Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system

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Incompressible Navier-Stokes system

Caffarelli, Kohn, and Nirenberg [1982]

$$\mathrm{div}_x \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \Delta \mathbf{u}$$

Energy inequality

$$p \in L^{3/2}(0, T) \times \Omega$$

$$\partial_t |\mathbf{u}|^2 + \nabla_x |\mathbf{u}|^2 \cdot \mathbf{u} + 2\mathrm{div}_x(p\mathbf{u}) + 2|\nabla_x \mathbf{u}|^2 \leq \Delta |\mathbf{u}|^2$$

Compressible Navier-Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$\mathbf{u}|_{\partial\Omega} = 0$$

$$\mathbb{S} = \mu\left(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I}$$

Germain [2010]:

- weak-strong results in the framework of “better” weak solutions
- suitable conditions formulated in terms of hypothetical smooth solutions
- periodic boundary conditions

Finite-energy weak solutions

EQUATION OF CONTINUITY

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

MOMENTUM EQUATION

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Energy inequality

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right)(\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + H(\varrho_0) \right) \ \mathrm{d}x$$

$$H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z$$

Relative entropy

Germain [2010], Berthelin and Vasseur [2005]

$$E(\varrho, r) = H(\varrho) - H'(r)(\varrho - r) - H(r)$$

$$H(\varrho) \equiv \varrho \int_1^{\varrho} \frac{p(z)}{z^2} \, \mathrm{d}z, \ \ P \equiv H'$$

$$\int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r) \right) \, \mathrm{d}x$$

Suitable weak solutions

Renormalized equation of continuity

$$\partial_t b(\varrho) + \mathrm{div}_x (b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

Momentum equation

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Generalized energy inequality

$$\int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + E(\varrho, r) \right) (\tau, \cdot) \ \mathrm{d}x$$

$$+ \int_0^\tau \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})] : \nabla_x (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0 - \mathbf{U}(0, \cdot)|^2 + E(\varrho_0, r(0, \cdot)) \right) \ \mathrm{d}x$$

$$+ \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \ \mathrm{d}t \text{ for a.a. } \tau \in (0, T),$$

for any smooth $r > 0$, $\mathbf{U}|_{\partial\Omega} = 0$

Remainder term

$$\mathcal{R}\left(\varrho, \mathbf{u}, r, \mathbf{U}\right)$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})(\mathbf{u} - \mathbf{U}) \right) \ \mathrm{d}x$$

$$+ \int_\Omega \Big((r - \varrho) \partial_t P(r) + \nabla_x P(r) \cdot \Big(r\mathbf{U} - \varrho \mathbf{u} \Big) -$$

$$\mathrm{div}_x \mathbf{U} \Big(\varrho \Big(P(\varrho) - P(r) \Big) - E(\varrho, r) \Big) \Big) \ \mathrm{d}x$$

Global existence

[E.F., A. Novotný, Y. Sun, Indiana Univ. Math. J., to appear]

Theorem

Let $\Omega \subset R^3$ be a bounded smooth domain. Let the pressure p be a continuously differentiable function satisfying

$$p(0) = 0, \ p'(\varrho) > 0 \text{ for all } \varrho > 0, \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma - 1}} = a > 0$$

for a certain $\gamma > 3/2$. Assume that the initial data satisfy

$$\varrho_0 \geq 0, \ \varrho_0 \not\equiv 0, \ \varrho_0 \in L^\gamma(\Omega), \varrho_0 |\mathbf{u}_0|^2 \in L^1(\Omega).$$

Then the compressible Navier-Stokes system possesses a suitable weak solution on $(0, T) \times \Omega$.

Weak-strong uniqueness

Strong solutions:

$$0 < \underline{\varrho} \leq \tilde{\varrho}(t,x) \leq \overline{\varrho}, \ |\tilde{\mathbf{u}}(t,x)| \leq \overline{u} \quad (1)$$

$$\nabla_x \tilde{\varrho} \in L^2(0,T;L^q(\Omega;R^3)), \ \nabla_x^2 \tilde{\mathbf{u}} \in L^2(0,T;L^q(\Omega;R^{3\times3\times3})) \quad (2)$$

$$q > \max\{3, \frac{3}{\gamma - 1}\}.$$

Theorem

Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. In addition to hypotheses of existence theorem, suppose that p is twice continuously differentiable on the open interval $(0, \infty)$. Assume that the Navier-Stokes system admits a weak solution $\tilde{\varrho}$, $\tilde{\mathbf{u}}$ in $(0, T) \times \Omega$ belonging to the regularity class specified through (1), (2).
Then $\tilde{\varrho} \equiv \varrho$, $\tilde{\mathbf{u}} \equiv \mathbf{u}$, where ϱ, $\mathbf{u}$ is the suitable weak solution of the Navier-Stokes system emanating from the same initial data.

Conditional regularity

Using the result of Sun, Wang and Zhang [2010] we have:

Theorem

Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. Let ϱ_0, $\mathbf{u}_0$ be given such that

$$\varrho_0 \in W^{1,6}(\Omega),\ 0 < \underline{\varrho} \leq \varrho_0(x) \leq \overline{\varrho} \text{ for all } x \in \Omega,$$

$$\mathbf{u}_0 \in W^{2,2}(\Omega; R^3) \cap W_0^{1,2}(\Omega; R^3).$$

Suppose that the pressure p satisfies the hypotheses of existence theorem, and that

$$\mu > 0,\ \eta = 0.$$

Let ϱ, $\mathbf{u}$ be a suitable weak solution of the Navier-Stokes system in $(0, T) \times \Omega$.
If, in addition,

$$\mathrm{ess} \sup_{(0,T)\times\Omega} \varrho < \infty,$$

then ϱ, $\mathbf{u}$ is the unique (strong) solution of the Navier-Stokes.

Corollary

Let Ω and the initial data ϱ_0, $\mathbf{u}_0$ be the same as in the previous theorem. Assume that ϱ, $\mathbf{u}$ is a suitable weak solution of the Navier-Stokes system such that

$$\mathrm{ess}\inf_{x \in \Omega} \varrho(\tau, x) = 0 \text{ for a certain } \tau \in (0, T).$$

The there exists $0 < \tau_0 \leq \tau$ such that

$$\limsup_{t \to \tau_0} \left[\mathrm{ess}\sup_{x \in \Omega} \varrho(t, x) \right] = \infty.$$

Stability

Theorem

Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. In addition to hypotheses of existence theorem, suppose that p is twice continuously differentiable on the open interval $(0, \infty)$. Assume that the Navier-Stokes system admits a (strong) solution $\tilde{\varrho}$, $\tilde{\mathbf{u}}$. In addition, let

$$\varrho_{0,\varepsilon} \to \tilde{\varrho}_0 \text{ in } L^\gamma(\Omega), \ \varrho_{0,\varepsilon} \geq 0, \ \int_\Omega \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon} - \tilde{\mathbf{u}}_0|^2 \ \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0.$$

Then

$$\sup_{\tau\in[0,T]} \|\varrho_\varepsilon(\tau,\cdot) - \tilde{\varrho}(\tau,\cdot)\|_{L^\gamma(\Omega)} \to 0$$

$$\sup_{\tau\in[0,T]} \|\varrho_\varepsilon \mathbf{u}_\varepsilon(\tau,\cdot) - \tilde{\varrho}\tilde{\mathbf{u}}(\tau,\cdot)\|_{L^1(\Omega;R^3)} \to 0,$$

and

$$\mathbf{u}_\varepsilon \to \tilde{\mathbf{u}} \text{ in } L^2(0,T;W_0^{1,2}(\Omega;R^3)),$$

where ϱ_ε, $\mathbf{u}_\varepsilon$ is a suitable weak solution of the Navier-Stokes system emanating from the initial data $\varrho_{0,\varepsilon}$, $\mathbf{u}_{0,\varepsilon}$.

Finite energy solutions are suitable

[E.F., A.Novotný, Bum Ja Jin, Preprint 2011]

Finite energy weak solutions are suitable weak solutions

- general domains (bounded, unbounded, irregular)
- general boundary conditions (no-slip, complete slip, Navier's slip)
- general driving force