Relative entropy of cone measures and L_p centroid bodies *

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Abstract

Let K be a convex body in $\mathbb{R}^n$. We introduce a new affine invariant, which we call Ω_K, that can be found in three different ways:

as a limit of normalized L_p-affine surface areas,

as the relative entropy of the cone measure of K and the cone measure of K°,

as the limit of the volume difference of K and L_p-centroid bodies.

We investigate properties of Ω_K and of related new invariant quantities. In particular, we show new affine isoperimetric inequalities and we show a "information inequality" for convex bodies.

1 Introduction

The starting point of our investigation was the study of the asymptotic behavior of the volume of L_p centroid bodies as p tends to infinity. This study resulted in the discovery of a new affine invariant, Ω_K. We then showed that the quantity Ω_K is the relative entropy of the cone measure of K and the cone measure of K°. Cone measures have been intensively studied in recent years (see e.g. Barthe/Guedon/Mendelson/Naor [8], Gromov/Milman [18], Naor [44] and Naor/Romik [45] and Schechtmann Zinn [50]) Finally, to our surprise, Ω_K appeared again naturally in a third way, namely as a limit of normalized L_p-affine surface areas.

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Thus, the invariant Ω_K introduces a novel idea -relative entropy- into the theory of convex bodies and links concepts from classical convex geometry like L_p centroid bodies and L_p-affine surface area with concepts from information theory. Such links have already been established. Guleryuz, Lutwak, Yang and Zhang [20, 37, 38, 39, 40]) use L_p Brunn Minkowski theory to develop certain entropy inequalities. Also, classical Brunn Minkowski theory is related to information theoretic concepts (see e.g. [3, 4, 5, 6, 14, 15].

An important affine invariant quantity in convex geometric analysis is the affine surface area, which, for a convex body $K \in \mathbb{R}^n$ is defined as

$$as_1(K) = \int_{\partial K} \kappa^{\frac{1}{n+1}}(x) d\mu(x). \tag{1.1}$$

$\kappa(x) = \kappa_K(x)$ is the generalized Gaussian curvature at the boundary point x of K and $\mu = \mu_K$ is the surface area measure on the boundary ∂K. Originally a basic affine invariant from the field of affine differential geometry, it has recently attracted increased attention(e.g. [7, 33, 41, 52, 58]). It is fundamental in the theory of valuations (see e.g., [1, 2, 24, 29]), in approximation of convex bodies by polytopes (e.g., [19, 31, 53]) and it is the subject of the affine Plateau problem solved in $\mathbb{R}^3$ by Trudinger and Wang [57, 60].

The definition (1.1), at least for convex bodies in $\mathbb{R}^2$ and $\mathbb{R}^3$ with sufficiently smooth boundary, goes back to Blaschke [9] and was extended to arbitrary convex bodies by e.g. [28, 33, 41, 52]. Schütt and Werner showed in [52] that the affine surface area equals

$$as_1(K) = \lim_{\delta \to 0} c_n \frac{|K| - |K_\delta|}{\delta^{\frac{2}{n+1}}}.$$

c_n is a constant depending only on n, $|K|$ denotes the n-dimensional volume of K and K_δ is the *convex floating body* of K [52]: the intersection of all halfspaces H^+ whose defining hyperplanes H cut off a set of volume δ from K.

It was shown by Milman and Pajor [43] that for "big" δ K_δ is homothetic, up to a constant depending on δ, to the dual of the Binet ellipsoid from classical mechanics and consequently K_δ° is homothetic to the Binet ellipsoid.

Lutwak and Zhang [36] generalized the notion of Binet ellipsoid and introduced the L_p *centroid bodies*: For a convex body K in $\mathbb{R}^n$ of volume 1 and $1 \leqslant p \leqslant \infty$, the L_p centroid body $Z_p(K)$ is this convex body that has support function

$$h_{Z_p(K)}(\theta) = \left(\int_K |\langle x, \theta \rangle|^p dx \right)^{1/p}. \tag{1.2}$$

Note that in [36] a different notation and normalization was used for the centroid body. In the present paper we will follow the notation and normalization that appeared in [46].

The results of this paper deal mostly with centrally symmetric convex bodies K. Symmetry is assumed mainly because the L_p centroid bodies are symmetric by definition (1.2) and used to approximate the convex bodies K. There exists a non-symmetric definition of L_p centroid bodies in [30] (see also [21]). Using this definition, we feel the results of the paper can be carried over to non-symmetric convex bodies.

In Proposition 2.2 we generalize the result by Milman and Pajor mentioned above and show that the floating body K_δ is - up to a universal constant - homothetic to the centroid body $Z_{\log \frac{1}{\delta}}(K)$.

L_p-affine surface area, an extension of affine surface area, was introduced by Lutwak in the ground breaking paper [34] for $p > 1$ and for general p by Schütt and Werner [54]. It is now at the core of the rapidly developing L_p Brunn Minkowski theory. Contributions here include new interpretations of L_p-affine surface areas [42, 53, 54, 58, 59], the study of solutions of nontrivial ordinary and, respectively, partial differential equations (see e.g. Chen [12], Chou and Wang [13], Stancu [55, 56]), the study of the L_p Christoffel-Minkowski problem by Hu, Ma and Shen [23], characterization theorems by Ludwig and Reitzner [29] and the study of L_p affine isoperimetric inequalities by Lutwak [34] and Werner and Ye [58, 59].

From now on we will always assume that the centroid of a convex body K in $\mathbb{R}^n$ is at the origin. We write $K \in C^2_+$, if K has C^2 boundary with everywhere strictly positive Gaussian curvature κ_K. For real $p \neq -n$, we define the L_p-affine surface area $as_p(K)$ of K as in [34] ($p > 1$) and [54] ($p < 1, p \neq -n$) by

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x) \tag{1.3}$$

and

$$as_{\pm\infty}(K) = \int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n} d\mu_K(x), \tag{1.4}$$

provided the above integrals exist. $N_K(x)$ is the outer unit normal vector at x to ∂K, the boundary of K, and μ_K is the usual surface area measure on ∂K. In particular, for $p = 0$

$$as_0(K) = \int_{\partial K} \langle x, N_K(x)\rangle \, d\mu_K(x) = n|K|.$$

For $p = 1$ we get the classical affine surface area (1.1) which is independent of the position of K in space.

We use the L_p-affine surface area to define a new affine invariant in Section 3:

$$\Omega_K = \lim_{p\to\infty} \left(\frac{as_p(K)}{n|K^\circ|}\right)^{n+p}. \tag{1.5}$$

This is a first way how Ω_K appears. We describe properties of this new invariant. E.g., in Corollary 3.9 we prove the following remarkable identity (1.6), which is the second way how Ω_K appears: It shows that the invariant Ω_K is the exponential of the relative entropy or Kullback-Leibler divergence D_{KL} of the cone measures cm_K and cm_{K° of K and K°.

$$\Omega_K^{1/n} = \frac{|K^\circ|}{|K|} \exp\Big(- D_{KL}(N_K N_{K^\circ}^{-1} cm_{\partial K^\circ} \| cm_{\partial K})\Big). \tag{1.6}$$

N_K^{-1} is the inverse of the Gauss map. We refer to Section 3 for its definition and that of the relative entropy and the cone measures. See also Gromov/Milman [18] and Naor [44] and Naor/Romik [45] for further information on cone measures.

We show that the information inequality [14] for the relative entropy of the cone measures implies an "information inequality" for convex bodies

$$\Omega_K \leqslant \left(\frac{|K|}{|K^\circ|}\right)^n$$

with equality if and only if K is an ellipsoid. Independently, we can derive this inequality from properties of the L_p-affine surface areas.

The next proposition gives a sample of some inequalities that hold for the affine invariant Ω_K, among them an isoperimetric inequality. More can be found in Proposition 3.5.

Proposition *Let K be a convex body with centroid at the origin.*

(i) $\Omega_{K^\circ} \leqslant \Omega_{(\widetilde{B_2^n})^\circ}$

(ii) For all $p \geq 0$, $\Omega_K \leqslant \left(\frac{as_p(K)}{n|K^\circ|}\right)^{n+p}$.

(iii) $\Omega_K \leqslant \left(\frac{|K|}{|K^\circ|}\right)^n$.

If K is in addition in C^2_+, then equality holds if and only if K is an ellipsoid. with equality holding in (i), (ii) and (iii) if and only if K is an ellipsoid.

Proposition 2.2 states that the floating body K_δ is - up to a universal constant - homothetic to the centroid body $Z_{\log\frac{1}{\delta}}(K)$. This, and the geometric interpretations of L_p-affine surface areas in terms of variants of the floating bodies [54, 58, 59], led us to investigate the L_p centroid bodies also in the context of affine surface area. Note the similarities in bahavior of the floating body and the L_p centroid body. Both "approximate" K as $\delta \to 0$ respectively $p \to \infty$: If K is symmetric and of volume 1, $Z_p(K) \to K$ as $p \to \infty$.

We found an amazing connection between the L_p centroid bodies and the new invariant Ω_K which is stated in the following theorem for convex bodies in C^2_+. A forthcoming paper will address general convex bodies.

Theorem 4.1 *Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1 that is in C^2_+. Then*

(i) $\lim_{p\to\infty} \frac{p}{\log p}\left(|Z_p^\circ(K)| - |K^\circ|\right) = \frac{n(n+1)}{2}\,|K^\circ|$.

(ii) $\lim_{p\to\infty} p\left(|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)}{2p}\log p\ |Z_p^\circ(K)|\right) =$

$$\lim_{p\to\infty} p\left(|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)}{2p}\log p\ |K^\circ|\right) =$$

$$-\frac{1}{2}\int_{S^{n-1}} h_K(u)^{-n}\ \log\left(2^{n+1}\pi^{n-1}h_K(u)^{n+1}f_K(u)\right)d\sigma(u) =$$

$$\frac{1}{2}\int_{\partial K}\frac{\kappa(x)}{\langle x, N(x)\rangle^n}\ \log\left(\frac{\kappa(x)}{2^{n+1}\pi^{n-1}\langle x, N(x)\rangle^{n+1}}\right)d\mu(x)$$

In view of Proposition 2.2, the first part of the Theorem 4.1 came as a surprise to us because it reveals a different behaviour of the bodies K_δ and $Z_{\log\frac{1}{\delta}}(K)$ when $\delta\to 0$. Indeed, it was shown in [42] that

$$\lim_{\delta\to 0}\ c_n\frac{|(K_\delta)^\circ| - |K^\circ|}{\delta^{\frac{2}{n+1}}} = as_{-n(n+2)}(K) = as_{-\frac{n}{n+2}}(K^\circ)$$

where c_n is a constant that depends on n only.

Even more surprising is the second part of Theorem 4.1. Indeed, Proposition 3.5 states that

$$\Omega_K = \exp\left(-\frac{1}{|K^\circ|}\int_{\partial K}\frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n}\log\frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n+1}}d\mu_K(x)\right).$$

This, together with Theorem 4.1 shows how the new invariant and the L_p centroid bodies are related, namely for a symmetric convex body K of volume 1 in C_+^2

$$\lim_{p\to\infty}\frac{2p}{n}\left(\frac{|Z_p^\circ(K)|}{|K^\circ|} - \left(1 - \frac{n(n+1)\log p}{2p}\right)\right) =$$

$$\lim_{p\to\infty}\frac{2p}{n}\left(\frac{(1-\frac{n(n+1)\log p}{2p})|Z_p^\circ(K)|}{|K^\circ|} - 1\right) =$$

$$-\frac{1}{2}\log\frac{\Omega_K^{\frac{1}{n}}}{2^{n+1}\pi^{n-1}}.$$

This is the third way how Ω_K appears.

Further notation. We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write σ for the rotationally invariant surface measure on S^{n-1}.

A convex body is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is 0 - symmetric, if $x\in C$ implies that $-x\in C$. We say that C has centre of mass at the origin if $\int_C\langle x,\theta\rangle dx = 0$ for every $\theta\in S^{n-1}$. The support function $h_C:\mathbb{R}^n\to\mathbb{R}$ of

C is defined by $h_C(x) = \max\{\langle x, y\rangle : y \in C\}$. The polar body C° of C is $C^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in C\}$.

Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. We refer to the books [49] and [51] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

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2 Comparison of Floating bodies and L_p centroid bodies

It is well known from mechanics that the body $Z_2(K)$ is an ellipsoid. Its polar body $Z_2^\circ(K)$ is called the Binet ellipsoid of inertia. $Z_1(K) = Z(K)$ is the classical centroid body and it is a zonoid by definition (see [17, 51]).

The isotropic contant L_K of a convex body $K \in \mathbb{R}^n$ is defined as

$$L_K = \left(\frac{|Z_2(K)|}{|B_2^n|}\right)^{1/n}$$

L_K is an affine invariant and $L_K \geq L_{B_2^n}$.

A major open problem in convex geometry asks if there exists a universal constant $C > 0$ such that $L_K \leq C$. The best -up to date- known result is due to Klartag [25] and states that $L_K \leq C n^{\frac{1}{4}}$, improving by a factor of logarithm an earlier result by Bourgain [10].

Let us briefly state some of the known properties of the L_p centroid bodies. For the proofs and further references see [46].

Let $T \in SL(n)$, i.e. $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator with determinant 1. Let T^* denote its adjoint. Then

$$h_{Z_p(TK)}(\theta) = \left(\int_{TK} |\langle x, \theta\rangle|^p dx\right)^{1/p} = \left(\int_K |\langle x, T^\star(\theta)\rangle|^p dx\right)^{1/p} = h_{Z_p(K)}(T^\star(\theta))$$

or

$$h_{Z_p(TK)}(\theta) = h_{T(Z_p(K))}(\theta)$$

By Hölder's inequality, we have for $1 \leqslant p \leqslant q \leqslant \infty$ that

$$Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K) = K. \tag{2.7}$$

As an application of the Brunn-Minkowski inequality, one has for $1 \leqslant p \leqslant q < \infty$ that

$$Z_q(K) \subseteq c \, \frac{q}{p} \, Z_p(K). \tag{2.8}$$

$c > 0$ is a universal constant.

Inequality (2.8) is sharp with the right constant for the l_n^1-ball [16].

By Brunn's principle we get for $p \geq n$ and a (new) absolute constant $c > 0$ (e.g., [47])

$$Z_p(K) \supseteq c\, K. \tag{2.9}$$

Lutwak, Yang and Zhang [35] and Lutwak and Zhang [36] proved the following L_p versions of Blaschke Santaló inequality and Busemann-Petty inequality. See also Campi and Gronchi [11] for an alternative proof.

Theorem 2.1. *[35, 36] Let K be a convex body in $\mathbb{R}^n$ of volume 1. Then for every $1 \leqslant p \leqslant \infty$*

$$|Z_p^\circ(K)| \leq |Z_p^\circ(\widetilde{B_2^n})|$$
$$|Z_p(K)| \geq |Z_p(\widetilde{B_2^n})|$$

with equality if and only if K is an ellipsoid.

A computation shows that $|Z_p(\widetilde{B_2^n}|^{1/n} \simeq \sqrt{\frac{p}{n+p}}$. Hence, the following inequality, proved in [46] for all $p \geq 1$ and a universal constant $c > 0$, can be viewed as an "Inverse Lutwak-Yang-Zhang inequality"

$$|Z_p(K)|^{1/n} \leq c\sqrt{\frac{p}{n+p}} L_K. \tag{2.10}$$

We now want to compare L_p centroid bodies and floating bodies. As K is symmetric and has volume 1, the floating body K_δ, for $\delta \in [0,1]$, may be defined in the following way [52]

$$K_\delta = \bigcap_{\theta \in S^{n-1}} \{x \in K : |\langle x, \theta\rangle| \leq t_\theta\} \tag{2.11}$$

where $t_\theta = \sup\{t > 0 : |\{x \in K : |\langle x, \theta\rangle| \leq t\}| = 1 - \delta\}$. Hence for every $\theta \in S^{n-1}$ one has that

$$h_{K_\delta}(\theta) = t_\theta. \tag{2.12}$$

Theorem 2.2. *Let K a symmetric convex body in $\mathbb{R}^n$ of volume 1. Let $\delta \in (0,1)$. Then we have for every $\theta \in S^{n-1}$*

$$c_1 h_{Z_{\log \frac{1}{\delta}}(K)}(\theta) \leqslant h_{K_\delta}(\theta) \leqslant c_2 h_{Z_{\log \frac{1}{\delta}}(K)}(\theta)$$

or, equivalently

$$c_1 Z_{\log\frac{1}{\delta}}(K) \subseteq K_\delta \subseteq c_2 Z_{\log\frac{1}{\delta}}(K),$$

where $c_1, c_2 > 0$ *are universal constants. Consequently*

$$\frac{1}{c_1} Z^\circ_{\log\frac{1}{\delta}}(K) \supseteq K^\circ_\delta \supseteq \frac{1}{c_2} Z^\circ_{\log\frac{1}{\delta}}(K)$$

Proof.

For $\delta \in (c_0, 1]$, c_0 appropriately chosen, the theorem was already shown in [43]. We assume that $\delta \leq c_0 < 1$. We apply Markov's inequality in (1.2) and get

$$|\{x \in K : |\langle x, \theta\}| \geq e h_{Z_p(K)}(\theta)\}| \leqslant e^{-p}.$$

Then (2.12) gives for all $p \geq 1$,

$$e h_{Z_p(K)}(\theta) \geq h_{K_{e^{-p}}}(\theta). \tag{2.13}$$

For the other side we will use the Paley-Zygmund inequality: If $Z \geq 0$ is a random variable with finite variance and $\lambda \in (0,1)$ then

$$Pr\{Z \geq \lambda E(Z)\} \geq (1-\lambda)^2 \frac{E(Z)^2}{E(Z^2)}.$$

Hence for $Z = |\langle x, \theta\rangle|^p$ we get

$$|\{x \in K : |\langle x, \theta\rangle|^p \geq \lambda \int_K |\langle x, \theta\rangle|^p dx\}| \geq (1-\lambda)^2 \frac{\left(\int_K |\langle x, \theta\rangle|^p dx\right)^2}{\int_K |\langle x, \theta\rangle|^{2p} dx}. \tag{2.14}$$

(2.8) implies that $h_{Z_{2p}(K)}(\theta) \leqslant 2c h_{Z_p(K)}(\theta)$, for all $\theta \in S^{n-1}$. So

$$\frac{\left(\int_K |\langle x, \theta\rangle|^p dx\right)^2}{\int_K |\langle x, \theta\rangle|^{2p} dx} \geq \left(\frac{1}{2c}\right)^{2p}.$$

Choose $\lambda = \frac{1}{2}$. Then (2.14) becomes

$$|\{x \in K : |\langle x, \theta\rangle| \geq \frac{1}{2} h_{Z_p(K)}(\theta)\}| \geq e^{-c_1 p}.$$

Now we use again (2.12) to get

$$\frac{1}{2} h_{Z_p(K)}(\theta) \leqslant h_{K_{e^{-c_1 p}}}(\theta)$$

or

$$h_{K_{e^{-p}}}(\theta) \geq \frac{1}{2} h_{Z_{\frac{p}{c_1}}(K)}(\theta) \geq c_2 h_{Z_p(K)}(\theta), \tag{2.15}$$

where we have used (2.8) again. (2.13) and (2.15) then imply that

$$c_2\ h_{Z_p(K)}(\theta) \leq h_{K_{e^{-p}}}(\theta) \leq e\ h_{Z_p(K)}(\theta).$$

Now choose $p = \log \frac{1}{\delta}$. This gives the theorem

One does not expect that floating bodies and L_q centroid bodies are identical in general. Indeed, observe that for $p < \infty$ the bodies $Z_p(K)$ are C^∞. However one can easily check that the floating body of the cube has points of non-differentiability on the boundary.

Theorem 2.2 allows us to "pass" results about L_p centroid bodies to floating bodies. In particular, (2.7) and (2.9) imply that for $\delta < e^{-n}$, K_δ is isomorphic to K:

$$K_\delta \subseteq K \subseteq c_1 K_\delta .$$

Moreover, (2.7) and (2.8) imply that

$$K_{\delta_2} \subseteq K_{\delta_1} \subseteq c_2 \frac{\log \frac{1}{\delta_1}}{\log \frac{1}{\delta_2}} K_{\delta_2} \ , \quad \text{for} \quad \delta_1 \leqslant \delta_2 \ ,$$

where $c_1, c_2 > 0$ are universal constants.

As a consequence we get the following corollary. There, $d(K, L)$, resp. $d_{BM}(K, L)$, mean the geometric, resp. Banach-Mazur distance of two convex bodies K and L

$$d(K, L) = \inf\{a \cdot b : \frac{1}{a} K \subset L \subset bK\}$$

$$d_{BM}(K, L) = \inf\{d\big(K, T(L)\big) : \text{T is a linear operator}\}$$

It is known that one may choose a $T \in SL(n)$ such that $T(K_{1/2})$ is isomorphic to B_2^n (see [43] for details).

Corollary 2.3. *Let K be a symmetric convex body of volume 1. Then for every $\delta \in (0, 1)$ one has*

$$d_{BM}\left(K_\delta, B_2^n\right) \leqslant c_1 \log \frac{1}{\delta} \ ,$$

and

$$d\left(K_\delta, K\right) \simeq d\left(K_\delta, K_{e^{-n}}\right) \leqslant c_2 \frac{n}{\log \frac{1}{\delta}} \ ,$$

where $c_1, c_2 > 0$ are universal constants.

Let us note that Theorem 2.1 and (2.10) imply sharp (up to L_K) bounds for the volume of K_δ. Namely, letting $c_\delta = \max\{\log\frac{1}{\delta}, 1\}$,

$$c_1 \sqrt{\frac{c_\delta}{n + c_\delta}} \leqslant |K_\delta|^{1/n} \leqslant c_2 \sqrt{\frac{c_\delta}{n + c_\delta}} \ L_K \ ,$$

where $c_1, c_2 > 0$ are universal constants.

Remark. The corollary is also true for non symmetric K.

In view of a result of R. Latala and J. Wojtaszczyk [27], Theorem 2.2 has another consequence: The floating body of a symmetric convex body K corresponds to a level set of the Legendre transform of the logarithmic Laplace transform on K.

Let $x \in \mathbb{R}^n$ and K a symmetric convex body of volume 1. Let

$$\Lambda^*_K(x) := \sup_{u \in \mathbb{R}^n} \left\{ \langle x, u\rangle - \log \int_K e^{\langle x,u\rangle} dx \right\} .$$

be the Legendre transform of the logarithmic Laplace transform on K. For any $r > 0$, let $B_r(K)$ be the convex body defined as

$$B_r(K) := \{x \in \mathbb{R}^n : \Lambda^*_K(x) \leqslant r\}.$$

It was proved in [27] that $B_p(K)$ is isomorphic to $Z_p(K)$,

$$c_1 Z_p(K) \subseteq B_p(K) \subseteq c_2 Z_p(K),$$

where $c_1, c_2 > 0$ are universal constants.

We combine this with Theorem 2.2 and get the following

Proposition 2.4. *Let K a symmetric convex body of volume 1 in $\mathbb{R}^n$. Then for every $\delta \in (0, \frac{1}{2})$ one has that*

$$c_1 \left\{ x \in \mathbb{R}^n : \Lambda^*_K(x) \leqslant \log \frac{1}{\delta} \right\} \subseteq K_\delta \subseteq c_2 \left\{ x \in \mathbb{R}^n : \Lambda^*_K(x) \leqslant \log \frac{1}{\delta} \right\}.$$

$c_1, c_2 > 0$ are universal constants.

3 Relative entropy of cone measures and related inequalities

Let K be a convex body in $\mathbb{R}^n$ with its centroid at the origin. For real $p \neq -n$, L_p-affine surface area $as_p(K)$ of K was defined in (1.3) and (1.4) in the introduction.

If K is in C^2_+, then (1.3) and (1.4) can be written as integrals over the boundary $\partial B^n_2 = S^{n-1}$ of the Euclidean unit ball B^n_2 in $\mathbb{R}^n$

$$as_p(K) = \int_{S^{n-1}} \frac{f_K(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}} d\sigma(u)$$

and

$$as_{\pm\infty}(K) = \int_{S^{n-1}} \frac{1}{h_K(u)^n} d\sigma(u) = n|K^\circ|. \tag{3.16}$$

$f_K(u)$ is the curvature function, i.e. the reciprocal of the Gauss curvature $\kappa(x)$ at that point x in ∂K that has u as outer normal.

We recall first results proved in [58].

Proposition 3.1. *[58] Let K be a convex body in $\mathbb{R}^n$ such that $\mu\{x \in \partial K : \kappa(x) = 0\} = 0$. Let $p \neq -n$ be a real number. Then*

(i) The function $p \to \left(\frac{as_p(K)}{as_\infty(K)}\right)^{n+p}$ is decreasing in $p \in (-n, \infty)$.

(ii) The function $p \to \left(\frac{as_p(K)}{n|K^\circ|}\right)^{n+p}$ is decreasing in $p \in (-n, \infty)$.

(iii) The function $p \to \left(\frac{as_p(K)}{n|K|}\right)^{\frac{n+p}{p}}$ is increasing in $p \in (-n, \infty)$.

(iv) $as_p(K) = as_{\frac{n^2}{p}}(K^\circ)$.

Remarks.

(i) It was shown in [22] that for $p > 0$ (iv) holds without any assumptions on the boundary of K.

(ii) Also, it follows from the proof in [58] that (i), (ii) and (iii) hold without assumptions on the boundary of K if $p \geq 0$.

(iii) Proposition 3.1 (ii) is not explicitly stated in [58], but follows (without any assumptions on the boundary of K if $p \geq 0$) from e.g., inequality (4.20) of [58] and the following fact (see [54]): Let K be a convex body in $\mathbb{R}^n$. Then

$$as_\infty(K) \leq n|K^\circ| \tag{3.17}$$

with equality if K is in C^2_+.

(iv) Strict monotonicity and characterization of equality in Proposition 3.1 (i), (ii) and (iiii):
Proposition 3.1 (i), (ii) and (iii) -without equality characterization- was proved in [58] using Hölder's inequality. It follows immediately from the characterization of equality in Hölder's inequality, that strict monotonicity holds in 3.1 (i), (ii) and (iii) if and only if μ -a.e. on ∂K

$$\frac{\kappa(x)}{\langle x, N(x)\rangle^{n+1}} = c,$$

where $c > 0$ is a constant - unless $\kappa(x) = 0$ μ -a.e. on ∂K. If $\kappa(x) = 0$ μ -a.e. on ∂K, then for all $p > 0$, $\left(\frac{as_p(K)}{as_\infty(K)}\right)^{n+p} = \text{constant} = 0$, $\left(\frac{as_p(K)}{n|K^\circ|}\right)^{n+p} = \text{constant} = 0$ and $\left(\frac{as_p(K)}{n|K|}\right)^{\frac{n+p}{p}} = \text{constant} = 0$.

If K is in C^2_+, then the following theorem due to Petty [48] implies that we have strict monotonicity in 3.1 (i), (ii) and (iii) unless K is an ellipsoid, in which case the quantities in 3.1 (i), (ii) and (iii) are all constant equal to 1.

Theorem 3.2. *[48] Let K be a convex body in C^2_+. K is an ellipsoid if and only if for all x in ∂K*

$$\frac{\kappa(x)}{\langle x, N(x)\rangle^{n+1}} = c,$$

where $c > 0$ is a constant.

We now introduce new affine invariants.

Definition 3.3. (i) Let K a convex body in $\mathbb{R}^n$ with centroid at the origin. We define

$$\Omega_K = \lim_{p\to\infty}\left(\frac{as_p(K)}{n|K^\circ|)}\right)^{n+p},$$

(ii) Let $K_1, \dots, K_n$ be convex bodies in $\mathbb{R}^n$, all with centroid at the origin. We define

$$\Omega_{K_1,\dots K_n} = \lim_{p\to\infty}\left(\frac{as_p(K_1,\dots,K_n)}{as_\infty(K_1,\dots,K_n)}\right)^{n+p}.$$

Here

$$as_p(K_1,\dots,K_n) = \int_{S^{n-1}}\left[h_{K_1}(u)^{1-p}f_{K_1}(u)\cdots h_{K_n}^{1-p}f_{K_n}(u)\right]^{\frac{1}{n+p}} d\sigma(u)$$

is the mixed p-affine surface area introduced for $1 \le p < \infty$ in [34] and for general p in [59].

$$\begin{aligned} as_\infty(K_1,\dots,K_n) &= \int_{S^{n-1}} \frac{1}{h_{K_1}(u)}\cdots\frac{1}{h_{K_n}(u)}\, d\sigma(u) \\ &= n\tilde{V}(K_1^\circ,\cdots,K_n^\circ) \end{aligned}$$

is the dual mixed volume of $K_1^\circ, \cdots, K_n^\circ$, introduced by Lutwak in [32].

We will concentrate on describing the properties of Ω_K. The analogue properties for the invariant $\Omega_{K_1,\dots K_n}$ also hold and are proved similarly using results about the mixed p-affine surface areas proved in [59]. For instance, the analogue to Proposition 3.5 (ii) holds: For all $p \ge 0$

$$\Omega_{K_1,\dots K_n} \le \left(\frac{as_p(K_1,\dots,K_n)}{as_\infty(K_1,\dots,K_n)}\right)^{n+p}.$$

This follows from a monotonicity behavior of $\left(\frac{as_p(K_1,\dots,K_n)}{as_\infty(K_1,\dots,K_n)}\right)^{n+p}$ which was shown in [59]. And the analogue to Proposition 3.6 (ii) holds

$$\Omega_{K_1,\dots K_n} = \exp\left(\frac{1}{as_\infty(K_1,\dots,K_n)}\int_{S^{n-1}}\frac{\sum_{i=1}^n \log\left[f_{K_i}h_{K_i}^{n+1}\right]}{\prod_{i=1}^n h_{K_i}} d\sigma\right)$$

Remarks.

(i) If $\mu\{x \in \partial K : \kappa(x) = 0\} = 0$, then $\Omega_K > 0$. If $\kappa(x) = 0$ μ -a.e. on ∂K, then $\Omega_K = 0$. In particular, $\Omega_P = 0$ for all polytopes P.

(ii) If K is in C^2_+, then, by (3.17), $as_\infty(K) = n|K^\circ|$ and thus we then also have

$$\Omega_K = \lim_{p\to\infty} \left(\frac{as_p(K)}{as_\infty(K)} \right)^{n+p}. \tag{3.18}$$

(ii) As for all $p \neq -n$ and for all linear, invertible transformations T, $as_p(T(K)) = |\det(T)|^{\frac{n-p}{n+p}} as_p(K)$ (see [54]) and $as_p(T(K_1), \dots, T(K_n)) = |\det(T)|^{\frac{n-p}{n+p}} as_p(K_1, \dots, K_n)$ [59], we get that

$$\Omega_{T(K)} = |\det(T)|^{2n}\, \Omega_K. \tag{3.19}$$

and

$$\Omega_{(T(K_1),\dots,T(K_n))} = |\det(T)|^{2n}\, \Omega_{K_1,\dots K_n}.$$

In particular, Ω_K and $\Omega_{K_1,\dots K_n}$ are invariant under linear transformations T with $|\det(T)| = 1$.

Corollary 3.4. *Let K be a convex body $\mathbb{R}^n$ with centroid at the origin. Then*

$$\Omega_K = \lim_{p\to 0} \left(\frac{as_p(K^\circ)}{n|K^\circ|} \right)^{\frac{n(n+p)}{p}}.$$

Proof. By Proposition 3.1 (iv) and Remark (i) after it

$$\begin{aligned}\Omega_K &= \lim_{p\to\infty} \left(\frac{as_p(K)}{n|K^\circ|} \right)^{n+p} = \lim_{p\to\infty} \left(\frac{as_{\frac{n^2}{p}}(K^\circ)}{n|K^\circ|} \right)^{n+p} \\ &= \lim_{q\to 0} \left(\frac{as_q(K^\circ)}{n|K^\circ|} \right)^{n+\frac{n^2}{q}} = \lim_{q\to 0} \left(\frac{as_q(K^\circ)}{n|K^\circ|} \right)^{\frac{n(n+q)}{q}}\end{aligned}$$

Example.

For $1 \leq r < \infty$, let $B^n_r = \{x \in \mathbb{R}^n : (\sum_{i=1}^n |x_i|^r)^{\frac{1}{r}} \leq 1\}$ and let $B^n_\infty = \{x \in \mathbb{R}^n : \max_{1\leq i\leq n}|x_i| \leq 1\}$. Then a straightforward, but tedious calculation gives

$$\Omega_{B^n_r} = \frac{\exp\left(-\frac{n^2(r-2)}{r} \left(\frac{\Gamma'(\frac{r-1}{r})}{\Gamma(\frac{r-1}{r})} - \frac{\Gamma'(n\frac{r-1}{r})}{\Gamma(n\frac{r-1}{r})} \right) \right)}{(r-1)^{n(n-1)}}. \tag{3.20}$$

Indeed, it was shown in [54] that $as_p(B^n_r) = \frac{2^n (r-1)^{\frac{p(n-1)}{n+p}}}{r^{n-1}} \frac{\left(\Gamma(\frac{n+rp-p}{r(n+p)})\right)^n}{\Gamma(\frac{n(n+rp-p)}{r(n+p)})}$. Therefore

$$\frac{as_p(B^n_r)}{n|(B^n_r)^\circ|} = \frac{1}{(r-1)^{\frac{n(n-1)}{n+p}}} \frac{\left(\Gamma(\frac{n+rp-p}{r(n+p)})\right)^n}{\Gamma(\frac{n(n+rp-p)}{r(n+p)})} \frac{\Gamma(\frac{n(r-1)}{r})}{\left(\Gamma(\frac{r-1}{r})\right)^n}$$

and

$$\Omega_{B^n_r} = \left(\frac{as_p(B^n_r)}{n|(B^n_r)^\circ|}\right)^{n+p} = \frac{\exp\left(-\frac{n^2(r-2)}{r}\left(\frac{\Gamma'(\frac{r-1}{r})}{\Gamma(\frac{r-1}{r})} - \frac{\Gamma'(n\frac{r-1}{r})}{\Gamma(n\frac{r-1}{r})}\right)\right)}{(r-1)^{n(n-1)}}$$

The next propositions describe more properties of Ω_K. Some were already stated in the introduction.

Proposition 3.5. *Let K be a convex body with centroid at the origin.*

(i) For all $p > 0$,

$$\Omega_K \leqslant \left(\frac{as_p(K^\circ)}{n|K^\circ|}\right)^{\frac{n(n+p)}{p}}.$$

If K is in addition in C^2_+, then equality holds if and only if K is an ellipsoid.

(ii) For all $p \geq 0$

$$\Omega_K \leqslant \left(\frac{as_p(K)}{n|K^\circ|}\right)^{n+p}.$$

If K is in addition in C^2_+, then equality holds if and only if K is an ellipsoid.

(iii) $\Omega_K \leqslant \left(\frac{|K|}{|K^\circ|}\right)^n$. If K is in addition in C^2_+, then equality holds if and only if K is an ellipsoid.

(iv) $\Omega_K \Omega_{K^\circ} \leqslant 1$. If K is in addition in C^2_+, then equality holds if and only if K is an ellipsoid.

Proof.

(i) The first part follows from Corollary 3.4, Proposition 3.1 (iii) and the Remark (ii) after it. The second part follows from Corollary 3.4, Proposition 3.1 (iii) and the Remark (iv) after it.

(ii) The first part follows from the definition of Ω_K, Proposition 3.1 (ii) and the Remark (ii) after it. The second part follows from the definition of Ω_K, Proposition 3.1 (ii) and the Remark (iv) after it.

(iii) By (ii), $\Omega_K \leqslant \left(\frac{as_0(K)}{n|K^\circ|}\right)^n = \left(\frac{|K|}{|K^\circ|}\right)^n$.

(iv) is immediate from (iii).

Proposition 3.6. *Let K be a convex body $\mathbb{R}^n$ with centroid at the origin.*

(i) $\Omega_K = \exp\left(\frac{1}{|K^\circ|}\int_{\partial K^\circ}\langle x, N_{K^\circ}(x)\rangle \log \frac{\kappa_{K^\circ}(x)}{\langle x, N_{K^\circ}(x)\rangle^{n+1}} d\mu_{K^\circ}(x)\right)$.

If K is in addition in C^2_+, then

(ii) $\Omega_K = \exp\Big(-\frac{1}{|K^\circ|}\int_{\partial K}\frac{\kappa_K(x)}{\langle x,N_K(x)\rangle^n}\log\frac{\kappa_K(x)}{\langle x,N_K(x)\rangle^{n+1}}d\mu_K(x)\Big)$.

(iii)
$$\frac{1}{|K|}\int_{\partial K}\langle x,N_K(x)\rangle\log\frac{\kappa_K(x)}{\langle x,N_K(x)\rangle^{n+1}}d\mu_K(x) \ \le\ n\log\frac{|K^\circ|}{|K|}\ \le \ \frac{1}{|K^\circ|}\int_{\partial K}\frac{\kappa_K(x)}{\langle x,N_K(x)\rangle^n}\log\frac{\kappa_K(x)}{\langle x,N_K(x)\rangle^{n+1}}d\mu_K(x).$$

Proof.

(i) By Corollary 3.4,

$$\begin{aligned}
\log\Omega_K &= \log\left(\lim_{p\to 0}\left(\frac{as_p(K^\circ)}{n|K^\circ|}\right)^{\frac{n(n+p)}{p}}\right) = \log\left(\lim_{p\to 0}\left(\frac{as_p(K^\circ)}{n|K^\circ|}\right)^{\frac{n^2}{p}}\right)\\
&= \lim_{p\to 0}\frac{n^2}{p}\log\frac{as_p(K^\circ)}{n|K^\circ|} = n^2\lim_{p\to 0}\frac{\frac{d}{dp}\left(as_p(K^\circ)\right)}{as_p(K^\circ)}\\
&= n^2\lim_{p\to 0}\frac{n(n+p)^{-2}}{as_p(K^\circ)}\int_{\partial K^\circ}\frac{\kappa_{K^\circ}(x)^{\frac{p}{n+p}}}{\langle x,N_{K^\circ}(x)\rangle^{\frac{n(p-1)}{n+p}}}\\
&\qquad\times\log\frac{\kappa_{K^\circ}(x)}{\langle x,N_{K^\circ}(x)\rangle^{n+1}}d\mu_{K^\circ}(x)\\
&= \frac{1}{|K^\circ|}\int_{\partial K^\circ}\langle x,N_{K^\circ}(x)\rangle\log\frac{\kappa_{K^\circ}(x)}{\langle x,N_{K^\circ}(x)\rangle^{n+1}}d\mu_{K^\circ}(x).
\end{aligned}$$

(ii) If K is in C^2_+, we have by (3.18) that

$$\begin{aligned}
\log\Omega_K &= \log\left(\lim_{p\to\infty}\left(\frac{as_p(K)}{as_\infty(K)}\right)^{n+p}\right) = \lim_{p\to\infty}\frac{\log\left(\frac{as_p(K)}{as_\infty(K)}\right)}{(n+p)^{-1}}\\
&= -\lim_{p\to\infty}\frac{(n+p)^2\frac{d}{dp}\left(as_p(K)\right)}{as_p(K)}\\
&= -\lim_{p\to\infty}\frac{(n+p)^2}{as_p(K)}\int_{\partial K}\frac{d}{dp}\Bigg(\exp\bigg(\log\left(\kappa_K(x)\right)\frac{p}{n+p}\\
&\qquad -\log\left(\langle x,N_K(x)\rangle\right)\frac{n(p-1)}{n+p}\bigg)\Bigg)d\mu_K(x)\\
&= -\lim_{p\to\infty}\frac{(n+p)^2}{as_p(K)}\int_{\partial K}\frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x,N_K(x)\rangle^{\frac{n(p-1)}{n+p}}}\bigg(\frac{n}{(n+p)^2}\log\left(\kappa_K(x)\right)\\
&\qquad -\frac{n(n+1)}{(n+p)^2}\log\left(\langle x,N_K(x)\rangle\right)\bigg)d\mu_K(x)\\
&= -\lim_{p\to\infty}\frac{n}{as_p(K)}\int_{\partial K}\frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x,N_K(x)\rangle^{\frac{n(p-1)}{n+p}}}\log\frac{\kappa_K(x)}{\langle x,N_K(x)\rangle^{n+1}}d\mu_K(x)\\
&= -\frac{n}{as_\infty(K)}\int_{\partial K}\frac{\kappa_K(x)}{\langle x,N_K(x)\rangle^n}\log\frac{\kappa_K(x)}{\langle x,N_K(x)\rangle^{n+1}}d\mu_K(x).
\end{aligned}$$

(iii) Combine Proposition 3.5 (iii) with (i) and (ii).

Let (X, μ) be a measure space and let $dP = pd\mu$ and $dQ = qd\mu$ be probability measures on X that are absolutely continuous with respect to the measure μ. The *Kullback-Leibler divergence* or *relative entropy* from P to Q is defined as [14]

$$D_{KL}(P\|Q) = \int_X p \log \frac{p}{q} d\mu. \tag{3.21}$$

The information inequality [14] holds for the Kullback-Leibler divergence: Let P and Q be as above. Then

$$D_{KL}(P\|Q) \geq 0, \tag{3.22}$$

with equality if and only if $P = Q$.

The invariant Ω_K is related to relative entropies on K and a corresponding information inequality holds, which is exactly the inequality of Proposition 3.5 (iii).

Proposition 3.7. *Let K a convex body in $\mathbb{R}^n$ that is C^2_+. Let*

$$p(x) = \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n \ n|K^\circ|}\,, \quad q(x) = \frac{\langle x, N_K(x)\rangle}{n\ |K|}. \tag{3.23}$$

Then $P = p\ \mu$ and $Q = q\ \mu$ are probability measures on ∂K that are absolutely continuous with respect to μ_K and

$$D_{KL}(P\|Q) = \log\left(\frac{|K|}{|K^\circ|}\Omega_K^{-\frac{1}{n}}\right) \tag{3.24}$$

and

$$D_{KL}(Q\|P) = \log\left(\frac{|K^\circ|}{|K|}\Omega_{K^\circ}^{-\frac{1}{n}}\right). \tag{3.25}$$

Moreover Gibb's inequality implies that

$$\Omega_K \leqslant \left(\frac{|K|}{|K^\circ|}\right)^n$$

with equality if and only if K is an ellipsoid.

Proof of Proposition 3.7.

As

$$n|K| = \int_{\partial K} \langle x, N_K\rangle d\mu_K(x) \quad \text{and} \quad n|K^\circ| = \int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n} d\mu_K(x),$$

$\int_{\partial K} p\ d\mu_K = \int_{\partial K} q\ d\mu_K = 1$ and hence P and Q are probability measures that are absolutely continuous with respect to μ_K on K.

(3.24) resp. (3.25) follow from the definition of the relative entropy (3.21) and Proposition 3.6 (ii) resp. Proposition 3.6 (i).

By (3.22), equality holds in the inequality of the proposition, if and only if for all $x \in \partial K$

$$\frac{\kappa_K(x)}{\langle x, N_K(x)\rangle} = \frac{|K|}{|K^\circ|} = \text{constant}$$

which holds, by the above mentioned theorem of Petty [48] if and only if K is an ellipsoid.

Let K be a convex body in $\mathbb{R}^n$. Recall that the normalized cone measure cm_K on ∂K is defined as follows: For every measurable set $A \subseteq \partial K$

$$cm_K(A) = \frac{1}{|K|}|\{ta : \ a \in A, t \in [0,1]\}|. \tag{3.26}$$

For more information about cone measures we refer to e.g., [8], [18], [44] and [45].

The next proposition is well known. It shows that the measures P and Q defined in Proposition 3.7 are the cone measures of K and K°. We include the proof for completeness. $N_K : \partial K \to S^{n-1}$, $x \to N_K(x)$ is the Gauss map.

Proposition 3.8. *Let K a convex body in $\mathbb{R}^n$ that is C^2_+. Let P and Q be the probability measures on ∂K defined by (3.23). Then*

$$P = N_K^{-1} N_{K^\circ} cm_{K^\circ} \quad and \quad Q = cm_K,$$

or, equivalently, for every measurable subset A in ∂K

$$P(A) = cm_{K^\circ}\left(N_{K^\circ}^{-1}\big(N_K(A)\big)\right) \quad and \quad Q(A) = cm_K(A).$$

Proof.

$$Q(A) = \frac{1}{n|K|}\int_A \langle x, N_K(x)\rangle d\mu_K(x) = cm_K(A).$$

Also

$$P(A) = \int_A \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n}\frac{d\mu_K(x)}{n|K^\circ|} = \frac{1}{n|K^\circ|}\int_{N_K(A)} \frac{1}{h_K^n(u)} d\sigma(u).$$

Let $B \subseteq \partial K^\circ$. Then

$$cm_{K^\circ}(B) = \frac{1}{|K^\circ|}\left|\left\{x \in \mathbb{R}^n : \|x\|_{K^\circ} \leqslant 1, \ \frac{x}{\|x\|_2} \in N_{K^\circ}(B)\right\}\right| .$$

Let $\Delta = \left\{x \in \mathbb{R}^n : \|x\|_{K^\circ} \leqslant 1, \ \frac{x}{\|x\|_2} \in N_{K^\circ}(B)\right\}$. We have (see [45])

$$\begin{aligned} cm_{K^\circ}(B) &= \frac{|\Delta|}{|K^\circ|} = \frac{1}{|K^\circ|}\int_0^\infty \int_{S^{n-1}} r^{n-1} 1_\Delta(r\theta) dr d\sigma(\theta) \\ &= \frac{1}{|K^\circ|}\int_{N_{K^\circ}(B)} \int_0^{\frac{1}{\|\theta\|_{K^\circ}}} r^{n-1} dr d\sigma(\theta) \\ &= \frac{1}{n|K^\circ|}\int_{N_{K^\circ}(B)} \frac{1}{h_K^n(\theta)} d\sigma(\theta). \end{aligned}$$

Let $B \in \partial K^\circ$ be such that $N_{K^\circ}(B) = N_K(A)$. This means that $B = N_{K^\circ}^{-1}\big(N_K(A)\big)$. Then $P(A) = cm_{K^\circ}\bigg(N_{K^\circ}^{-1}\big(N_K(A)\big)\bigg)$, which completes the proof.

Therefore, with P and Q defined as in (3.23),

$$D_{KL}(P\|Q) = D_{KL}\big(N_K N_{K^\circ}^{-1} cm_{K^\circ} \| cm_K\big) \tag{3.27}$$

and we get as a corollary to Proposition 3.6 that the invariant Ω_K is the exponential of the relative entropy of the cone measures of K and K°.

Corollary 3.9. *Let K be a convex body in C^2_+. Then*

$$\Omega_K^{1/n} = \frac{|K^\circ|}{|K|} \exp\bigg(- D_{KL}(N_K N_{K^\circ}^{-1} cm_{K^\circ} \| cm_K)\bigg) .$$

Finally, an isoperimetric inequality holds for the affine invariant Ω_K:

Proposition 3.10. *Let K be a convex body in C^2_+ of volume 1. Then*

$$\Omega_{K^\circ} \leqslant \Omega_{(\widetilde{B^2_n})^\circ}$$

with equality if and only if $K = \widetilde{B^2_n}$.

Proof.

The proof follows from the above information inequality for convex bodies together with the Blaschke Santaló inequality and the fact that $\Omega_{(\widetilde{B^2_n})^\circ} = |B^2_n|^{2n}$.

In the next section we show that the invariant Ω_K is related to the L_p centroid bodies.

4 $Z_p(K)$ for K in C^2_+

In this section we show how Ω_K is related to the L_p centroid bodies. The main theorem of the section is Theorem 4.1. We assume there that K is symmetric, mainly because the bodies $Z_p(K)$ are symmetric by definition. Also, throughout this section we assume that K is of volume 1.

Theorem 4.1. *Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1 that is in C^2_+. Then*

(i) $\displaystyle \lim_{p\to\infty} \frac{p}{\log p}\big(|Z_p^\circ(K)| - |K^\circ|\big) = \frac{n(n+1)}{2}\, |K^\circ|.$

$$(ii)\ \lim_{p\to\infty} p\left(|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)}{2p}\log p\ |Z_p^\circ(K)|\right) =$$
$$\lim_{p\to\infty} p\left(|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)}{2p}\log p\ |K^\circ|\right) =$$
$$-\frac{1}{2}\int_{S^{n-1}} h_K(u)^{-n}\ \log\left(2^{n+1}\pi^{n-1}h_K(u)^{n+1}f_K(u)\right)d\sigma(u) =$$
$$\frac{1}{2}\int_{\partial K}\frac{\kappa(x)}{\langle x,N(x)\rangle^n}\ \log\left(\frac{\kappa(x)}{2^{n+1}\pi^{n-1}\langle x,N(x)\rangle^{n+1}}\right)d\mu_K(x)$$

Thus Theorem 4.1 shows that if K is a symmetric convex body in C^2_+ of volume 1, then

$$\lim_{p\to\infty} p\left(|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)\log p}{2p}|Z_p^\circ(K)|\right) =$$
$$\lim_{p\to\infty} p\left(|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)}{2p}\log p\ |K^\circ|\right) =$$
$$\frac{1}{2}\int_{\partial K}\frac{\kappa_K(x)}{\langle x,N(x)\rangle^n}\log\left(2^{n+1}\pi^{n-1}\frac{\kappa_K(x)}{\langle x,N(x)\rangle^{n+1}}\right)d\mu_K(x) =$$
$$\frac{\log\left(2^{n+1}\pi^{n-1}\right)}{2}\int_{\partial K}\frac{\kappa_K(x)}{\langle x,N(x)\rangle^n}d\mu_K(x)$$
$$+\frac{1}{2}\int_{\partial K}\frac{\kappa_K(x)}{\langle x,N(x)\rangle^n}\log\left(\frac{\kappa_K(x)}{\langle x,N(x)\rangle^{n+1}}\right)d\mu_K(x) =$$
$$\log\left(2^{n+1}\pi^{n-1}\right)\frac{n|K^\circ|}{2} - \frac{|K^\circ|}{2}\log\Omega_K = -\frac{|K^\circ|}{2}\log\frac{\Omega_K}{2^{n(n+1)}\pi^{n(n-1)}}$$

or

$$\lim_{p\to\infty} p\left(\frac{|Z_p^\circ(K)|}{|K^\circ|} - \left(1 - \frac{n(n+1)\log p}{2p}\right)\right)$$
$$= \lim_{p\to\infty} p\left((1 - \frac{n(n+1)\log p}{2p})\frac{|Z_p^\circ(K)|}{|K^\circ|} - 1\right) = -\frac{1}{2}\log\frac{\Omega_K}{2^{n(n+1)}\pi^{n(n-1)}}. \quad (4.28)$$

So we have the following

Corollary 4.2. *Let K and C be symmetric convex bodies of volume 1 in C^2_+. Then*

$$(i)\ \lim_{p\to\infty}\frac{2p}{n}\left(\frac{(1-\frac{n(n+1)\log p}{2p})|Z_p^\circ(K)|}{|K^\circ|} - 1\right) =$$
$$\lim_{p\to\infty}\frac{2p}{n}\left(\frac{|Z_p^\circ(K)|}{|K^\circ|} - \left(1 - \frac{n(n+1)\log p}{2p}\right)\right) = -\frac{1}{2}\log\frac{\Omega_K^{\frac{1}{n}}}{2^{n+1}\pi^{n-1}}$$
$$= (n+1)\log\left(\frac{2\pi^{\frac{n-1}{n+1}}}{|K^\circ|}\right) + D_{KL}\left(N_K N_{K^\circ}^{-1} cm_{K^\circ}\|cm_K\right)$$

$$(ii)\ \lim_{p\to\infty} p\left((1-\frac{n(n+1)\log p}{2p})\frac{|Z_p^\circ(K)|}{|K^\circ|}-1\right)\geq \frac{1}{2}\log\left(2^{n(n+1)}\pi^{n(n-1)}\frac{|K^\circ|}{|K|}\right).$$

The corresponding statement for $\lim_{p\to\infty} p\left(\frac{|Z_p^\circ(K)|}{|K^\circ|}-\left(1-\frac{n(n+1)\log p}{2p}\right)\right)$ *also holds.*

$$(iii)\ \lim_{p\to\infty} p\left(1-\frac{n(n+1)\log p}{2p}\right)\left(\frac{|Z_p^\circ(K)|}{|K^\circ|}-\frac{|Z_p^\circ(C)|}{|C^\circ|}\right)=\frac{1}{2n}\log\frac{\Omega_C}{\Omega_K}.$$

Proof.

(i) follows from (4.28) and Corollary 3.9, (ii) follows from Proposition 3.5 (iii) and (iii) follows from (4.28).

The remainder of the section is devoted to the proof of Theorem 4.1. We need several lemmas and notations.

Let $x, y > 0$. Let $\Gamma(x)=\int_0^\infty \lambda^{x-1}e^{-\lambda}d\lambda$ be the Gamma function and let $B(x,y)=\int_0^1 \lambda^{x-1}(1-\lambda)^{y-1}d\lambda=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ be the Beta function.

We write $f(p)=g(p)\pm o(p)$, if there exists a function $h(p)$ such that $f(p)=g(p)+h(p)$ and $\lim_{p\to\infty} ph(p)=0$, i.e. $h(p)$ has terms of order $\frac{1}{p^2}$ and higher. Similarly, $f(p)=g(p)\pm o(p^2)$, if there exists a function $h(p)$ such that $f(p)=g(p)+h(p)$ and $\lim_{p\to\infty} p^2h(p)=0$, i.e. $h(p)$ has terms of order $\frac{1}{p^3}$ and higher. We write$f(p)=g(p)\pm O(p)$, if there exists a function $h(p)$ such that $f(p)=g(p)+h(p)$ and $\lim_{p\to\infty} h(p)=0$

Lemma 4.3. *Let $p>0$. Then*

$$\begin{aligned}(i)\ \left(B\left(p+1,\frac{n+1}{2}\right)\right)^{\frac{n}{p}} &= 1-\frac{n(n+1)}{2p}\log p+\frac{n}{p}\log\left(\Gamma(\frac{n+1}{2})\right)+\\ &\frac{n^2(n+1)^2}{8p^2}(\log p)^2-\frac{n^2(n+1)}{2p^2}\log\left(\Gamma(\frac{n+1}{2})\right)\log p+\\ &\frac{n}{2p^2}\left[n\left(\log\left(\Gamma(\frac{n+1}{2})\right)\right)^2-\frac{n+1}{4}\left(n(n+1)+2(n+3)\right)\right]\\ &\pm o(p^2).\end{aligned}$$

(ii) Let $0\leq a\leq 1$. Then

$$\left(\int_0^1 u^p(1-u)^{\frac{n-1}{2}}\left(1-a\left(1-u\right)\right)^{\frac{n-1}{2}}du\right)^{\frac{n}{p}}=1-\frac{n(n+1)}{2p}\log p+$$
$$\frac{n}{p}\log\left(\Gamma(\frac{n+1}{2})\right)+\frac{n^2(n+1)^2}{8p^2}(\log p)^2-\frac{n^2(n+1)}{2p^2}\log\left(\Gamma(\frac{n+1}{2})\right)\log p+$$
$$\frac{n}{2p^2}\left[n\left(\log\left(\Gamma(\frac{n+1}{2})\right)\right)^2-\frac{(n+1)\left(n^2+3n+6\right)}{4}-(n+1)\binom{\frac{n-1}{2}}{1}\ a\right]\pm o(p^2).$$

The proof of Lemma 4.3 is in the Appendix.

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a C^2 log-concave function with $\int_{\mathbb{R}_+} f(t)dt < \infty$ and let $p \geq 1$. Let $g_p(t) = t^p f(t)$ and let $t_p = t_p(f)$ the unique point such that $g'(t_p) = 0$. We make use of the following Lemma due to B. Klartag [26] (Lemma 4.3 and Lemma 4.5).

Lemma 4.4. *Let f be as above. For every $\varepsilon \in (0,1)$,*

$$\int_0^\infty t^p f(t)dt \leq \left(1 + Ce^{-cp\varepsilon^2}\right) \int_{t_p(1-\varepsilon)}^{t_p(1+\varepsilon)} t^p f(t)dt$$

where $C > 0$ and $c > 0$ are universal constants.

We think that the next lemma is well known. We give a proof for completeness.

Lemma 4.5. *Let $u \in S^{n-1}$. Let f and t_p be as above and f also such that it is decreasing and a probability density on $[0, h(u)]$. Then*

$$\lim_{p\to\infty} t_p = h(u).$$

Proof.
We only have to show that $\lim_{p\to\infty} t_p \geq h(u)$. By Hölder, $\left(\int_0^{h(u)} t^p f(t)dt\right)^{\frac{1}{p}} \to h(u)$. Thus, for $\varepsilon > 0$ given, there exists p_ε such that for all $p \geq p_\varepsilon$,

$$\int_0^{h(u)} t^p f(t)dt \geq \big(h(u) - \varepsilon\big)^p$$

By Lemma 4.4, for all $0 < \delta < 1$, $\int_0^\infty t^p f(t)dt \leq \left(1 + Ce^{-cp\delta^2}\right) \int_{t_p(1-\delta)}^{t_p(1+\delta)} t^p f(t)dt$. We choose $\delta = \frac{1}{p^{\frac{1}{4}}}$ with $p > p_\varepsilon$ and get, using the monotonicity behavior of $t^p f$ on the respective intervals, that

$$\begin{aligned} \big(h(u) - \varepsilon\big)^p &\leq \left(1 + Ce^{-cp\sqrt{p}}\right) \left[\int_{t_p(1-\delta)}^{t_p} t^p f(t)dt + \int_{t_p}^{t_p(1+\delta)} t^p f(t)dt\right] \\ &\leq \left(1 + Ce^{-cp\sqrt{p}}\right)\, p^{\frac{1}{4}} t_p f(t_p)\, t_p^p. \end{aligned}$$

As f is decreasing, $f(t_p) \leq f(0)$. Moreover $t_p \leq h(u)$. Thus, for $p \geq p_\varepsilon$ large enough, $\left(p^{\frac{1}{4}} t_p f(t_p)\right)^{\frac{1}{p}} \leq 1 + \varepsilon$ and hence $h(u) - \varepsilon < (1 + \varepsilon)\ t_p$

Remark

We will apply Lemma 4.4 to the function $f(t) = |K \cap (u^\perp + tu)|$, $u \in S^{n-1}$. We show below that f is C^2. Thus t_p is well defined and Lemma 4.4 holds. Also, t_p is an increasing function of p and by the above Lemma, 4.5, $\lim_{p\to\infty} t_p = h_K(u)$.

We also think that the following lemma is well known but we could not find a proof in the literature. Therefore we include a proof.

Lemma 4.6. *Let K be a convex body in in C^2_+. Let $u \in S^{n-1}$ and let H_t be the hyperplane orthogonal to u at distance t from the origin. Let $f(t) = |K \cap H_t|$. Then f is C^2. In fact,*

$$f'(t) = -\int_{\partial K \cap H_t} \frac{\langle u, N_K(x)\rangle}{\left(1 - \langle u, N_K(x)\rangle^2\right)^{\frac{1}{2}}} \, d\mu_{\partial K \cap H_t}(x)$$

and

$$f''(t) =$$

$$-\int_{\partial K \cap H_t} \left[\frac{\kappa(x_t)^{\frac{1}{n-1}}}{(1 - \langle N_K(x_t), u\rangle^2)^{\frac{3}{2}}} - \frac{(n-2)\ \langle N_K(x_t), u\rangle^2}{\langle N_{K\cap H_t}(x_t), x_t\rangle\ \ (1 - \langle N_K(x_t), u\rangle^2)}\right] d\mu_{\partial K \cap H_t}(x_t).$$

Proof.
We assume that $\mathrm{int}(K) \cap H_t \neq \emptyset$. To show that $f \in C^2$, we compute the derivates of f. We first show that

$$f'(t) = -\int_{\partial K \cap H_t} \frac{\langle u, N_K(x)\rangle}{\left(1 - \langle u, N_K(x)\rangle^2\right)^{\frac{1}{2}}} \, d\mu_{\partial K \cap H_t}(x).$$

Indeed, for $x \in \partial K \cap H_t$, let $\alpha(x)$ be the (smaller) angle formed by $N_K(x)$ and u. Then $\cos\, \alpha(x) = \langle u, N_K(x)\rangle$ and

$$\begin{aligned} f'(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(|K \cap H_{t+\varepsilon}| - |K \cap H_t| \right) &= -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\int_{\partial K \cap H_t} \varepsilon \cot\, \alpha(x)\ d\mu_{\partial K \cap H_t}(x) \right) \\ &= -\int_{\partial K \cap H_t} \frac{\langle u, N_K(x)\rangle}{\left(1 - \langle u, N_K(x)\rangle^2\right)^{\frac{1}{2}}} \, d\mu_{\partial K \cap H_t}(x). \end{aligned}$$

We show next that

$$f''(t) =$$

$$-\int_{\partial K \cap H_t} \left[\frac{\kappa(x_t)^{\frac{1}{n-1}}}{(1 - \langle N_K(x_t), u\rangle^2)^{\frac{3}{2}}} - \frac{(n-2)\ \langle N_K(x_t), u\rangle^2}{\langle N_{K\cap H_t}(x_t), x_t\rangle\ \ (1 - \langle N_K(x_t), u\rangle^2)}\right] d\mu_{\partial K \cap H_t}(x_t).$$

By definition

$$\begin{aligned} f''(t) = -\lim_{\varepsilon \to 0}\ \frac{1}{\varepsilon} \Bigg(&\int_{\partial K \cap H_{t+\varepsilon}} \frac{\langle u, N_K(y_{t+\varepsilon})\rangle}{\left(1 - \langle u, N_K(y_{t+\varepsilon})\rangle^2\right)^{\frac{1}{2}}} \, d\mu_{\partial K \cap H_{t+\varepsilon}}(y_{t+\varepsilon}) \\ &- \int_{\partial K \cap H_t} \frac{\langle u, N_K(x_t)\rangle}{\left(1 - \langle u, N_K(x_t)\rangle^2\right)^{\frac{1}{2}}} \, d\mu_{\partial K \cap H_t}(x_t) \Bigg) \end{aligned}$$

We project $K \cap H_{t+\varepsilon}$ onto $K \cap H_t$ and we want to integrate both expressions over $\partial K \cap H_t$. To do so, we fix - after the projection - an interior point x_0 in $K \cap H_{t+\varepsilon}$. For $x_t \in \partial K \cap H_t$, let $[x_0, x_t]$ be the line segment from x_0 to x_t and let $x_{t+\varepsilon} = \partial K \cap H_{t+\varepsilon} \cap [x_0, x_t]$. Now observe that

$$d\mu_{\partial K \cap H_{t+\varepsilon}} = \frac{1}{\langle N_{K\cap H_t}(x_t), N_{K\cap H_{t+\varepsilon}}(x_{t+\varepsilon})\rangle} \ \left(\frac{\|x_{t+\varepsilon}\|}{\|x_t\|}\right)^{n-2} \ d\mu_{\partial K \cap H_t},$$

where $N_{K\cap H_t}(x_t)$ is the outer normal in x_t to the boundary of the $n-1$ dimensional convex body $K\cap H_t$ and similarly, $N_{K\cap H_{t+\varepsilon}}(x_{t+\varepsilon})$ is the outer normal in $x_{t+\varepsilon}$ to the boundary of the $n-1$ dimensional convex body $K\cap H_{t+\varepsilon}$.

Notice further that

$$\|x_t\| - \|x_{t+\varepsilon}\| = \frac{\varepsilon \, \langle N_K(x_t), u\rangle \, \|x_t\|}{\langle N_{K\cap H_t}(x_t), x_t\rangle \ \left(1-\langle N_K(x_t),u\rangle^2\right)^{\frac{1}{2}}} + \text{ higher order terms in } \varepsilon.$$

Therefore

$$\begin{aligned}\left(\frac{\|x_{t+\varepsilon}\|}{\|x_t\|}\right)^{n-2} &= \left(1-\frac{\varepsilon \, \langle N_K(x_t), u\rangle}{\langle N_{K\cap H_t}(x_t), x_t\rangle \ \left(1-\langle N_K(x_t),u\rangle^2\right)^{\frac{1}{2}}}\right)^{n-2} \\ &= 1-\frac{(n-2)\, \varepsilon \, \langle N_K(x_t), u\rangle}{\langle N_{K\cap H_t}(x_t), x_t\rangle \ \left(1-\langle N_K(x_t),u\rangle^2\right)^{\frac{1}{2}}} \\ &+ \quad \text{higher order terms in } \varepsilon\end{aligned}$$

Thus

$$\begin{aligned} f''(t) &= -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\partial K\cap H_t}\Bigg[\frac{\langle u, N_K(y_{t+\varepsilon})\rangle}{\langle N_{K\cap H_t}(x_t), N_{K\cap H_{t+\varepsilon}}(x_{t+\varepsilon})\rangle\ \left(1-\langle u, N_K(y_{t+\varepsilon})\rangle^2\right)^{\frac{1}{2}}} \\ &\times \left(1-\frac{(n-2)\,\varepsilon\,\langle N_K(x_t),u\rangle}{\langle N_{K\cap H_t}(x_t), x_t\rangle\ \left(1-\langle N_K(x_t),u\rangle^2\right)^{\frac{1}{2}}} + \text{ higher order terms in } \varepsilon\right) \\ &- \frac{\langle u, N_K(x_t)\rangle}{\left(1-\langle u, N_K(x_t)\rangle^2\right)^{\frac{1}{2}}}\Bigg]\, d\mu_{\partial K\cap H_t}(x_t) \\ &= -\int_{\partial K\cap H_t}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Bigg[\frac{\langle u, N_K(y_{t+\varepsilon})\rangle}{\langle N_{K\cap H_t}(x_t), N_{K\cap H_{t+\varepsilon}}(x_{t+\varepsilon})\rangle\ \left(1-\langle u, N_K(y_{t+\varepsilon})\rangle^2\right)^{\frac{1}{2}}} \\ &\times \left(1-\frac{(n-2)\,\varepsilon\,\langle N_K(x_t),u\rangle}{\langle N_{K\cap H_t}(x_t), x_t\rangle\ \left(1-\langle N_K(x_t),u\rangle^2\right)^{\frac{1}{2}}} + \text{ higher order terms in } \varepsilon\right) \\ &- \frac{\langle u, N_K(x_t)\rangle}{\left(1-\langle u, N_K(x_t)\rangle^2\right)^{\frac{1}{2}}}\Bigg]\, d\mu_{\partial K\cap H_t}(x_t).\end{aligned}$$

We can interchange integration and limit using Lebegue's theorem as the functions under the integral are uniformly (in t) bounded by a constant.

Denote $g_x(t) = \frac{\langle N_K(x_t),u\rangle}{\left(1-\langle u,N_K(x_t)\rangle^2\right)^{\frac{1}{2}}}$. Then the expression under the integral becomes

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\ \Bigg[\frac{g_y(t+\varepsilon)}{\langle N_{K\cap H_t}(x_t), N_{K\cap H_{t+\varepsilon}}(x_{t+\varepsilon})\rangle}\Bigg(1-\frac{(n-2)\,\varepsilon\,\langle N_K(x_t),u\rangle}{\langle N_{K\cap H_t}(x_t), x_t\rangle\ \left(1-\langle N_K(x_t),u\rangle^2\right)^{\frac{1}{2}}} \\ &\qquad + \text{higher order terms in } \varepsilon\Bigg) - g_x(t)\Bigg] \\ &\qquad = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\ \Big[g_y(t+\varepsilon)-g_x(t)\Big] - \frac{(n-2)\ \langle N_K(x_t),u\rangle^2}{\langle N_{K\cap H_t}(x_t), x_t\rangle\ \left(1-\langle N_K(x_t),u\rangle^2\right)}.\end{aligned}$$

Here we have also used that, as $\varepsilon\to 0$, $x_{t+\varepsilon}\to x_t$, $N_{K\cap H_{t+\varepsilon}}(x_{t+\varepsilon})\to N_{K\cap H_t}(x_t)$ and $g_y(t+\varepsilon)\to g_x(t)$.

To compute $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[g_y(t+\varepsilon) - g_x(t)\right]$, we approximate the boundary of ∂K in x_t by an ellipsoid. This can be done as ∂K is C^2_+ by assumption (see Lemma 4.8 below). To simplify the computations, we assume that the approximating ellipsoid is a Euclidean ball. The case of the ellipsoid is treated similarly, the computations are just slightly more involved. As the expression under the integral depends only on the angles between the vectors involved, we can put the origin so that the approximating Euclidean ball is centered at 0. Let $r = \kappa(x_t)^{\frac{-1}{n-1}}$ be its radius. Then

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[g_y(t+\varepsilon) - g_x(t)\right] = \frac{1}{r\ (1-\langle N_K(x_t), u\rangle^2)^{\frac{3}{2}}} = \frac{\kappa(x_t)^{\frac{1}{n-1}}}{(1-\langle N_K(x_t), u\rangle^2)^{\frac{3}{2}}}.$$

Alltogether

$$f''(t) = -\int_{\partial K \cap H_t}\left[\frac{\kappa(x_t)^{\frac{1}{n-1}}}{(1-\langle N_K(x_t), u\rangle^2)^{\frac{3}{2}}} - \frac{(n-2)\langle N_K(x_t), u\rangle^2}{\langle N_{K\cap H_t}(x_t), x_t\rangle\,(1-\langle N_K(x_t), u\rangle^2)}\right] d\mu_{\partial K \cap H_t}(x_t).$$

Lemma 4.7. *Let K be a symmetric convex body of volume 1 in C^2_+.*

(i) The functions

$$\frac{p}{log(p)}\frac{1}{h_{Z_p(K)}(u)^n}\left(1 - \frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right)$$

are uniformly (in p) bounded by a function that is integrable on S^{n-1}.

(ii) The functions

$$\frac{p}{h_{Z_p(K)}(u)^n}\left(1 - \frac{h_{Z_p(K)}(u)^n}{h_K(u)^n} - \frac{n(n+1)}{2}\ \frac{log(p)}{p}\ \frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right)$$

are uniformly (in p) bounded by a function that is integrable on S^{n-1}.

Proof.

(i) Let $u \in S^{n-1}$. Let $x \in \partial K$ be such that $N_K(x) = u$. As K is in C^2_+, by the Blaschke rolling theorem (see [51]), there exists a ball with radius r_0 that rolls freely in K: for all $x \in \partial K$, $B^n_2(x - r_0 N(x), r_0) \subset K$. As K is symmetric,

$$\begin{aligned} h_{Z_p}(u)^n &= \left(2\int_0^{h_K(u)} t^p |\{y \in K : \langle u, y\rangle = t\}| dt\right)^{\frac{n}{p}} \\ &\geq \left(2\int_{h_K(u)-r}^{h_K(u)} t^p |\{y \in B^n_2\big(x - r_0\ u, r_0\big) : \langle u, y\rangle = t\}|\ dt\right)^{\frac{n}{p}} \\ &= 2^{\frac{n}{p}}\ |B^{n-1}_2|^{\frac{n}{p}}\left(\int_{h_K(u)-r_0}^{h_K(u)} t^p\left(2r_0\big(h_K(u)-t\big)\left[1 - \frac{h_K(u)-t}{2r_0}\right]\right)^{\frac{n-1}{2}} dt\right)^{\frac{n}{p}} \end{aligned}$$

The equality holds as the $(n-1)$-dimensional Euclidean ball

$$B_2^n(x - r_0\ u, r) \cap \{y \in \mathbb{R}^n : \langle u, y\rangle = t\}$$

has radius $\left(2r_0\big(h_K(u)-t\big)\left[1-\frac{h_K(u)-t}{2r_0}\right]\right)^{\frac12}$. Now - where, to abbreviate, we write h_K, $h_{Z_p(K)}$, instead of $h_K(u)$, $h_{Z_p(K)}(u)$ - and where we use that $\frac12 \le 1-\frac{h_K(u)-t}{2r_0}$,

$$\begin{aligned} h_{Z_p}(u)^n &\ge 2^{\frac{n}{p}}\ |B_2^{n-1}|^{\frac{n}{p}}\big(r_0\ h_K\big)^{\frac{n(n-1)}{2p}}\left(\int_{h_K-r_0}^{h_K} t^p\left(1-\frac{t}{h_K}\right)^{\frac{n-1}{2}} dt\right)^{\frac{n}{p}} \\ &= h_K^n\ \left(2\ |B_2^{n-1}|\ h_K^{\frac{n+1}{2}}\ r_0^{\frac{n-1}{2}}\right)^{\frac{n}{p}}\ \left(\int_{1-\frac{r_0}{h_K}}^{1} w^p(1-w)^{\frac{n-1}{2}}dw\right)^{\frac{n}{p}}. \qquad (4.29) \end{aligned}$$

As K is symmetric, $r_0 \le h_K(u)$. If $r_0 = h_K(u)$, then

$$\frac{h^n_{Z_p(K)}}{h_K^n} \ge \left(2\ r_0^{\frac{n-1}{2}}\ h_K^{\frac{n+1}{2}}\ |B_2^{n-1}|\right)^{\frac{n}{p}}\left(\int_0^1 w^p(1-w)^{\frac{n-1}{2}}dw\right)^{\frac{n}{p}}.$$

If $r_0 < h_K(u)$, we apply Lemma 4.4 to the function $f(w) = (1-w)^{\frac{n-1}{2}}$. We choose ε so small and p_0 so large that $\varepsilon + (1+\varepsilon)\frac{n-1}{2p_0} \le \frac{r_0}{h_K}$. Then Lemma 4.4 holds and we get for all $p \ge p_0$

$$\begin{aligned} \frac{h^n_{Z_p(K)}}{h_K^n} &\ge \left(2\ r_0^{\frac{n-1}{2}}\ h_K^{\frac{n+1}{2}}\ |B_2^{n-1}|\right)^{\frac{n}{p}}\left(\int_{1-\frac{r_0}{h_K}}^1 w^p(1-w)^{\frac{n-1}{2}}dw\right)^{\frac{n}{p}} \\ &\ge \left(\frac{2\ r_0^{\frac{n-1}{2}}\ h_K^{\frac{n+1}{2}}\ |B_2^{n-1}|}{1+C\ e^{-cp\varepsilon^2}}\right)^{\frac{n}{p}}\ \left(\int_0^1 w^p(1-w)^{\frac{n-1}{2}}dw\right)^{\frac{n}{p}} \\ &= \left(\frac{2\ r_0^{\frac{n-1}{2}}\ h_K^{\frac{n+1}{2}}\ |B_2^{n-1}|}{1+C\ e^{-cp\varepsilon^2}}\right)^{\frac{n}{p}}\ \left(B(p+1,\frac{n+1}{2})\right)^{\frac{n}{p}}. \end{aligned}$$

As

$$\left(2\ r_0^{\frac{n-1}{2}}\ h_K^{\frac{n+1}{2}}\ |B_2^{n-1}|\right)^{\frac{n}{p}} = 1+\frac{n}{p}\log\left[2\ r_0^{\frac{n-1}{2}}\ h_K^{\frac{n+1}{2}}\ |B_2^{n-1}|\right] \pm o(p),$$

respectively

$$\left(\frac{2\ r_0^{\frac{n-1}{2}}\ h_K^{\frac{n+1}{2}}\ |B_2^{n-1}|}{1+C\ e^{-cp\varepsilon^2}}\right)^{\frac{n}{p}} = 1+\frac{n}{p}\log\left[\frac{2\ r_0^{\frac{n-1}{2}}\ h_K^{\frac{n+1}{2}}\ |B_2^{n-1}|}{1+C\ e^{-cp\varepsilon^2}}\right] \pm o(p)$$

we get, together with Lemma 4.3 (i)

$$\begin{aligned} \frac{h^n_{Z_p(K)}}{h_K^n} &\ge 1-\frac{n(n+1)}{2p}\log p+\frac{n}{p}\ \log\left[2\ r_0^{\frac{n-1}{2}}\ h_K^{\frac{n+1}{2}}\ |B_2^{n-1}|\ \Gamma\left(\frac{n+1}{2}\right)\right] \pm o(p) \\ &\ge 1-\frac{n(n+1)}{2p}\log p+\frac{n}{2p}\ \log\left[4\ r_0^{n-1}\pi^{n-1}h_K^{n+1}\right] \pm o(p) \qquad (4.30) \end{aligned}$$

respectively

$$\begin{aligned}
\frac{h^n_{Z_p(K)}}{h^n_K} &\geq 1-\frac{n(n+1)}{2p}\log p+\frac{n}{p}\ \log\left[\frac{2\ r_0^{\frac{n-1}{2}}\ \ h_K^{\frac{n+1}{2}}\ |B_2^{n-1}|\ \Gamma\left(\frac{n+1}{2}\right)}{1+C\ e^{-cp\varepsilon^2}}\right]\pm o(p) \qquad (4.31)\\
&\geq 1-\frac{n(n+1)}{2p}\log p+\frac{n}{2p}\ \log\left[\frac{4\ r_0^{n-1}\pi^{n-1}h_K^{n+1}}{(1+C\ e^{-cp\varepsilon^2})^2}\right]\pm o(p) \qquad (4.32)
\end{aligned}$$

Now notice that there is $\alpha>0$ such that

$$B_2^n(0,\alpha)\subset K\subset B_2^n(0,\frac{1}{\alpha}).$$

This implies that for all $u\in S^{n-1}$ $\alpha\leq h_K\leq\frac{1}{\alpha}$. Moreover we can choose α so small that we have for all $p\geq p_0>1$

$$B_2^n(0,\alpha)\subset Z_p(K)\subset K\subset B_2^n(0,\frac{1}{\alpha}),$$

which implies that for all $u\in S^{n-1}$, for all $p\geq p_0$,

$$\alpha\leq h_{Z_p(K)}\leq\frac{1}{\alpha}. \qquad (4.33)$$

On the one hand, as $Z_p(K)\subset K$,

$$\frac{p}{\log(p)}\frac{1}{h_{Z_p(K)}(u)^n}\left(1-\frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right)\geq 0$$

On the other hand, we get by (4.30), (4.31) and (4.33) with a constant c

$$\begin{aligned}
\frac{p}{\log(p)}\frac{1}{h_{Z_p(K)}(u)^n}\left(1-\frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right) &\leq \frac{cn}{\alpha^n}\left(n+1-\frac{1}{\log p}\log\left(4r_0^{n-1}\pi^{n-1}h_K^{n+1}\right)\right)\\
&\leq \frac{cn}{\alpha^n}\left(n+1+\frac{1}{\log p_0}\left|\log\left(\frac{4r_0^{n-1}\pi^{n-1}}{\alpha^{n+1}}\right)\right|\right)
\end{aligned}$$

respectively

$$\begin{aligned}
\frac{p}{\log(p)}\frac{1}{h_{Z_p(K)}(u)^n}\left(1-\frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right) &\leq \frac{cn}{\alpha^n}\left(n+1-\frac{1}{\log p}\log\left(\frac{4r_0^{n-1}\pi^{n-1}h_K^{n+1}}{(1+C\ e^{-cp\varepsilon^2})^2}\right)\right)\\
&\leq \frac{cn}{\alpha^n}\left(n+1+\frac{1}{\log p_0}\left|\log\left(\frac{4r_0^{n-1}\pi^{n-1}}{\alpha^{n+1}}\right)\right|\right)
\end{aligned}$$

The right hand side is a constant and hence integrable.

(ii) As K is in C^2_+, there is $R\geq r_0>0$ such that for all $x\in\partial K$, $K\subset B_2^n(x-RN(x),R)$. Then we show similarly to (4.29) that

$$h_{Z_p}(u)^n\ \leq\ h_K^n\ \left(2^{\frac{n-1}{2}}\ |B_2^{n-1}|\ h_K^{\frac{n+1}{2}}\ R^{\frac{n-1}{2}}\right)^{\frac{n}{p}}\ \left(\int_0^1 w^p(1-w)^{\frac{n-1}{2}}dw\right)^{\frac{n}{p}}.$$

and thus, similar to (4.30)

$$\frac{h^n_{Z_p(K)}}{h^n_K} \le 1 - \frac{n(n+1)}{2p} \log p + \frac{n}{2p} \ \log\left[2^{n+1}\ R^{n-1}\pi^{n-1} h_K^{n-1}\right] \pm o(p)$$

Hence, together with (4.30) respectively (4.31)

$$-\frac{n}{2\ h_{Z_p(K)^n}}\ \log\left[2^{n+1}\ R^{n-1}\pi^{n-1}\ h_K^{n-1}\right] \pm O(p) \le$$

$$\frac{p}{h_{Z_p(K)}(u)^n}\left(1 - \frac{h_{Z_p(K)}(u)^n}{h_K(u)^n} - \frac{n(n+1)}{2}\ \frac{\log(p)}{p}\ \frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right)$$

$$\le -\frac{n}{2\ h_{Z_p(K)^n}}\ \log\left[4\ r_0^{n-1}\pi^{n-1}\ h_K^{n+1}\right] \pm O(p).$$

respectively

$$-\frac{n}{2\ h_{Z_p(K)^n}}\ \log\left[2^{n+1}\ R^{n-1}\pi^{n-1}\ h_K^{n-1}\right] \pm O(p) \le$$

$$\frac{p}{h_{Z_p(K)}(u)^n}\left(1 - \frac{h_{Z_p(K)}(u)^n}{h_K(u)^n} - \frac{n(n+1)}{2}\ \frac{\log(p)}{p}\ \frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right)$$

$$\le -\frac{n}{2\ h_{Z_p(K)^n}}\ \log\left[\frac{4\ r_0^{n-1}\pi^{n-1}\ h_K^{n+1}}{(1+C\ e^{-cp\varepsilon^2})^2}\right] \pm O(p).$$

Hence, using (4.33), we get with an absolute constant c for all $p \ge p_0$

$$\left|\ \frac{p}{h_{Z_p(K)}(u)^n}\left(1 - \frac{h_{Z_p(K)}(u)^n}{h_K(u)^n} - \frac{n(n+1)}{2}\ \frac{\log(p)}{p}\ \frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right)\ \right|$$

$$\le \frac{cn}{\alpha^n}\left|\log\left[\frac{2^{n+1}\ R^{n-1}\pi^{n-1}}{\alpha^{n-1}}\right]\right|$$

Again, the right hand side is a constant and therefore integrable.

As $K \in C^2_+$, the indicatrix of Dupin at every $x \in \partial K$ is an ellipsoid. Since the quantities considered in the above Theorem 4.1 are affine invariant, we can assume that the indicatrix is a Euclidean ball. We have (see [52])

Lemma 4.8. *Let $K \subset \mathbb{R}^n$ be a convex body in C^2_+. We assume that the indicatrix of Dupin at $x \in \partial K$ is a Euclidean ball. Let $r = r(x) = \kappa(x)^{-\frac{1}{n-1}}$ and put $u = N_K(x)$. $B(x - ru, r)$ is the Euclidean ball with center at $x - ru$ and radius r. Then for every $\varepsilon > 0$ there exists $\Delta_\varepsilon > 0$ such that for all $\Delta \le \Delta_\varepsilon$,*

$$B\big(x - (1-\varepsilon)ru, (1-\varepsilon)r\big) \cap H(x - \Delta u, u)^-$$

$$\subset K \cap H(x - \Delta u, u)^- \subset B\big(x - (1+\varepsilon)ru, (1+\varepsilon)r\big) \cap H(x - \Delta u, u)^-.$$

$H(x - \Delta u, u)$ is the hyperplane with normal u through $x - \Delta u$ and $H(x - \Delta u, u)^-$ is the half space determined by this hyperplane into which u points.

Proof of Theorem 4.1

(i)

$$|Z_p^\circ(K)| - |K^\circ| = \frac{1}{n} \int_{S^{n-1}} \left(\frac{1}{h^n_{Z_p(K)}(u)} - \frac{1}{h^n_K(u)} \right) d\sigma(u)$$

Hence

$$\begin{aligned} \lim_{p\to\infty} \frac{p}{\log p} \left(|(Z_p^\circ(K))| - |K^\circ|\right) &= \frac{1}{n} \lim_{p\to\infty} \frac{p}{\log p} \int_{S^{n-1}} \frac{1}{h^n_{Z_p(K)}(u)} \left(1 - \frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right) d\sigma(u) \\ &= \frac{1}{n} \int_{S^{n-1}} \lim_{p\to\infty} \frac{p}{\log p} \frac{1}{h^n_{Z_p(K)}(u)} \left(1 - \frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right) d\sigma(u), \end{aligned}$$

where we have used Lemma 4.7 (i) and Lebegue's theorem to interchange integration and limit. Let $u \in S^{n-1}$. Let $x \in \partial K$ be such that $N_K(x) = u$. As K is in C^2_+, $\kappa = \kappa_K(x) > 0$ and we can assume that the indicatrix of Dupin at x is a Euclidean ball with radius $r = r(x) = \kappa(x)^{\frac{-1}{n-1}}$.

$$\begin{aligned} h^n_{Z_p(K)}(u) &= \left(\int_K |\langle y, u\rangle|^p dy\right)^{\frac{n}{p}} = \left(2\int_0^{h_K(u)} t^p |\{y \in K : \langle u, y\rangle = t\}| dt\right)^{\frac{n}{p}} \\ &\geq \left(2\int_{(1-\varepsilon)(h_K(u)-\Delta_\varepsilon)}^{h_K(u)} t^p |\{y \in K : \langle u, y\rangle = t\}|\, dt\right)^{\frac{n}{p}} \\ &\geq \left(2\int_{(1-\varepsilon)(h_K(u)-\Delta_\varepsilon)}^{h_K(u)} t^p |\{y \in B\big(x-(1-\varepsilon)r\, u, (1-\varepsilon)r\big) : \langle u, y\rangle = t\}| dt\right)^{\frac{n}{p}}, \end{aligned}$$

where we have applied Lemma 4.8. In addition, we also choose Δ_ε of Lemma 4.8 so that $\Delta_\varepsilon \leq \min\{\varepsilon, (1-\varepsilon)r\}$.

$B(x-(1-\varepsilon)r\, u, (1-\varepsilon)r) \cap \{y \in \mathbb{R}^n : \langle u, y\rangle = t\}$ is a $(n-1)$-dimensional Euclidean ball with radius

$$\left(2(1-\varepsilon)r\big(h_K(u)-t\big)\left[1 - \frac{h_K(u)-t}{2(1-\varepsilon)r}\right]\right)^{\frac{1}{2}},$$

which, by choice of Δ_ε is bigger or equal than

$$\left(2(1-\varepsilon)r\big(h_K(u)-t\big)\left[1 - \frac{\varepsilon\,(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]\right)^{\frac{1}{2}}.$$

Hence

$$h^n_{Z_p(K)}(u) = \left(\int_K |\langle y,u\rangle|^p dy\right)^{\frac{n}{p}} \geq$$

$$\left(\frac{2\ |B_2^{n-1}|\ [2(1-\varepsilon)\ r\ h_K(u)]^{\frac{n-1}{2}}}{\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{\frac{n-1}{2}}}\right)^{\frac{n}{p}} \left(\int_{(1-\varepsilon)(h_K(u)-\Delta_\varepsilon)}^{h_K(u)} t^p \left(1-\frac{t}{h_K(u)}\right)^{\frac{n-1}{2}} dt\right)^{\frac{n}{p}} =$$

$$\left(\frac{|B_2^{n-1}|\,((1-\varepsilon)\ r)^{\frac{n-1}{2}}\ \ [2h_K(u)]^{\frac{n+1}{2}}}{\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{\frac{n-1}{2}}}\right)^{\frac{n}{p}} h_K(u)^n \left(\int_{(1-\varepsilon)(1-\frac{\Delta_\varepsilon}{h_K(u)})}^{1} v^p (1-v)^{\frac{n-1}{2}} dv\right)^{\frac{n}{p}}$$

Now we apply Lemma 4.4 to the function $f(v) = (1-v)^{\frac{n-1}{2}}$. f is C^2 and $v_p = \frac{1}{1+\frac{n-1}{2p}}$. Thus Lemma 4.4 holds. v_p of Lemma 4.4 is an increasing function of p and $\lim_{p\to\infty} v_p = 1$. Hence, for $\varepsilon > 0$ given there exists $p_\varepsilon = p_{\varepsilon,\Delta_\varepsilon}$ namely $p_\varepsilon \geq \frac{(n-1)(h_K(u)-\Delta_\varepsilon)}{2\Delta_\varepsilon}$, such that for all $p \geq p_\varepsilon$, $v_p \geq \frac{h_K(u)-\Delta_\varepsilon}{h_K(u)}$. In addition, we also choose p_ε so large so that $p_\varepsilon \geq \frac{1}{\varepsilon^3}$. Thus

$$\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)} \quad \geq \quad \left(\frac{|B_2^{n-1}|\,((1-\varepsilon)\ r)^{\frac{n-1}{2}}\ [2h_K(u)]^{\frac{n+1}{2}}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)\ \left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{\frac{n-1}{2}}}\right)^{\frac{n}{p}} \left(\int_0^1 v^p(1-v)^{\frac{n-1}{2}} dv\right)^{\frac{n}{p}}.$$

Now

$$\left(\frac{|B_2^{n-1}|\,((1-\varepsilon)\ r)^{\frac{n-1}{2}}\ [2h_K(u)]^{\frac{n+1}{2}}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)\ \left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{\frac{n-1}{2}}}\right)^{\frac{n}{p}} = 1 + \frac{n}{p}\log\left(\frac{|B_2^{n-1}|\,((1-\varepsilon)\ r)^{\frac{n-1}{2}}\ [2h_K(u)]^{\frac{n+1}{2}}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)\ \left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{\frac{n-1}{2}}}\right)$$
$$+\frac{1}{2}\left(\frac{n}{p}\log\left(\frac{|B_2^{n-1}|\,((1-\varepsilon)\ r)^{\frac{n-1}{2}}\ [2h_K(u)]^{\frac{n+1}{2}}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)\ \left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{\frac{n-1}{2}}}\right)\right)^2 \pm o(p^2). \qquad (4.34)$$

Together with Lemma 4.3 (ii) (for $a = 0$) we then get: For $\varepsilon > 0$ given, there exists p_ε such that for all $p \geq p_\varepsilon$

$$
\begin{aligned}
&\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)} \geq \\
&1-\frac{n(n+1)}{2p}\log p+\frac{n}{2p}\log\left(\frac{\pi^{n-1}\left((1-\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)^2\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{n-1}}\right)+\\
&\frac{n^2(n+1)^2}{8p^2}(\log p)^2-\frac{n^2(n+1)}{2p^2}\log\left(\frac{\pi^{n-1}\left((1-\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)^2\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{n-1}}\right)\log p\\
&-\frac{n(n+1)}{2p^2}\left[\frac{\left(n^2+3n+6\right)}{4}\right]+\\
&\frac{n^2}{2p^2}\left[\left(\log\left(\Gamma(\frac{n+1}{2})\right)\right)^2+2\log\left(\frac{\pi^{n-1}\left((1-\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)^2\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{n-1}}\right)\right]+\\
&\frac{n^2}{2p^2}\left[\left(\log\left(\frac{|B_2^{n-1}|\left((1-\varepsilon)\ r\right)^{\frac{n-1}{2}}\left[2h_K(u)\right]^{\frac{n+1}{2}}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{\frac{n-1}{2}}}\right)\right)^2\right]\pm o(p^2). \qquad (4.35)
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{p}{\log p}\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)\leq\\
&\frac{n(n+1)}{2}-\frac{n}{2\log p}\log\left(\frac{\pi^{n-1}\left((1-\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)^2\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{n-1}}\right)\pm o(p). \quad (4.36)
\end{aligned}
$$

On the other hand, by Lemma 4.6, the function $f(t)=|K\cap(u^\perp+tu)|$ satisfies the assumptions of Lemma 4.4 and t_p is well defined. Also, t_p is an increasing function of p and by Lemma 4.5, $\lim_{p\to\infty}t_p=h_K(u)$. Hence, for $\varepsilon>0$ given there exists $p_\varepsilon=p_{\varepsilon,\Delta_\varepsilon}$ such that for all $p\geq p_\varepsilon$, $t_p\geq h_K(u)-\Delta_\varepsilon$. In addition, we also choose p_ε so large so that $p_\varepsilon\geq\frac{1}{\varepsilon^3}$. Thus

$$
\begin{aligned}
h^n_{Z_p(K)}(u) &= \left(2\int_0^{h_K(u)}t^p|\{y\in K:\langle u,y\rangle=t\}|dt\right)^{\frac{n}{p}}\\
&\leq \left(2\left(1+Ce^{-c\varepsilon^2p}\right)\int_{t_p(1-\varepsilon)}^{h_K(u)}t^p|\{y\in K:\langle u,y\rangle=t\}|dt\right)^{\frac{n}{p}}\\
&\leq \left(2\left(1+Ce^{-\frac{c}{\varepsilon}}\right)\int_{(1-\varepsilon)(h_K(u)-\Delta_\varepsilon)}^{h_K(u)}t^p|\{y\in K:\langle u,y\rangle=t\}|dt\right)^{\frac{n}{p}}
\end{aligned}
$$

$$\leq \left(2\left(1+Ce^{-\frac{c}{\varepsilon}}\right)\int_{(1-\varepsilon)(h_K(u)-\Delta_\varepsilon)}^{h_K(u)} t^p|\{y\in B\big(x-(1+\varepsilon)r\ u,(1+\varepsilon)r\big):\langle u,y\rangle=t\}|dt\right)^{\frac{n}{p}}.$$

In the last inequality we have used Lemma 4.8. The latter is

$$\leq \left(2\left(1+Ce^{-\frac{c}{\varepsilon}}\right)\int_{0}^{h_K(u)} t^p|\{y\in B\big(x-(1+\varepsilon)r\ u,(1+\varepsilon)r\big):\langle u,y\rangle=t\}|dt\right)^{\frac{n}{p}}.$$

As above, we notice that $B(x-(1+\varepsilon)r\ u,(1+\varepsilon)r)\cap\{y\in\mathbb{R}^n:\langle u,y\rangle=t\}$ is a $(n-1)$-dimensional Euclidean ball with radius

$$\left(2(1+\varepsilon)r\big(h_K(u)-t\big)\left[1-\frac{h_K(u)-t}{2(1+\varepsilon)r}\right]\right)^{\frac{1}{2}}$$

which is smaller than or equal

$$\left(2(1+\varepsilon)r\big(h_K(u)-t\big)\right)^{\frac{1}{2}}$$

We continue similar to above and get that there exists (a new) p_ε (chosen larger than the ones previously chosen and larger than $\frac{1}{\varepsilon^3}$) such that for all $p\geq p_\varepsilon$

$$\begin{aligned}
&\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\leq 1-\frac{n(n+1)}{2p}\log p+\frac{n}{2p}\log\left(\frac{\pi^{n-1}\left((1+\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{(1+Ce^{-\frac{c}{\varepsilon}})^{-2}}\right)+\\
&\frac{n^2(n+1)^2}{8p^2}(\log p)^2-\frac{n^2(n+1)}{2p^2}\log\left(\frac{\pi^{n-1}\left((1+\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{(1+Ce^{-\frac{c}{\varepsilon}})^{-2}}\right)\ \log p\\
&-\frac{n(n+1)}{2p^2}\left[\frac{(n^2+3n+6)}{4}\right]+\\
&\frac{n^2}{2p^2}\left[\left(\log\left(\Gamma(\frac{n+1}{2})\right)\right)^2+2\log\left(\frac{\pi^{n-1}\left((1+\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{(1+Ce^{-\frac{c}{\varepsilon}})^{-2}}\right)\right]+\\
&\frac{n^2}{2p^2}\left[\left(\log\left(\frac{|B_2^{n-1}|\left((1+\varepsilon)\ r\right)^{\frac{n-1}{2}}\left[2h_K(u)\right]^{\frac{n+1}{2}}}{(1+Ce^{-\frac{c}{\varepsilon}})^{-1}}\right)\right)\right]\pm o(p^2).
\end{aligned}\tag{4.37}$$

Thus

$$\begin{aligned}
&\frac{p}{\log p}\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)\geq\\
&\frac{n(n+1)}{2}-\frac{n}{2\log p}\log\left(\frac{\pi^{n-1}\left((1+\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{(1+Ce^{-\frac{c}{\varepsilon}})^{-2}}\right)\pm o(p).
\end{aligned}\tag{4.38}$$

(4.36) and (4.38) give that

$$\lim_{p\to\infty}\frac{p}{\log p}\left(1-\frac{h^n_{Z_p(K)}(u)}{h_K(u)^n}\right)=\frac{n(n+1)}{2}.$$

Hence, also using that, since $|K|=1$, $h_{Z_p(K)}(u) \to h_K(u)$,

$$\begin{aligned}
\lim_{p\to\infty} \frac{p}{\log p}\left(|Z_p^\circ(K)| - |K^\circ|\right) &= \frac{1}{n}\int_{S^{n-1}} \lim_{p\to\infty} \frac{p}{\log p}\,\frac{1}{h^n_{Z_p(K)}(u)}\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)d\sigma(u)\\
&= \frac{1}{n}\int_{S^{n-1}} \lim_{p\to\infty} \frac{1}{h^n_{Z_p(K)}(u)}\lim_{p\to\infty}\frac{p}{\log p}\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)d\sigma(u)\\
&= \frac{n+1}{2}\int_{S^{n-1}} \frac{1}{h^n_K(u)}d\sigma(u)\\
&= \frac{n(n+1)}{2}|K^\circ|.
\end{aligned}$$

This finishes (i).

(ii)

$$|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)\log p}{2p}|K^\circ| =$$
$$\frac{1}{n}\int_{S^{n-1}}\left(\frac{1}{h^n_{Z_p(K)}(u)} - \frac{1}{h^n_K(u)} - \frac{n(n+1)}{2}\,\frac{\log(p)}{p}\,\frac{1}{h^n_K(u)}\right)d\sigma(u) =$$
$$\frac{1}{n}\int_{S^{n-1}}\frac{1}{h^n_{Z_p(K)}(u)}\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)} - \frac{n(n+1)}{2}\,\frac{\log(p)}{p}\,\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)d\sigma(u).$$

Hence

$$\lim_{p\to\infty} p\left(|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)\log p}{2p}|K^\circ|\right) =$$
$$\frac{1}{n}\int_{S^{n-1}}\lim_{p\to\infty}\frac{p}{h^n_{Z_p(K)}(u)}\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)} - \frac{n(n+1)}{2}\,\frac{\log(p)}{p}\,\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)d\sigma(u),$$

where we have used Lemma 4.7 (ii) and Lebegue's theorem to interchange integration and limit. By (4.35) we have for all $p \geq p_\varepsilon$

$$\left(1 - \frac{h^n_{Z_p(K)}(u)}{h^n_K(u)} - \frac{n(n+1)}{2}\ \frac{\log(p)}{p}\ \frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right) \leq$$

$$-\frac{n}{2p}\log\left(\frac{\pi^{n-1}\left((1-\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)^2\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{n-1}}\right) + \frac{n^2(n+1)^2}{8p^2}(\log p)^2$$

$$+\ \frac{n(n+1)}{2p^2}\left[\frac{(n^2+3n+6)}{4}\right] -$$

$$\frac{n^2}{2p^2}\left[\left(\log\left(\Gamma(\frac{n+1}{2})\right)\right)^2 + 2\log\left(\frac{\pi^{n-1}\left((1-\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)^2\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{n-1}}\right)\right] -$$

$$\frac{n^2}{2p^2}\left[\left(\log\left(\frac{|B^{n-1}_2|\left((1-\varepsilon)\ r\right)^{\frac{n-1}{2}}\left[2h_K(u)\right]^{\frac{n+1}{2}}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{\frac{n-1}{2}}}\right)\right)^2\right] \pm o(p^2)$$

Thus

$$p\ \left(1 - \frac{h^n_{Z_p(K)}(u)}{h^n_K(u)} - \frac{n(n+1)}{2}\ \frac{\log(p)}{p}\ \frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right) \leq$$

$$-\frac{n}{2}\log\left(\frac{\pi^{n-1}\left((1-\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)^2\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{n-1}}\right) + \frac{n^2(n+1)^2}{8p}(\log p)^2$$

$$+\ \frac{n(n+1)}{2p}\left[\frac{(n^2+3n+6)}{4}\right] -$$

$$\frac{n^2}{2p}\left[\left(\log\left(\Gamma(\frac{n+1}{2})\right)\right)^2 + 2\log\left(\frac{\pi^{n-1}\left((1-\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)^2\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{n-1}}\right)\right] -$$

$$\frac{n^2}{2p}\left[\left(\log\left(\frac{|B^{n-1}_2|\left((1-\varepsilon)\ r\right)^{\frac{n-1}{2}}\left[2h_K(u)\right]^{\frac{n+1}{2}}}{\left(1+Ce^{-\frac{c}{\varepsilon}}\right)\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]^{\frac{n-1}{2}}}\right)\right)^2\right] \pm o(p) \tag{4.39}$$

Similarly, using (4.37), we get for all $p \geq p_\varepsilon$

$$p \left(1 - \frac{h^n_{Z_p(K)}(u)}{h^n_K(u)} - \frac{n(n+1)}{2} \frac{\log(p)}{p} \frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right) \geq$$
$$-\frac{n}{2} \log\left(\frac{\pi^{n-1}\left((1+\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{(1+Ce^{-\frac{c}{\varepsilon}})^{-2}}\right) + \frac{n^2(n+1)^2}{8p}(\log p)^2$$
$$+ \frac{n(n+1)}{2p}\left[\frac{\left(n^2+3n+6\right)}{4}\right] -$$
$$\frac{n^2}{2p}\left[\left(\log\left(\Gamma(\frac{n+1}{2})\right)\right)^2 + 2\log\left(\frac{\pi^{n-1}\left((1+\varepsilon)r\right)^{n-1}\left[2h_K(u)\right]^{n+1}}{(1+Ce^{-\frac{c}{\varepsilon}})^{-2}}\right)\right] -$$
$$\frac{n^2}{2p}\left[\left(\log\left(\frac{|B_2^{n-1}|\left((1+\varepsilon)\ r\right)^{\frac{n-1}{2}}\left[2h_K(u)\right]^{\frac{n+1}{2}}}{(1+Ce^{-\frac{c}{\varepsilon}})^{-1}}\right)\right)^2\right] \pm o(p) \tag{4.40}$$

(4.39) and (4.40) give that

$$\lim_{p\to\infty} p \left(1 - \frac{h^n_{Z_p(K)}(u)}{h^n_K(u)} - \frac{n(n+1)}{2} \frac{\log(p)}{p} \frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)$$
$$= -\frac{n}{2}\log\left(\pi^{n-1}r^{n-1}\left[2h_K(u)\right]^{n+1}\right).$$

The limit $\lim_{p\to\infty} p\left(|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)}{2p}\log p\ |Z_p^\circ(K)|\right)$ is computed similarly.

5 Applications

The fact that Ω_K can be expressed in different ways allows us to compute the integral in the next proposition.

Proposition 5.1. *Let $1 < r < \infty$ and let B_r^n be the l_r^n- unit ball and let $(B_r^{n-1})^+$ be the set of all vectors in B_r^{n-1} having nonnegative coordinates. Then*

$$\int_{(B_r^{n-1})^+} \prod_{i=1}^{n-1}|x_i|^{r-2}\ \log\left[(r-1)^{n-1}\prod_{i=1}^{n}|x_i|^{r-2}\right] x_n^{-1}dx_1\dots dx_{n-1} =$$
$$\frac{n}{r^{n-1}}\frac{\left(\Gamma(\frac{r-1}{r})\right)^n}{\Gamma(\frac{n(r-1)}{r})}\left[\frac{n(r-2)}{r}\left(\left(\frac{\Gamma'(\frac{r-1}{r})}{\Gamma(\frac{r-1}{r})} - \frac{\Gamma'(n\frac{r-1}{r})}{\Gamma(n\frac{r-1}{r})}\right)\right) + (n-1)\ \log r\right]$$

Proof.

In Chapter 3 it was shown that

$$\log\Omega_K = -\frac{n}{as_\infty(K)}\int_{\partial K}\frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n}\log\frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n+1}}d\mu_K(x).$$

We apply this formula to $K = B_r^n$, $1 < r < \infty$. It was also shown in Chapter 3 that

$$\log \Omega_{B_r^n} = -n \left[\frac{n(r-2)}{r} \left(\frac{\Gamma'(\frac{r-1}{r})}{\Gamma(\frac{r-1}{r})} - \frac{\Gamma'(n\frac{r-1}{r})}{\Gamma(n\frac{r-1}{r})} \right) + (n-1) \ \log r \right]$$

The curvature at a boundary point of B_r^n is (see [54])

$$\kappa(x) = \frac{(r-1)^{n-1} \prod_{i=1}^n |x_i|^{r-2}}{(\sum_{i=1}^n |x_i|^{2r-2})^{\frac{n+1}{2}}}$$

and the normal is (see [54])

$$N_{\partial B_r^n}(x) = \frac{(\operatorname{sgn}(x_1)|x_1|^{r-1}, \dots, \operatorname{sgn}(x_n)|x_n|^{r-1})}{(\sum_{i=1}^n |x_i|^{2r-2})^{\frac{1}{2}}}.$$

Thus we get - where $B_{r'}^n$ is the polar of B_r^n, i.e. r' is the conjugate exponent of r -

$$n \left[\frac{n(r-2)}{r} \left(\frac{\Gamma'(\frac{r-1}{r})}{\Gamma(\frac{r-1}{r})} - \frac{\Gamma'(n\frac{r-1}{r})}{\Gamma(n\frac{r-1}{r})} \right) + (n-1) \ \log r \right] |B_{r'}^n| =$$

$$\int_{\partial B_r^n} \frac{((r-1)^{n-1} \prod_{i=1}^n |x_i|^{r-2}}{(\sum_{i=1}^n |x_i|^{2r-2})^{\frac{1}{2}}} \ \log \left[(r-1)^{n-1} \prod_{i=1}^n |x_i|^{r-2} \right] d\mu_{\partial B_r^n}(x).$$

Now we integrate with respect to the variables $x_1, \dots, x_{n-1}$. The volume of a surface element in the plane of the first $n-1$ coordinates equals the volume of the corresponding surface element on ∂B_r^n times

$$| < e_n, N_{\partial B_r^n}(x) > | = \frac{|x_n|^{r-1}}{(\sum_{i=1}^n |x_i|^{2r-2})^{\frac{1}{2}}}.$$

Thus, with $(B_r^{n-1})^+$ being the set of all vectors in B_r^{n-1} having nonnegative coordinates,

$$2^n (r-1)^{n-1} \int_{(B_r^{n-1})^+} \prod_{i=1}^n |x_i|^{r-2} \ \log \left[(r-1)^{n-1} \prod_{i=1}^n |x_i|^{r-2} \right] x_n^{1-r} dx_1 \dots dx_{n-1}$$

$$= 2^n (r-1)^{n-1} \int_{(B_r^{n-1})^+} \prod_{i=1}^{n-1} |x_i|^{r-2} \ \log \left[(r-1)^{n-1} \prod_{i=1}^n |x_i|^{r-2} \right] x_n^{-1} dx_1 \dots dx_{n-1}$$

$$= 2^n (r-1)^{n-1} \frac{n}{r^{n-1}} \frac{\left(\Gamma(\frac{r-1}{r})\right)^n}{\Gamma(\frac{n(r-1)}{r})} \left[\frac{n(r-2)}{r} \left(\frac{\Gamma'(\frac{r-1}{r})}{\Gamma(\frac{r-1}{r})} - \frac{\Gamma'(n\frac{r-1}{r})}{\Gamma(n\frac{r-1}{r})} \right) + (n-1) \ \log r \right],$$

where we have also used that

$$|B_{r'}^n| = \frac{2^n (r-1)^{n-1}}{n \ r^{n-1}} \frac{\left(\Gamma(\frac{r-1}{r})\right)^n}{\Gamma(\frac{n(r-1)}{r})}.$$

There are still other ways how Ω_K can be expressed. Similar to Theorem 4.1, Ω_K appears in the asymptotic behavior of the volume of certain surface bodies and illumination surface bodies [59]. We show the result for the surface bodies. For the illumination surface bodies it is done similarly.

The surface bodies, a variant of the floating bodies, were introduced in [53, 54] as follows

Definition

Let $s \geq 0$ and $f : \partial K \to \mathbb{R}$ be a nonnegative, integrable function. The *surface body* $K_{f,s}$ is the intersection of all the closed half-spaces H^+ whose defining hyperplanes H cut off a set of $f\mu_K$-measure less than or equal to s from ∂K. More precisely,

$$K_{f,s} = \bigcap_{\int_{\partial K \cap H^-} f d\mu_K \leq s} H^+.$$

Proposition 5.2. *Let K be a symmetric convex body in $\mathbb{R}^n$ that is in C^2_+.*

$$d_n lim_{s\to 0} \frac{|K| - |K_{f,s}|}{s^{\frac{2}{n-1}}} = \int_{\partial K} \frac{\kappa(x)}{\langle x, N(x)\rangle^n} \ \log\left(\frac{\kappa(x)}{\langle x, N(x)\rangle^{n+1}}\right) d\mu(x) = |K^\circ| \ \log \frac{1}{\Omega_K}.$$

where $K_{f,s}$ is the surface body of K for the function

$$f = \frac{\langle x, N_K(x)\rangle^{\frac{n(n-1)}{2}}}{\kappa^{\frac{n-2}{2}}} \left(\log \ (\frac{\kappa}{\langle x, N_K(x)\rangle^{n+1}})\right)^{-\frac{n-1}{2}}$$

and where $d_n = 2\left(|B_2^{n-1}|\right)^{\frac{2}{n-1}}$.

Proof.

The proof follows immediately from the following formula which was proved in [54] (Theorem 14)

$$d_n \lim_{s\to 0} \frac{|K| - |K_{f,s}|}{s^{\frac{2}{n-1}}} = \int_{\partial K} \frac{\kappa^{\frac{1}{n-1}}}{f^{\frac{2}{n-1}}} d\mu_{\partial K}.$$

6 Appendix: Calculations with Γ-functions.

For $x, y > 0$, $\Gamma(x) := \int_0^\infty \lambda^{x-1} e^{-\lambda} d\lambda$ is the Gamma function and $B(x,y) := \int_0^1 \lambda^{x-1}(1-\lambda)^{y-1} d\lambda = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function.

Recall that we write $f(p) = g(p) \pm o(p)$, if there exists a function $h(p)$ such that $f(p) = g(p) + h(p)$ and $\lim_{p\to\infty} ph(p) = 0$ and similarly, $f(p) = g(p) \pm o(p^2)$, if there exists a function $h(p)$ such that $f(p) = g(p) + h(p)$ and $\lim_{p\to\infty} p^2 h(p) = 0$

We will frequently use: For $x \to \infty$,

$$\Gamma(x) = \sqrt{2\pi}\ x^{x-\frac{1}{2}}\ e^{-x}\ \left[1 + \frac{1}{12x} + \frac{1}{288x^2} \pm o(x^2)\right]. \tag{6.41}$$

For every $z, w > 0$

$$z^{1/p} = 1 + \frac{\log z}{p} + \frac{(\log z)^2}{2p^2} \pm o(p^2)$$

and

$$(p+z)^{w/p} = 1 + \frac{w}{p}\log p + \frac{w^2(\log z)^2}{2p^2} + \frac{wz}{p^2} \pm o(p^2).$$

Note that if $f(p)^2 = o(p)$ then $(1+f(p))(1-f(p)) = 1 \pm o(p)$, which means that

$$\frac{1}{1+f(p)} = 1 - f(p) \pm o(p).$$

Also

$$\frac{a}{p+b} = \frac{a}{p} - \frac{ab}{p^2} \pm o(p^2).$$

Proof of Lemma 4.3

(i) We use (6.41) and get

$$\left(B\big(p+1, \frac{n+1}{2}\big)\right)^{\frac{n}{p}} = \left(\frac{\Gamma(p+1)}{\Gamma(p+1+\frac{n+1}{2})}\Gamma(\frac{n+1}{2})\right)^{\frac{n}{p}}$$

$$= \left(\frac{\Gamma(\frac{n+1}{2})\ e^{\frac{n+1}{2}}\ (p+1)^{p+\frac{1}{2}}\left[1+\frac{1}{12(p+1)}+\frac{1}{288(p+1)^2}\pm o(p^2)\right]}{(p+1+\frac{n+1}{2})^{p+1+\frac{n}{2}}\left[1+\frac{1}{12(p+1+\frac{n+1}{2})}+\frac{1}{288(p+1+\frac{n+1}{2})^2}\pm o(p^2)\right]}\right)^{\frac{n}{p}}$$

$$= \left(\Gamma\left(\frac{n+1}{2}\right)\ e^{\frac{n+1}{2}}\right)^{\frac{n}{p}}\left(\frac{p+1}{p+1+\frac{n+1}{2}}\right)^{\frac{n}{p}(p+\frac{1}{2})}\left(\frac{1}{p+1+\frac{n+1}{2}}\right)^{\frac{n(n+1)}{2p}} \times$$
$$\left(\frac{1+\frac{1}{12(p+1)}+\frac{1}{288(p+1)^2}\pm o(p^2)}{1+\frac{1}{12(p+1+\frac{n+1}{2})}+\frac{1}{288(p+1+\frac{n+1}{2})^2}\pm o(p^2)}\right)^{\frac{n}{p}}$$

Note that

$$\left(\frac{1+\frac{1}{12(p+1)}+\frac{1}{288(p+1)^2}\pm o(p^2)}{1+\frac{1}{12(p+1+\frac{n+1}{2})}+\frac{1}{288(p+1+\frac{n+1}{2})^2}\pm o(p^2)}\right)^{\frac{n}{p}} = 1 \pm o(p^2).$$

Also

$$\left(\Gamma\left(\frac{n+1}{2}\right)\ e^{\frac{n+1}{2}}\right)^{\frac{n}{p}}$$
$$= 1 + \frac{n}{p}\left[\frac{n+1}{2} + \log\left(\Gamma(\frac{n+1}{2})\right)\right] + \frac{n^2}{2p^2}\left[\frac{n+1}{2} + \log\left(\Gamma(\frac{n+1}{2})\right)\right]^2 \pm o(p^2),$$

$$\left(\frac{1}{1+\frac{n+1}{2(p+1)}}\right)^{n(1+\frac{1}{2p})} = \left(\frac{1}{1+\frac{n+1}{2(p+1)}}\right)^{n} e^{-\frac{n}{2p}\log\left(1+\frac{n+1}{2p+2}\right)}$$
$$= 1-\frac{n(n+1)}{2p}+\frac{n(3+5n+3n^2+n^3)}{8p^2}\pm o(p^2)$$

and

$$\left(\frac{1}{p+1+\frac{n+1}{2}}\right)^{\frac{n(n+1)}{2p}} = e^{-\frac{n(n+1)}{2p}\log\left(p+\frac{n+3}{2}\right)}$$
$$=1-\frac{n(n+1)}{2p}\log p+\frac{n^2(n+1)^2}{8p^2}(\log p)^2-\frac{n(n+1)(n+3)}{4p^2}\pm o(p^2).$$

Hence

$$\left(B\big(p+1,\frac{n+1}{2})\right)^{\frac{n}{p}} = \left(1\pm o(p^2)\right)$$
$$\left(1+\frac{n}{p}\left[\frac{n+1}{2}+\log\left(\Gamma(\frac{n+1}{2})\right)\right]+\frac{n^2}{2p^2}\left[\frac{n+1}{2}+\log\left(\Gamma(\frac{n+1}{2})\right)\right]^2\pm o(p^2)\right)$$
$$\left(1-\frac{n(n+1)}{2p}+\frac{n(3+5n+3n^2+n^3)}{8p^2}\pm o(p^2)\right)$$
$$\left(1-\frac{n(n+1)}{2p}\log p+\frac{n^2(n+1)^2}{8p^2}(\log p)^2-\frac{n(n+1)(n+3)}{4p^2}\pm o(p^2)\right)$$
$$=1-\frac{n(n+1)}{2p}\log p+\frac{n}{p}\log\left(\Gamma(\frac{n+1}{2})\right)+\frac{n^2(n+1)^2}{8p^2}(\log p)^2$$
$$-\frac{n^2(n+1)}{2p^2}\log\left(\Gamma(\frac{n+1}{2})\right)\log p$$
$$+\frac{n}{2p^2}\left[n\left(\log\left(\Gamma(\frac{n+1}{2})\right)\right)^2-\frac{n+1}{4}\left(n(n+1)+2(n+3)\right)\right]\pm o(p^2).$$

(ii)

$$\left(\int_0^1 u^p(1-u)^{\frac{n-1}{2}}\left(1-a\left(1-u\right)\right)^{\frac{n-1}{2}}du\right)^{\frac{n}{p}}$$
$$=\left(\int_0^1 u^p(1-u)^{\frac{n-1}{2}}\left[1-\binom{\frac{n-1}{2}}{1}\,a\,(1-u)+\binom{\frac{n-1}{2}}{2}a^2(1-u)^2\pm\dots\right]du\right)^{\frac{n}{p}}$$
$$=\left(B\big(p+1,\frac{n+1}{2})\right)^{\frac{n}{p}}\left[1-\binom{\frac{n-1}{2}}{1}\,a\,B_3+\binom{\frac{n-1}{2}}{2}\,a^2\,B_5-\binom{\frac{n-1}{2}}{3}\,a^3\,B_7\pm\dots\right]^{\frac{n}{p}}$$
$$=\left(B\big(p+1,\frac{n+1}{2})\right)^{\frac{n}{p}}\exp\left\{\frac{n}{p}\log\left[1-\binom{\frac{n-1}{2}}{1}\,a\,B_3+\binom{\frac{n-1}{2}}{2}\,a^2\,B_5\pm\dots\right]\right\}$$
$$=\left(B\big(p+1,\frac{n+1}{2})\right)^{\frac{n}{p}}\times$$
$$\left[1-\frac{n}{p}\left\{\binom{\frac{n-1}{2}}{1}\,a\,B_3-\binom{\frac{n-1}{2}}{2}\,a^2\,B_5+\frac{1}{2}\left(\binom{\frac{n-1}{2}}{1}\right)^2a^2\,B_3^2\pm\dots\right\}\dots\right]$$

where for $3 \le k \le n-2$ and for a constant c

$$B_k = \frac{B\big(p+1, \frac{n+k}{2}\big)}{B\big(p+1, \frac{n+1}{2}\big)} = \frac{\Gamma(\frac{n+k}{2})}{\Gamma(\frac{n+1}{2})} \frac{1}{p^{\frac{k-1}{2}}} \left(1 + \frac{c}{p} \pm o(p)\right)$$

Hence, together with (i),

$$\left(\int_0^1 u^p (1-u)^{\frac{n-1}{2}} \left(1 - a\left(1-u\right)\right)^{\frac{n-1}{2}} du\right)^{\frac{n}{p}} = 1 - \frac{n(n+1)}{2p} \log p +$$
$$\frac{n}{p} \log\left(\Gamma(\frac{n+1}{2})\right) + \frac{n^2(n+1)^2}{8p^2} (\log p)^2 - \frac{n^2(n+1)}{2p^2} \log\left(\Gamma(\frac{n+1}{2})\right) \log p +$$
$$\frac{n}{2p^2}\left[n\left(\log\left(\Gamma(\frac{n+1}{2})\right)\right)^2 - \frac{(n+1)\left(n^2+3n+6\right)}{4} - 2\binom{\frac{n-1}{2}}{1}\, a\, \frac{\Gamma(\frac{n+3}{2})}{\Gamma(\frac{n+1}{2})}\right] +$$
$$\frac{n}{2p^2}\left[n\left(\log\left(\Gamma(\frac{n+1}{2})\right)\right)^2 - \frac{(n+1)\left(n^2+3n+6\right)}{4} - (n+1)\binom{\frac{n-1}{2}}{1}\, a\right]$$
$$\pm o(p^2).$$

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