. I show (amongst other similar results) that in certain circumstances the group can be broken into the product of two groups, one on which the two topologies are compatible, and the other along which the functions continuous in both topologies are only the constant functions.

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