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Relative Computational Power of Integrable and Nonintegrable Soliton Systems*

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1 Introduction

The present document is devoted to the question of whether effective computation can be performed by the interaction of solitons [24, 34] in a bulk medium. The resulting computational system would fulfill the promise of Toffoli's "programmable matter" [42] — offering computation that is very close to the underlying physics, and therefore potentially providing ultra-scale parallel processing.

The most immediate physical realization of such computation may be provided by solitons in an optical fiber [41, 20, 15]. Other media are also possible, including Josephson junctions [35] and electrical transmission lines [31, 21].

We should emphasize that using optical solitons in this way is quite different from what is commonly termed "optical computing" [19, 20], which uses optical solitons to construct gates that replace electronic gates, but which remains within the "lithographic" paradigm of laying out gates and wires. The idea here uses a completely homogeneous medium for computation — the entire computation is determined by an input stream of particles. A general version of the structure proposed is shown in Fig. 1.

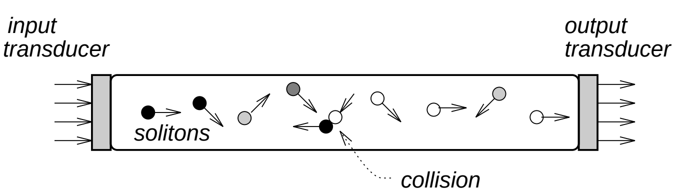

Figure 1: Computing with solitons in a bulk medium. Solitons are injected at the left of the diagram, computation takes place within the medium via the interaction of the pseudoparticles, and the results exit from the right of the diagram. The actual medium can be linear, planar, or three-dimensional.

The idea of using solitons in a homogeneous medium for "gateless" computation goes back at least to [39], where solitons in a cellular automaton (CA) [1] are used to build a carry-ripple adder. A general model called the *particle machine* (PM) for computation using collisions of particles was laid out and studied in [37, 38]. The present paper moves from the abstraction of CA to the physical realm represented by partial differential equations (PDE's) such as the nonlinear Schrödinger, Korteweg-de Vries, and sine-Gordon equations [34].

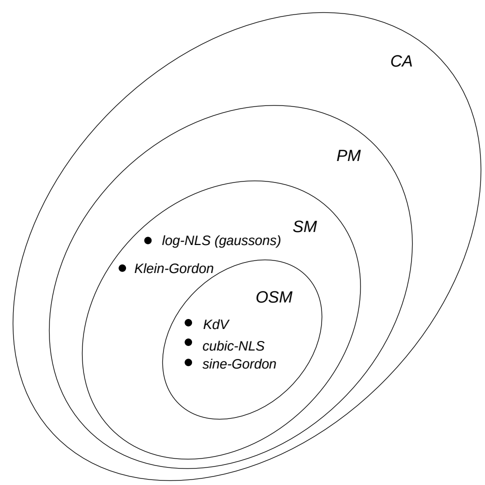

Figure 2: Hierarchy of computational systems in the world of cellular automata (CA). Particle machines (PM's) are CA designed to model particle-supporting physical media. Soliton machines (SM's) are restricted PM's that model general soliton systems, including PDE's such as the Klein-Gordon and log-NLS equations. Oblivious soliton machines (OSM's) are SM's that model integrable soliton systems, such as the KdV, cubic-NLS, and sine-Gordon equations. All these PDE's are described later in this document.

To use physical solitons for computation, we define restricted versions of the PM called *soliton machines (SM's)*. Both PM's and SM's are 1-d cellular automata that model motion and collision of particles in a uniform medium. *Oblivious soliton machines* (OSM's) are SM's further restricted to model a class of integrable soliton

*A full version of this document [22] has been submitted to *Complex Systems*.

[1] These solitons arise in the mathematical framework of a CA [30, 14, 11, 12], and have an entirely different origin than the physically based solitons we consider here. However, CA-based and PDE-based solitons display remarkably similar behavior. As far as we know, the connection between CA solitons and PDE solitons is unexplained, though some authors [1, 29] have juxtaposed discussions of both systems.

systems. The hierarchy of the computational systems we consider is shown in Fig. 2. In general, we abstract a physical system by modeling it first with PDE's, which we then model with CA, namely PM's and SM's, as shown in Fig. 3.

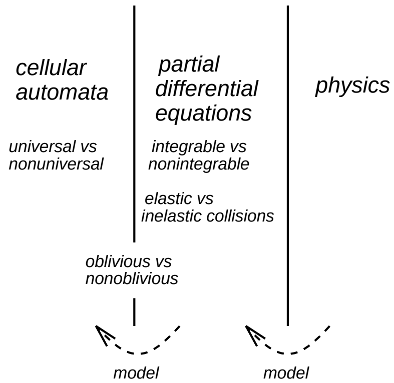

Figure 3: The three worlds considered in this paper. Notice that the property *oblivious* applies to both cellular automata and soliton solutions of partial differential equations, whereas the properties *integrable* and *having elastic collisions* apply only to soliton systems.

We will discuss the computational power of the ideal machines with which we model physical systems. Being able to simulate a Turing machine, or another universal model, is neither necessary nor sufficient for being able to perform useful computation. For example, certain particle machines can perform some very practical regular numerical computations, such as digital filtering, quite efficiently, and yet these PM's are not necessarily universal [37, 38]. Conversely, simulating a Turing machine is a very cumbersome and inefficient way to compute, and any practical application of physical phenomena to computing would require a more flexible computational environment. Nevertheless, universality serves as a guide to the inherent power of a particular machine model.

2 Particle Machines

The *particle machine* (PM) model of computation, introduced and shown to be universal in [37], is an abstract framework for computing with particles. The PM is a general model, not based on any specific physical system, but which tries to capture the properties of physical particles and particle-like phenomena.

Briefly, a PM is a CA with a next-state rule designed to support a set of *particles* propagating with constant velocities in an infinite 1-d medium. Two or more particles may collide; a set of *collision rules* specifies which particles are created, which are destroyed, and which are unaffected in collisions. A PM begins with a finite *initial configuration* of particles and evolves in discrete time steps. A PM, like a CA, can have a *periodic background*; that is, an infinite, periodic sequence of nonzero state values (particles) in the medium of the CA. Periodic backgrounds are sometimes used to add computational power to CA, as in [5].

Particle machines capture and abstract the behavior of particles in systems which may be used for computation. *Soliton machines*, which are restricted PM's that we will define later, bring the abstraction a step closer to physical reality by modeling systems governed by certain well known PDE's. We now describe a class of these PDE's and systems.

3 Integrable Soliton Systems

Certain integrable [2] nonlinear PDE's give rise to *solitons*, or particle-like solitary waves that propagate without decay in homogeneous media and survive collisions with shape and velocity intact. Systems such as the Korteweg-de Vries, sine-Gordon, and cubic nonlinear Schrödinger equations describe the motion and interaction of solitons in shallow water, electrical transmission lines, optical fibers, and other materials [34, 8]. In recent years much effort has been expended on analyzing the properties of solitons for purposes such as high-speed communications and optical computing gates [41, 19, 20]. We will examine issues involved in using solitons to implement SM's.

Non-integrable systems also support soliton-like waves, whose more complex behavior we will describe later. The integrable soliton-supporting equations that we consider in this section have exact soliton solutions, which may be obtained by the inverse scattering method [3]. Nonintegrable equations, and integrable equations with arbitrary initial conditions, must in general be solved numerically.

Later we will show that a certain class of integrable PDE's can do only limited computation using SM's; we conjecture that this is true of all integrable equations. The simple behavior of integrable soliton systems makes them unlikely candidates for useful computing media.

4 Oblivious Soliton Machines

The PM model is a convenient abstraction for computing with solitons. In practice, however, general PM's model much more general behavior than that exhibited by integrable systems. For example, the integrable soliton systems we have described do not support the creation of new solitons or the destruction of existing solitons, and soliton state changes due to collisions are very limited. Thus, we adopt a restricted model of a PM called an *oblivious soliton machine (OSM)*. Like a PM, an OSM is a CA designed

[2] The term *integrable*, referring to PDE's, is not used with perfect consistency throughout the literature. Here we use *integrable* to mean *solvable by the inverse scattering method* [3].

to support particles propagating through a homogeneous medium, but an OSM more closely models the integrable soliton systems under consideration.

4.1 The OSM Model

An OSM is a PM in which each particle has a constant identity and a variable state that are both vectors of real numbers. A particle's velocity is part of its identity. A typical state may consist of a *phase* and a *position relative to a Galilean frame of reference*, whereas a typical identity may include an *amplitude* in addition to a velocity. No particles can be created or destroyed in collisions, and the identities of particles are preserved. A function of the *identities* (not states) of the colliding particles determines particle state changes.

Immediately after a collision, particles are *displaced*. Displacement amounts are functions of the identities of the colliding particles, and particles must be displaced into distinct cells. In addition, we require that once two particles collide, the same particles can never collide again. This scheme models particle interaction in the integrable soliton systems described earlier.

4.2 OSM's Are Not Universal

We refer to OSM's as *oblivious* because the state changes in an OSM do not depend on the variable states of colliding particles, but only on their constant identities. *Oblivious* collisions in the OSM model correspond to *elastic* collisions in the integrable PDE's discussed here; however, it is an open question to the authors whether or not all elastic soliton collisions in all integrable systems are oblivious. The spatial displacements of OSM particles after collisions occur only in the constrained fashion described above. The result of these properties is that OSM's cannot compute universally. In [22] we prove the following theorem by calculating an upper bound on the time taken by an OSM to do any computation:

Theorem 1 *OSM's are not computation-universal, either with or without a periodic background. The maximum time that an OSM can spend performing useful computation is cubic in the size of the input.* ■

Corollary 1 *OSM-based computational systems governed by the KdV and sine-Gordon equations are not universal, given that positions are used as state. OSM-based systems governed by the cubic-NLS equation are not universal, given that positions and phases are used as state.* ■

Conjecture 1 *All integrable systems using any choice of state are non-universal using the OSM model.* ■

5 Soliton Machines

Intuitively, OSM's cannot compute universally because particles in an OSM do not transfer enough state information during collisions. We can make a simple modification to the OSM model so that universal computation becomes possible: We make the results of collisions depend on both the identities and *states* of colliding particles. In addition, we allow particle identities to change. We call the resulting model a *soliton machine* (SM). In the final section of this paper, we will describe non-integrable equations that support soliton-like waves which we believe may be capable of realizing the SM model.

Like an OSM, an SM is also a CA and a PM (see Fig. 2). The only difference between an SM and a PM is that no particles can be created or destroyed in an SM. However, we can use a periodic background of particles in special *inert* or *blank* states, and simulate creation and destruction of particles by choosing collision rules so that particles go into, and out of, these states.

5.1 Universality of SM's

SM's with a quiescent background have at least the computational power of Turing machines (TM's) with finite tapes, as we prove in [22]. The question of whether such SM's are universal is open, however. Still, these SM's are more powerful than any OSM, since OSM's can only do computation that requires at most cubic time, while problems exist that require more than cubic time on bounded-tape Turing machines. SM's with a periodic background can simulate an arbitrary TM, and are thus universal, as we show in [22].

Theorem 2 *SM's with a quiescent background are at least as powerful as Turing machines with bounded tapes.* ■

Theorem 3 *SM's with a periodic background are computation-universal.* ■

The class of algorithms that a finite-tape TM can implement depends on the specific function that bounds the size of the TM's tape; for instance, TM's with tapes of length polynomial in the input size can do any problem in $PSPACE$. Although not universal, such TM's can do almost any problem of practical significance.

5.2 Discussion

Theorems 1–3 suggest that we should look to nonintegrable systems for solitons that may support universal computation. It is an open question whether or not there exists such a soliton system. In what follows, we describe nonintegrable equations and explain the features that could enable them to encode a universal SM. Then we describe some preliminary experiments with a particular nonintegrable PDE, the logarithmically nonlinear Schrödinger (log-NLS) equation.

6 Nonintegrable Soliton Systems

Certain nonintegrable PDE's support soliton-like waves [3] with behavior more complex than that of integrable solitons. Examples include PDE's such as the Klein-Gordon [2] and logarithmically nonlinear Schrödinger (log-NLS) equations [6, 7, 26]. The solitons in these systems can change their velocities, as well as their phases, upon collisions, and new solitons may be created after collisions.

Soliton collisions in nonintegrable systems may be inelastic or *near-elastic*; that is, colliding solitons can dissipate their energy by producing varying amounts of *radiation* (see Fig. 4), which erodes other solitons and may eventually lead to complete decay of useful information in the system. To our knowledge, it is an open question whether or not there exists a nonintegrable system with perfectly elastic, or non-radiating, collisions. It also appears to be an open question whether or not perfect elasticity implies obliviousness in any system. A system with collisions that are both perfectly elastic and non-oblivious would offer promise for realizing the SM model using solitons. The system we describe next, the log-NLS equation, has very near-elastic, non-oblivious collisions, and may support perfectly elastic, non-oblivious collisions as well.

6.1 Gaussons in the log-NLS system

The log-NLS equation, which supports solitons called *gaussons*, was proposed as a nonlinear model of wave mechanics [6, 7, 28]. Gaussons are wave packets with gaussian-shaped envelopes and sinusoidal carrier waves. They are analogous to the wavefunctions of linear wave (quantum) mechanics; that is, the square of the amplitude of a gausson at a given point x can be interpreted as the probability that the particle described by the gausson is at x.

Our numerical simulations of gausson collisions verify a published report [28] that they range from deeply inelastic to near-elastic, and perhaps perfectly elastic, depending on the velocities of the colliding gaussons. In [28] an approximate range of velocities (the *resonance region*) is given for which collisions are apparently inelastic; outside this region, collisions are reportedly elastic. We confirmed these results numerically with the aid of the split-step Fourier method [9, 40], and investigated in more detail to find three distinct velocity regions in which gaussons behave very differently: Depending on the region and gausson phases, gausson collisions can result in amplitude and velocity changes, radiation, or no apparent interaction (see Fig. 4).

6.2 Soliton stability and elasticity

The inelastic and near-elastic soliton collisions we observed in regions 1 and 2 are non-oblivious, thus leaving open the possibility of using them for computation in SM's. We could use an approach similar to the techniques in [39]. As with the CA solitons in [39], we might first create a database of pairwise collisions of gaussons by running a series of numerical experiments; we would then search the database for useful collisions to encode a specific computation. This approach was used in [39] to implement a solitonic ripple-carry adder.

One problem with such a method is the potential connection between soliton stability and collision elasticity. We observed that inelastic collisions often resulted in *radiation ripples* emanating from collisions (Fig. 4) and eventual disintegration of gaussons in a cylindrical 1-d system. In region 2, these ripples and the resulting instability may make the system unsuitable for sustained computation. The more inelastic the collisions, the more quickly the system decayed. However, we do not know if stability and elasticity are necessarily correlated in general, nor do we know if elasticity and obliviousness (and thus lack of computation universality) are related. In fact, collisions of region-1 solitons in the log-NLS equation appear to be both elastic and strongly non-oblivious.

7 Summary and questions

We have explored certain well known soliton systems, with the goal of using them for computation in a 1-d homogeneous bulk medium. We defined soliton machines (SM's) to model integrable and non-integrable soliton systems, and found that a class of integrable PDE's cannot support universal computation under the OSM model. In addition, we proved that the SM model is universal in general, and suggested that gaussons in the log-NLS equation may be capable of realizing universal SM's.

Many open problems remain. Foremost among these is determining whether or not gaussons have behavior sufficiently complex and stable to implement a universal SM. We found three velocity regions in which gaussons have different behavior. Gaussons with low velocities (region 1) offer the most promise for realizing useful computation, since their collisions appear both elastic and non-oblivious. We may be able to use the phase-coding approach in [39] to implement useful computation with gausson interactions. Collisions of gaussons with higher velocities (regions 2 and 3) appear in general to be either oblivious or radiating, though for some combinations of

[3] In this section we refer to such waves as *solitons*, as is often done in the literature.

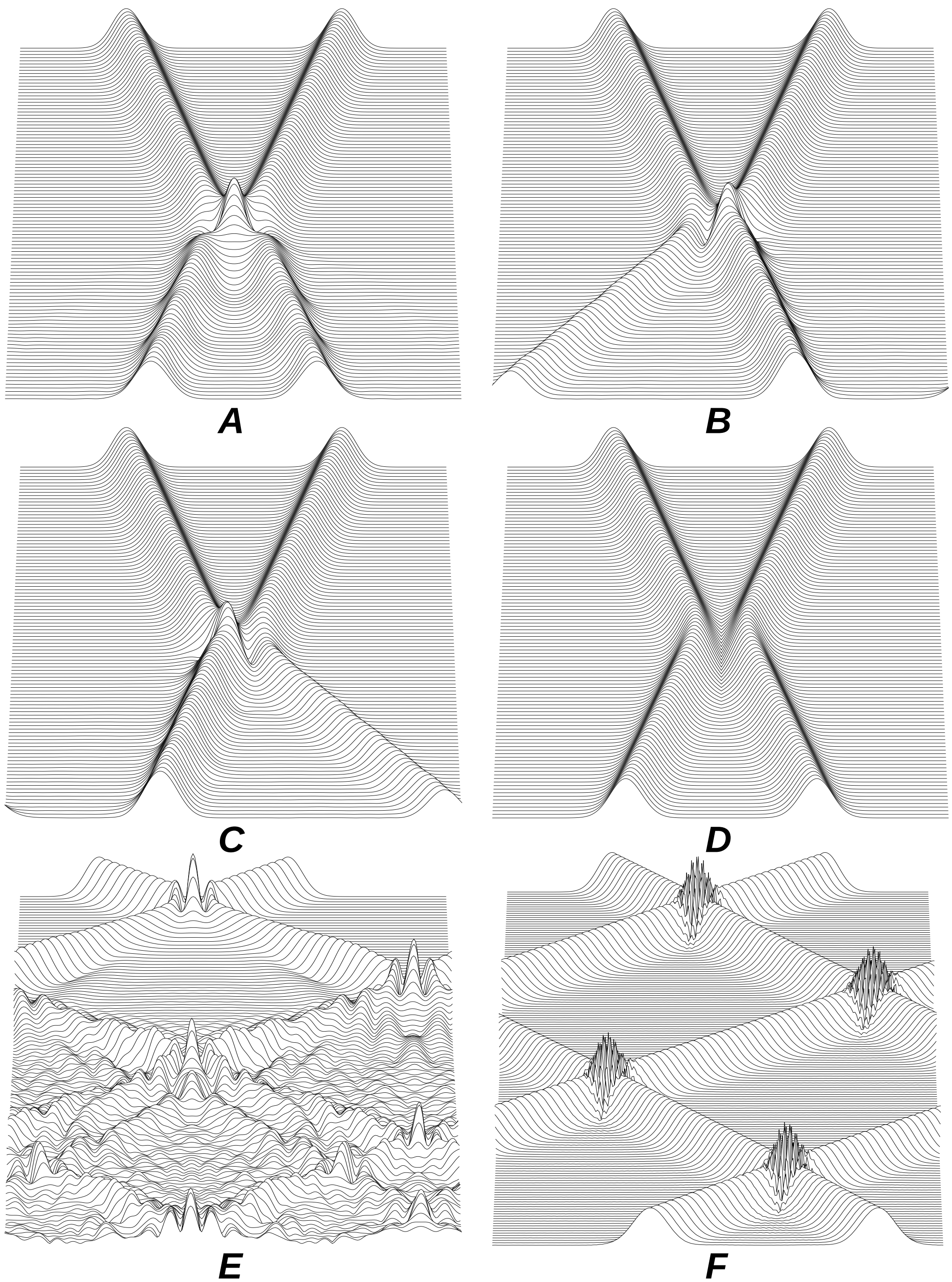

Figure 4: Gausson collisions, shown in graphs of space vs. time. Time increases from top to bottom, and space wraps around the left and right edges. The variable graphed is the gausson amplitude. Graphs A–D shows the effects of varying only gausson phases in region 1 (low gausson velocity). Graphs E and F show collisions in regions 2 (moderate velocity) and 3 (high velocity), respectively. For display purposes, time is scaled differently in graphs A–D, E, and F.

velocities and phases, these collisions are non-oblivious and very near-elastic. The search for answers is complicated by the necessity of numerical solution of the log-NLS equation.

Even if we were to show that the log-NLS equation can be used for universal computation, we would still be left with a gap: We know of no physical realization of this equation. But other nonintegrable nonlinear PDE's also offer possibilities for implementing SM's, and many of these do correspond to real physical systems. For example, the Klein-Gordon equation [2], the NLS equation with additional terms to model optical fiber loss and dispersion, and the coupled NLS equation for birefringent optical fibers [41, 20] all support soliton collisions with complex behavior potentially useful for encoding SM's. Optical solitons that arise from these more complicated equations exhibit gausson-like behavior, and are easily realizable in physical fibers; thus, such optical solitons may be particularly useful as practical means of computing using SM's. *Near-integrable* equations [25], or slightly altered versions of integrable equations, could also offer possibilities for implementing general SM's.

In addition, we may consider using solitons in two or three dimensions [10, 32]. Gaussons, for example, exist in any number of dimensions, and display behavior similar to that in one dimension. The added degrees of freedom of movement in two or more dimensions may enable implementation of universal systems such as the billiard ball computation model [27] or lattice gas models [36].

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