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# Relative Entropy for States of von Neumann Algebras II

By

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**Abstract**

Earlier definition of the relative entropy of two faithful normal positive linear functionals of a von Neumann algebra is generalized to non-faithful functionals. Basic properties of the relative entropy are proved for this generalization.

## § 1. Introduction

For two faithful normal positive linear functionals $\phi$ and $\psi$ of a von Neumann algebra $M$, the relative entropy $S(\psi|\phi)$ is defined and its properties are proved in an earlier paper [1].

When $M$ is a finite dimensional factor, it is given by

$$S(\psi|\phi) = \phi(\log \rho_\phi - \log \rho_\psi) \tag{1.1}$$

where $\rho_\phi$ and $\rho_\psi$ are density matrices for $\phi$ and $\psi$. If $\psi$ and $\phi$ are faithful, $\rho_\phi$ and $\rho_\psi$ are strictly positive and (1.1) clearly makes sense. However the first term of (1.1) always makes sense (under the convention $\lambda \log \lambda = 0$ for $\lambda = 0$) and the second term is either finite or infinite. Therefore (1.1) can be given an unambiguous finite or positive infinite value for every $\phi$ and $\psi$.

We shall make corresponding generalization for an arbitrary von Neumann algebra $M$ and any normal positive linear functionals $\psi$ and $\phi$. We shall also define the relative entropy of two positive linear functionals of a $C^*$-algebra $\mathfrak{A}$ and give an alternative proof of a result of [2]. For the latter case, we relate the conditional entropy introduced in [3] with our relative entropy.

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The relative entropy for non-faithful functionals will be shown to satisfy all properties proved for faithful functionals in [1]. Some of these properties will be applied to a discussion of local thermodynamical stability in [3].

For simplicity, we shall assume that $M$ has a faithful normal state although many of the results are independent of this assumption.

## § 2. Relative Modular Operator

Let $\Phi$ and $\Psi$ be vectors in a natural positive cone $V$([4], [5], [6]) for a von Neumann algebra $M$ on a Hilbert space $H$ and let $\phi$ and $\psi$ be the corresponding normal positive linear functionals of $M$. Let $s^R(\Omega)$ denote the $R$-support of a vector $\Omega$, where $R$ is a von Neumann algebra.

**Definition 2.1.** Operators $S_{\Phi,\Psi}$ and $F_{\Phi,\Psi}$ with their domains

$$D(S_{\Phi,\Psi}) = M\Psi + (\mathbf{1} - s^{M'}(\Psi))H,$$

$$D(F_{\Phi,\Psi}) = M'\Psi + (\mathbf{1} - s^{M}(\Psi))H,$$

are defined by

$$S_{\Phi,\Psi}\{x\Psi + \Omega\} = s^M(\Psi)x^*\Phi, \tag{2.1}$$

$$F_{\Phi,\Psi}\{x'\Psi + \Omega'\} = s^{M'}(\Psi)x'^*\Phi, \tag{2.2}$$

where $x \in M$, $x' \in M'$, $s^{M'}(\Psi)\Omega = 0$, $s^M(\Psi)\Omega' = 0$.

**Lemma 2.2.** *$S_{\Phi,\Psi}$ and $F_{\Phi,\Psi}$ are closable antilinear operators.*

*Proof*: If $x_1\Psi + \Omega_1 = x_2\Psi + \Omega_2$ for $x_1, x_2 \in M$ and $\Omega_1, \Omega_2 \in (\mathbf{1} - s^{M'}(\Psi))H$, then $\Omega_1 = \Omega_2$ and $(x_1 - x_2)s^M(\Psi) = \mathbf{0}$, so that $s^M(\Psi)x_1^*\Phi = s^M(\Psi)x_2^*\Phi$. This shows that $S_{\Phi,\Psi}$ is well-defined. Then it is clearly antilinear. Similarly $F_{\Phi,\Psi}$ is an antilinear operator.

Let $x \in M$, $x' \in M'$, $s^{M'}(\Psi)\Omega = s^M(\Psi)\Omega' = 0$. Then

$$(S_{\Phi,\Psi}\{x\Psi + \Omega\}, \{x'\Psi + \Omega'\}) = (x^*\Phi, x'\Psi)$$

$$= (F_{\Phi,\Psi}\{x'\Psi + \Omega\}, \{x\Psi + \Omega\}).$$

Since $S_{\Phi,\Psi}$ and $F_{\Phi,\Psi}$ have dense domains, this shows the closability of

$S_{\Phi,\Psi}$ and $F_{\Phi,\Psi}$.

**Definition 2. 3.** The *relative modular operator* $\Delta_{\Phi,\Psi}$ is defined by

$$\Delta_{\Phi,\Psi}=(S_{\Phi,\Psi})^{*}\bar{S}_{\Phi,\Psi} \tag{2. 3}$$

where the bar denotes the closure.

We denote by $J$ the modular conjugation operator associated with the natural positive cone V.

**Theorem 2. 4.**

(1) *The kernel of* $\Delta_{\Phi,\Psi}$ *is* $\mathbf{1}-s^{M'}(\Psi)s^{M}(\Phi)$.

(2) *The following formulas hold, where the bar denotes the closure.*

$$\bar{S}_{\Phi,\Psi}=J(\Delta_{\Phi,\Psi})^{1/2},\quad \bar{F}_{\Phi,\Psi}=(\Delta_{\Phi,\Psi})^{1/2}J, \tag{2. 4}$$

$$J\Delta_{\Psi,\Phi}J\Delta_{\Phi,\Psi}=\Delta_{\Phi,\Psi}J\Delta_{\Psi,\Phi}J=s^{M'}(\Psi)s^{M}(\Phi). \tag{2. 5}$$

(3) *If* $s^{M}(\Phi_1)\perp s^{M}(\Phi_2)$, then

$$\Delta_{\Phi_1+\Phi_2,\Psi}=\Delta_{\Phi_1,\Psi}+\Delta_{\Phi_2,\Psi}\,. \tag{2. 6}$$

*Proof*:

(1) and (2): First we prove Theorem for the special case $\Phi=\Psi$. The domain of $S_{\Psi,\Psi}$ is split into a direct sum of 3 parts:

$$D(S_{\Psi,\Psi})=s^{M}(\Psi)M\Psi+(\mathbf{1}-s^{M}(\Psi))M\Psi+(\mathbf{1}-s^{M'}(\Psi))H.$$

Accordingly, we split $S_{\Psi,\Psi}$ as a direct sum

$$S_{\Psi,\Psi}=\hat{S}_{\Psi,\Psi}\oplus\mathbf{0}\oplus\mathbf{0}$$

where $S_{\Psi,\Psi}$ is the operator on $s^{M}(\Psi)\ \ s^{M'}(\Psi)H$ defined by

$$\hat{S}_{\Psi,\Psi}x\Psi=x^{*}\Psi\,,\quad x\in s^{M}(\Psi)Ms^{M}(\Psi)$$

and the splitting of the Hilbert space is

$$H=s^{M}(\Psi)s^{M'}(\Psi)H\oplus(\mathbf{1}-s^{M}(\Psi))s^{M'}(\Psi)H\oplus(\mathbf{1}-s^{M'}(\Psi))H\,.$$

Since $\Psi$ is cyclic and separating relative to $s^{M}(\Psi)Ms^{M}(\Psi)$ in the subspace $s^{M}(\Psi)s^{M'}(\Psi)H$,

$$\Delta_{\Psi,\Psi}=\tilde{\Delta}_{\Psi,\Psi}\oplus\mathbf{0}\oplus\mathbf{0} \tag{2. 7}$$

where $\tilde{\Delta}_{\Psi,\Psi}$ is the modular operator of $\Psi$ relative to $s^M(\Psi)Ms^M(\Psi)$. Since[1] $s^{M'}(\Psi)=J\, s^M(\Psi)J$, $J$ commutes with $s^M(\Psi)\ s^{M'}(\Psi)$ and hence leaves $s^M(\Psi)s^{M'}(\Psi)H$ invariant. The restriction of $J$ to this subspace is the modular conjugation operator for $\Psi$, as can be checked by the characterization of $J$ given in [4]. Therefore the known property of the modular operator for a cyclic and separating vector implies (1) and (2) for the case $\Psi=\Phi$.

To prove (1) and (2) for the general case, we use the $2\times 2$ matrix method of Connes [7]. Let $\tilde{M}=M\otimes M_2$ with $M_2$ a type $I_2$ factor on a 4-dimensional space $K$, let $u_{ij}$ be a matrix unit of $M_2$, let $e_{ij}$ be an orthonormal basis of $K$ satisfying $u_{ij}e_{kl}=\delta_{jk}e_{il}$, let $J_K$ be the modular conjugation operator of $e_{11}+e_{22}$ (i.e. $J_Ke_{ij}=e_{ji}$), and let

$$\Omega=\sum \Omega_j\otimes e_{jj} \tag{2.8}$$

with $\Omega_1=\Psi$ and $\Omega_2=\Phi$. From definition, we obtain

$$\begin{aligned} s^{\tilde{M}}(\Omega) &=\sum s^M(\Omega_j)\otimes u_{jj}, \\ s^{\tilde{M}'}(\Omega) &=\sum s^{M'}(\Omega_j)\otimes J_Ku_{jj}J_K, \\ (\mathbf{1}\otimes u_{ii})S_{\Omega,\Omega}(\mathbf{1}\otimes u_{jj}) &=S_{\Omega_j,\Omega_i}\otimes u_{ii}J_Ku_{jj}. \end{aligned} \tag{2.9}$$

Since the modular conjugation operator $J$ for the natural positive cone of $M$ containing $V\otimes(e_1+e_2)$ is given by $J\otimes J_K$, we obtain

$$\Delta_{\Omega,\Omega}(\mathbf{1}\otimes J_Ku_{ii}J_Ku_{jj})=\Delta_{\Omega_j,\Omega_i}\otimes J_Ku_{ii}J_Ku_{jj}. \tag{2.10}$$

Hence (1) and (2) proved above for $\Delta_{\Omega,\Omega}$ imply the same for $\Delta_{\Phi,\Psi}$ and $\Delta_{\Psi,\Phi}$.

(3) If $s^M(\Phi_j)$ is mutually orthogonal for $j=1,2$, then the same holds for $s^{M'}(\Phi_j)=Js^M(\Phi_j)J$. By (1) and (2), the range projection of $S_{\Phi_j\Psi}$ is

$$Js^{M'}(\Psi)s^M(\Phi_j)J=s^M(\Psi)s^{M'}(\Phi_j)$$

and is mutually orthogonal for $j=1,2$. The same holds for the corange projection. From definition we obtain

$$S_{\Phi_1+\Phi_2,\Psi}=S_{\Phi_1,\Psi}+S_{\Phi_2,\Psi}. \tag{2.11}$$

Hence we obtain (2.6). Q.E.D.

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[1] This follows from $J\Psi=\Psi$.

## § 3. Relative Entropy for States of von Neumann Algebras

Let $M, \Psi, \Phi, \psi$ and $\phi$ be as in the previous section. Let $E_\lambda^{\Phi,\Psi}$ denote the spectral projections of $\Delta_{\Phi,\Psi}$, $s(\omega)$ denote the support of the positive linear functional $\omega$.

**Definition 3.1.** For $\phi \neq 0$, the relative entropy $S(\psi/\phi)$ is defined by

$$S(\psi/\phi) \begin{cases} = \int_{+0}^{\infty} \log \lambda d(\Phi, E_\lambda^{\Phi,\Psi}\Phi) & \text{if } s(\psi) \geqq s(\phi). \\ = +\infty & \text{otherwise}. \end{cases}$$

**Lemma 3.2.** $S(\psi/\phi)$ is well defined, takes finite value or $+\infty$ and satisfies

$$S(\psi/\phi) \geqq -\phi(\mathbf{1}) \log\{\psi(s(\phi))/\phi(\mathbf{1})\}. \tag{3.1}$$

*Proof*: First consider the case $s(\psi) \geqq s(\phi)$. Since $s^M(\Psi) = s(\psi) \geqq s(\phi) = s^M(\Phi)$, we have $S_{\Phi,\Psi}\Psi = \Phi$.

Since $J\Phi = \Phi$, we have $(\Delta_{\Phi,\Psi})^{1/2}\Psi = \Phi$. Hence

$$\begin{aligned} \int_{+0}^{\infty} \lambda^{-1} \mathrm{d}(\Phi, E_\lambda^{\Phi,\Psi}\Phi) &= (\Psi, (\mathbf{1} - E_{+0}^{\Phi,\Psi})\Psi) \\ &= (\Psi, s^{M'}(\Psi) s^M(\Phi)\Psi) \\ &= (\Psi, s(\phi)\Psi) = \psi(s(\phi)) \leqq 1. \end{aligned} \tag{3.2}$$

This implies that the integral defining $S(\psi/\phi)$ converges at the lower end. Hence it is well defined and takes either finite value or $+\infty$.

Since $s(\psi) \geqq s(\phi)$ implies

$$\int_{+0}^{\infty} \mathrm{d}(\Phi, E_\lambda^{\Phi,\Psi}\Phi) = (\Phi, s^{M'}(\Psi) s^M(\Phi)\Phi) = \phi(\mathbf{1}),$$

$\mathrm{d}(\Phi, E_\lambda^{\Phi,\Psi}\Phi)/\phi(\mathbf{1})$ is a probability measure on $(0, +\infty)$. By the concavity of the logarithm, we obtain

$$\begin{aligned} S(\psi/\phi) &= -\phi(\mathbf{1}) \int_{+0}^{\infty} \log(\lambda^{-1}) \mathrm{d}(\Phi, E_\lambda^{\Phi,\Psi}\Phi)/\phi(\mathbf{1}) \\ &\geqq -\phi(\mathbf{1}) \log\left\{\int_{+0}^{\infty} \lambda^{-1} \mathrm{d}(\Phi, E_\lambda^{\Phi,\Psi}\Phi)/\phi(\mathbf{1})\right\} \end{aligned}$$

$$= -\phi(\mathbf{1})\log\{\psi(s(\phi))/\phi(\mathbf{1})\}.$$

The statement of Lemma holds trivially for the case where $s(\psi) \geqq s(\phi)$ does not hold.

*Remark* 3.3. The definition of $S(\psi/\phi)$ uses the (unique) vector representatives $\Psi$ and $\Phi$ in a natural positive cone V. The value $S(\psi/\phi)$, however does not depend on the choice of the natural positive cone V because of the following reason. If V′ is another natural positive cone, then there exists a unitary $w' \in M'$ such that $V' = w'V$. $\Psi' = w'\Psi$ and $\Phi' = w'\Phi$ are representative vectors of $\psi$ and $\phi$ in $V'$. We then obtain $\Delta_{\Phi',\Psi'} = w'\Delta_{\Phi,\Psi}(w')^*$ and hence $S(\psi/\phi)$ is unchanged.

*Remark* 3.4. By Theorem 2.4 (2), we have

$$\{\log \Delta_{\Phi,\Psi} + J(\log \Delta_{\Psi,\Phi}) J\} s^M(\Phi) s^{M'}(\Psi) = \mathbf{0}.$$

Hence, for the case $s(\psi) \geqq s(\phi)$, we obtain the following expression ([1]):

$$S(\psi/\phi) = -(\Phi, \log \Delta_{\Psi,\Phi}\Phi). \tag{3.3}$$

*Remark* 3.5. If $s(\psi) = s(\phi)$, then $\Delta_{\Phi,\Psi}$ is $\mathbf{0}$ on $(\mathbf{1} - s(\psi)Js(\psi)J)H$ and coincides with the relative modular operator for $s(\psi)Ms(\psi)$ on the space $s(\psi)Js(\psi)JH$, where $\Phi$ and $\Psi$ are cyclic and separating for $s(\psi)Ms(\psi)$. Hence $S(\psi/\phi)$ in this case is the same as the relative entropy of two *faithful* normal positive linear functionals $\psi$ and $\phi$ of $s(\psi)Ms(\psi)$.

**Theorem 3.6.**

(1) *If* $\psi(\mathbf{1}) = \phi(\mathbf{1}) > 0$, *then* $S(\psi/\phi) \geqq 0$. *The equality* $S(\psi/\phi) = 0$ *holds if and only if* $\psi = \phi$.

(2) *If* $s(\phi_1) \perp s(\phi_2)$, *then*

$$S(\psi/\phi_1) + S(\psi/\phi_2) = S(\psi/\phi_1 + \phi_2). \tag{3.4}$$

(3) *For* $\lambda_1, \lambda_2 > 0$,

$$S(\lambda_1\psi/\lambda_2\phi) = \lambda_2 S(\psi/\phi) - \lambda_2\phi(\mathbf{1}) \log(\lambda_1/\lambda_2). \tag{3.5}$$

(4) *If* $\psi_1 \geqq \psi_2$, *then*

(3. 6) $$S(\psi_1/\phi)\leqq S(\psi_2/\phi).$$

*Proof.*

(1) Since $\psi(s(\phi))\leqq\psi(\mathbf{1})$, the assumption $\psi(\mathbf{1})=\phi(\mathbf{1})$ and (3. 1) imply $S(\psi/\phi)\geqq 0$. Furthermore, the equality $S(\psi/\phi)=0$ holds only if $s(\psi)\geqq s(\phi)$ and $\psi(s(\phi))=\psi(\mathbf{1})$. We then have $s(\phi)=s(\psi)$; hence Remark 3. 5 and the strict positivity of $S(\psi/\phi)$ for faithful $\psi$ and $\phi$ ([1]) imply $\phi=\psi$ also in the present case. Conversely $\phi=\psi$ implies $S(\psi/\phi)=0$.

(2) (3. 4) follows from (2. 6) and Definition 3. 1.

(3) The vector representatives for $\lambda_1\psi$ and $\lambda_2\phi$ differs from those for $\psi$ and $\phi$ by factors $(\lambda_1)^{1/2}$ and $(\lambda_2)^{1/2}$ respectively. Hence this induces a change of $S_{\Phi,\Psi}$ by a factor $(\lambda_2/\lambda_1)^{1/2}$ and a change of $\Delta_{\Phi,\Psi}$ by a factor $(\lambda_2/\lambda_1)$. The latter proves (3. 5).

(4) If $s(\psi_2)\geqq s(\phi)$ does not hold, then (3. 6) is trivially true. Hence we assume $s(\psi_2)\geqq s(\phi)$. Since $\psi_1\geqq\psi_2$ implies $s(\psi_1)\geqq s(\psi_2)$, we also have $s(\psi_1)\geqq s(\phi)$. The following proof is then the same as that for the case of faithful $\psi$'s and $\phi$:

Denoting representative vectors of $\psi_1$, $\psi_2$ and $\phi$ in the natural positive cone by $\Psi_1$, $\Psi_2$ and $\Phi$, respectively, we obtain

$$\|(\Delta_{\Psi_1,\Phi})^{1/2}x\Phi\|^2=\|S_{\psi_1,\Phi}x\Phi\|^2=\|s(\phi)x^*\Psi_1\|^2$$
$$=\psi_1(xs(\phi)x^*)\geqq\psi_2(xs(\phi)x^*)=\|(\Delta_{\Psi_2,\Phi})^{1/2}x\Phi\|^2,$$

for all $x\in M$. Since both $(\Delta_{\Psi_j,\Phi})^{1/2}$ vanish on $(s^{M'}(\Phi)H)^{\perp}$ and since $M\Phi+(\mathbf{1}-s^{M'}(\Phi))H$ is the core of $(\Delta_{\Psi_1,\Phi})^{1/2}$, it follows that the domain of $(\Delta_{\Psi_1,\Phi})^{1/2}$ is contained in the domain of $(\Delta_{\Psi_2,\Phi})^{1/2}$ and for all $\Omega$ in the domain of $(\Delta_{\Psi_1,\Phi})^{1/2}$

$$\|(\Delta_{\Psi_1,\Phi})^{1/2}\Omega\|\geqq\|(\Delta_{\Psi_2,\Phi})^{1/2}\Omega\|.$$

Hence

$$\|(\Delta_{\Psi_1,\Phi}+r)^{1/2}\Omega\|^2\geqq\|(\Delta_{\Psi_2,\Phi}+r)^{1/2}\Omega\|^2$$

for all such $\Omega$ and $r>0$. Taking $\Omega=(\Delta_{\Psi_1,\Phi}+r)^{-1/2}\Omega'$ with an arbitrary $\Omega'$, we find

$$\|(\Delta_{\Psi_2,\Phi}+r)^{1/2}(\Delta_{\Psi_1,\Phi}+r)^{-1/2}\|\leqq 1\,.$$

Taking adjoint operator acting on $\Omega=(\Delta_{\Psi_2,\Phi}+r)^{-1/2}\Omega'$ with an arbitrary $\Omega'$, we find

$$\|(\Delta_{\Psi_1,\Phi}+r)^{-1/2}\Omega'\|^2\leqq\|(\Delta_{\Psi_2,\Phi}+r)^{-1/2}\Omega'\|^2$$

and hence

$$(3.7)\qquad (\Delta_{\Psi_1,\Phi}+r)^{-1}\leqq(\Delta_{\Psi_2,\Phi}+r)^{-1}\,.$$

By (3. 3) we have

$$(3.8)\quad S(\psi_j/\phi)=-\int_0^\infty\left\{\int_0^\infty[(1+r)^{-1}-(\lambda+r)^{-1}]\mathrm{d}r\right\}\mathrm{d}(\Phi,E_\lambda^{\Psi_j,\Phi}\Phi)$$

$$=\int_0^\infty(\Phi,[(r+\Delta_{\Psi_j,\Phi})^{-1}-(1+r)^{-1}]\Phi)\,dr$$

where $E_\lambda^{\Psi_j,\Phi}$ is the spectral projection of $\Delta_{\Psi_j,\Phi}$ and the interchange of $r$- and $\lambda$- integrations are allowed because the double integral is definite in the Lebesque sense (finite or $+\infty$) due to

$$\int_0^\infty\lambda\mathrm{d}(\Phi,E_\lambda^{\Psi_j,\Phi}\Phi)=\|(\Delta_{\Psi_j,\Phi})^{1/2}\Phi\|^2=\|s(\phi)\Psi_j\|^2<\infty\,.$$

The equations (3. 8) and (3. 7) imply (3. 6). Q.E.D.

The following Theorem describes the continuity property of $S(\psi/\phi)$ as a function of $\psi$ and $\phi$. (It is the same as the case of faithful $\psi$ and $\phi$.)

**Theorem 3. 7.**

*Assume that* $\lim\|\phi_\alpha-\phi\|=\lim\|\psi_\alpha-\psi\|=0$.

(1) $\liminf S(\psi_\alpha/\phi_\alpha)\geqq S(\psi/\phi)$ *(the lower semicontinuity).*

(2) *If* $\lambda\psi_\alpha\geqq\phi_\alpha$ *for a fixed* $\lambda>0$, *then*

$$\lim S(\psi_\alpha/\phi_\alpha)=S(\psi/\phi)\,.$$

(3) *If* $\psi_\alpha$ *is monotone decreasing, then*

$$\lim S(\psi_\alpha/\phi)=S(\psi/\phi)\,.$$

We shall give proof of this Theorem in the next section. Using this theorem in an approximation argument, we obtain the next theorem from the same theorem ([1]) for faithful functionals.

**Theorem 3.8.**

(1) *$S(\psi/\phi)$ is jointly convex in $\psi$ and $\phi$.*

(2) *Let $N$ be a von Neumann subalgebra of $M$ and $E_N\omega$ denotes the restriction of a functional $\omega$ to $N$. Then*

$$S(E_N\psi/E_N\phi)\leqq S(\psi/\phi) \tag{3.9}$$

*if $N$ is any one of the following type:*

($\alpha$) *$N=\mathfrak{A}'\cap M$ for a finite dimensional abelian $*$-subalgebra $\mathfrak{A}$ of $M$.*

($\beta$) *$M=N\otimes N_1$.*

($\gamma$) *$N$ is approximately finite.*

*Proof.*

(1) We have to prove the following

$$S(\sum_{j=1}^{n}\lambda_j\psi_j/\sum_{j=1}^{n}\lambda_j\phi_j)\leqq\sum_{j=1}^{n}\lambda_jS(\psi_j/\phi_j) \tag{3.10}$$

for $\lambda_j>0$, $\sum\lambda_j=1$. Let $\psi=\sum\lambda_j\psi_j$, $\phi=\sum\lambda_j\phi_j$, $\omega=\psi+\phi$. By Remark 3.5,

$$S(\psi+\varepsilon\omega/\phi+\eta\omega)\leqq\sum_{j=1}^{n}\lambda_jS(\psi_j+\varepsilon\omega/\phi_j+\eta\omega)$$

follows from the convexity of $S(\psi_0/\phi_0)$ for faithful $\psi_0$ and $\phi_0$. We first take the limit $\eta\to+0$ using Theorem 3.7 (2) and then take the limit $\varepsilon\to+0$ using Theorem 3.7 (3) to obtain (3.10).

(2) Let $\omega_0$ be a faithful normal state of $M$ and let $\omega=\omega_0+\phi+\psi$. Then

$$S(E_N(\psi+\varepsilon\omega)/E_N(\phi+\eta\omega))\leqq S(\psi+\varepsilon\omega/\phi+\eta\omega).$$

Again Theorem 3.7 (2) and (3) yield (3.9). Q.E.D.

The following Theorem describe some continuity property of $S(E_N\psi/E_N\phi)$ on $N$.

**Theorem 3.9.** *Let $N_\alpha$ be monotone increasing net of von Neumann subalgebras of $M$ generating $M$.*

(1) $\liminf S(E_{N_\alpha}\psi/E_{N_\alpha}\phi)\geqq S(\psi/\phi)$.

(2) *If $N_\alpha$ is an AF algebra for all $\alpha$, then*

$$\lim S(E_{N_\alpha}\psi/E_{N_\alpha}\phi)=S(\psi/\phi).$$

Proof of (1) and (2) will be given in the next section. (2) follows from (1) and Theorem 3. 8 (2) ($\gamma$).

Let $\psi$ be a faithful normal positive linear functional of $M$ corresponding to a cyclic and separating vector $\Psi$ and $h=h^*\in M$. Let $\Psi(h)$ denote the perturbed vector defined by (4. 1) in [8]. Let $\psi^h$ denote the perturbed state defined by

$$\psi^h(x)=(\Psi(h),x\Psi(h)),\quad x\in M.$$

**Theorem 3. 10.**

$$S(\psi^h/\phi)=-\phi(h)+S(\psi/\phi),$$

$$S(\phi/\psi^h)=\psi^h(h)+S(\phi^h/\psi^h).$$

## § 4. Some Continuity Properties

We first prove some continuity properties of the relative modular operators.

**Lemma 4. 1.** *If* $\lim\|\phi_\alpha-\phi\|=\lim\|\psi_\alpha-\psi\|=0$, *then*

(4. 1) $$\lim(r+(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2})^{-1}s^{M'}(\Psi)=(r+(\Delta_{\Phi,\Psi})^{1/2})^{-1}s^{M'}(\Psi)$$

*for* $r>0$ *and the convergence is uniform in* $r$ *if* $r$ *is restricted to any compact subset of* $(0,\infty)$, *where* $\Phi_\alpha$, $\Psi_\alpha$, $\Phi$ *and* $\Psi$ *are the representative vectors of* $\phi_\alpha$, $\psi_\alpha$, $\phi$ *and* $\psi$ *in the positive natural cone, respectively.*

*Proof.* The condition $\lim\|\phi_\alpha-\phi\|=\lim\|\psi_\alpha-\psi\|=0$ implies (Theorem 4(8) in [4])

(4. 2) $$\lim\|\Phi_\alpha-\Phi\|=\lim\|\Psi_\alpha-\Psi\|=0.$$

For $x'\in M'$, we have

$$\lim\|s^M(\Psi_\alpha)x'\Psi-x'\Psi\|$$
$$=\lim\|s^M(\Psi_\alpha)x'(\Psi-\Psi_\alpha)+x'(\Psi_\alpha-\Psi)\|=0.$$

Hence

$$\lim s^M(\Psi_\alpha) s^M(\Psi) = s^M(\Psi). \tag{4.3}$$

For $x \in M\ s^M(\Psi)$, we have

$$\begin{aligned}
&\|(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2} x\Psi_\alpha - (\Delta_{\Phi,\Psi})^{1/2} x\Psi\| \\
&\quad = \|J(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2} x\Psi_\alpha - J(\Delta_{\Phi,\Psi})^{1/2} x\Psi\| \\
&\quad = \|s^M(\Psi_\alpha) x^*\Phi_\alpha - x^*\Phi\| \\
&\quad \leqq \|(s^M(\Psi_\alpha) s^M(\Psi) - s^M(\Psi)) x^*\Phi\| + \|s^M(\Psi_\alpha) x^*(\Phi_\alpha - \Phi)\| \to 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|\{[\mathbf{1} + (\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2}]^{-1} - [\mathbf{1} + (\Delta_{\Phi,\Psi})^{1/2}]^{-1}\} [\mathbf{1} + (\Delta_{\Phi,\Psi})^{1/2}] x\Psi\| \\
&\quad = \|[\mathbf{1} + (\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2}]^{-1} \\
&\qquad \times \{x(\Psi - \Psi_\alpha) + [(\Delta_{\Phi,\Psi})^{1/2} x\Psi - (\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2} x\Psi_\alpha]\} + x(\Psi_\alpha - \Psi)\| \\
&\quad \leqq 2\|x(\Psi_\alpha - \Psi)\| + \|(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2} x\Psi_\alpha - (\Delta_{\Phi,\Psi})^{1/2} x\Psi\| \to 0.
\end{aligned}$$

Since $M\ s^M(\Psi)\Psi + (\mathbf{1} - s^{M'}(\Psi))H$ is a core for $(\Delta_{\Phi,\Psi})^{1/2}$, the vectors

$$(\mathbf{1} + (\Delta_{\Phi,\Psi})^{1/2}) x\Psi, \quad x \in M s^M(\Psi)$$

are dense in $s^{M'}(\Psi)H$. Since

$$\|[\mathbf{1} + (\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2}]^{-1}\| \leqq 1$$

is uniformly bounded, we obtain

$$\lim [\mathbf{1} + (\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2}]^{-1} s^{M'}(\Psi) = [\mathbf{1} + (\Delta_{\Phi,\Psi})^{1/2}]^{-1} s^{M'}(\Psi).$$

The rest of the proof is standard. For $r > 0$ and $\Delta_\alpha = \Delta_\alpha^* \geqq 0$,

$$(r + \Delta_\alpha)^{-1} = R_r(\Delta_\alpha)(\mathbf{1} + \Delta_\alpha)^{-1} \tag{4.4}$$

with

$$R_r(\Delta_\alpha) = \{\mathbf{1} + (r-1)(\mathbf{1} + \Delta_\alpha)^{-1}\}^{-1},$$

$$\|R_r(\Delta_\alpha)\| \leqq \max\{1, r^{-1}\}.$$

If $\Delta = \Delta^* \geqq 0$, $\lim (\mathbf{1} + \Delta_\alpha)^{-1} s = (\mathbf{1} + \Delta)^{-1} s$ for a projection $s$ commuting with $\Delta$, then the formula

$$\begin{aligned}
(r + \Delta_\alpha)^{-1} - (r + \Delta)^{-1} &= R_r(\Delta_\alpha)\{(\mathbf{1} + \Delta_\alpha)^{-1} - (\mathbf{1} + \Delta)^{-1}\} \\
&\quad - R_r(\Delta_\alpha)(r-1)\{(\mathbf{1} + \Delta_\alpha)^{-1} - (\mathbf{1} + \Delta)^{-1}\} R_r(\Delta)(\mathbf{1} + \Delta)^{-1}
\end{aligned}$$

implies

$$\lim\{(r+\Delta_\alpha)^{-1}-(r+\Delta)^{-1}\}s=0,$$

where the convergence is uniform if $r$ is restricted to any compact subset of $(0,\infty)$. By applying this result to $\Delta_\alpha=(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2}$, $\Delta=(\Delta_{\Phi,\Psi})^{1/2}$ and $s=s^{M'}(\Psi)$, we obtain the Lemma. Q.E.D.

*Proof of Theorem* 3.7 (1). We divide our proof into several steps. Obviously we may omit those $\alpha$ for which $s(\psi_\alpha)\geqq s(\phi_\alpha)$ does not hold out of our consideration so that we may assume $s(\psi_\alpha)\geqq s(\phi_\alpha)$ for all $\alpha$ without loss of generality.

(a) *The case where* $\psi$ *is faithful*: Due to $s(\psi)=\mathbf{1}$, we have $s^{M'}(\Psi)=Js^M(\Psi)J=\mathbf{1}$. Hence (4.2) and Lemma 4.1 imply

$$\lim\int_\varepsilon^L dr(\Phi_\alpha,\{(\mathbf{1}+r)^{-1}-[r+(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2}]^{-1}\}\Phi_\alpha) \tag{4.5}$$

$$=\int_\varepsilon^L dr(\Phi,\{(1+r)^{-1}-[r+(\Delta_{\Phi,\Psi})^{1/2}]^{-1}\}\Phi)$$

for all $0<\varepsilon<L<\infty$. (Note that

$$\|[r+(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2}]^{-1}\|\leqq r^{-1}$$

is uniformly bounded.)

We also have the following estimates:

$$\int_0^\varepsilon dr\left|(\Phi_\alpha,\{(1+r)^{-1}-[r+(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2}]^{-1}\}\Phi_\alpha)\right| \tag{4.6}$$

$$=\int_0^\varepsilon dr\left|\int_0^\infty(1+r)^{-1}(1+r\lambda^{-1/2})^{-1}(1-\lambda^{-1/2})\,d(\Phi_\alpha,E_\lambda^{\Phi_\alpha,\Psi_\alpha}\Phi_\alpha)\right|$$

$$\leqq\int_0^\varepsilon dr\int_0^\infty\max(1,\lambda^{-1})\,d(\Phi_\alpha,E_\lambda^{\Phi_\alpha,\Psi_\alpha}\Phi_\alpha)$$

$$\leqq\varepsilon(\phi_\alpha(\mathbf{1})+\psi_\alpha(\mathbf{1}))$$

due to (3.2), where $E_\lambda^{\Phi_\alpha,\Psi_\alpha}$ is the spectral projection of $\Delta_{\Phi_\alpha,\Psi_\alpha}$.

$$\int_L^\infty dr\left|\int_0^1\{(1+r)^{-1}-(r+\lambda^{1/2})^{-1}\}\,d(\Phi_\alpha,E_\lambda^{\Phi_\alpha,\Psi_\alpha}\Phi_\alpha)\right| \tag{4.7}$$

$$=\int_L^\infty dr\int_0^1(1+r)^{-1}(1+r\lambda^{-1/2})^{-1}(\lambda^{-1/2}-1)\,d(\Phi_\alpha,E_\lambda^{\Phi_\alpha,\Psi_\alpha}\Phi_\alpha)$$

$$\leqq \int_L^\infty \mathrm{d}r(1+r)^{-1}r^{-1}\int_0^1 \mathrm{d}(\Phi_\alpha, E_\lambda^{\Phi_\alpha,\Psi_\alpha}\Phi_\alpha)$$

$$\leqq \log(1+L^{-1})\phi_\alpha(\mathbf{1}).$$

Finally

$$\int_L^\infty \mathrm{d}r \int_1^\infty \{(1+r)^{-1}-(r+\lambda^{1/2})^{-1}\}\mathrm{d}(\Phi_\alpha, E_\lambda^{\Phi_\alpha,\Psi_\alpha}\Phi_\alpha)\geqq 0. \tag{4.8}$$

Hence

$$\liminf \int_0^\infty \mathrm{d}r(\Phi_\alpha,\{(1+r)^{-1}-[r+(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2}]^{-1}\}\Phi_\alpha) \tag{4.9}$$

$$\geqq \int_\varepsilon^L \mathrm{d}r(\Phi, \{(1+r)^{-1}-[r+(\Delta_{\Phi,\Psi})^{1/2}]^{-1}\}\Phi)$$

$$-\varepsilon(\phi(\mathbf{1})+\psi(\mathbf{1}))-\{\log(1+L^{-1})\}\phi(\mathbf{1}).$$

We now use the following formula, which holds if $s(\psi)\geqq s(\phi)$.

$$S(\psi/\phi)=2\int_0^\infty\left(\int_0^\infty\{(1+r)^{-1}-(r+\lambda^{1/2})^{-1}\}\mathrm{d}r\right)\mathrm{d}(\Phi, E_\lambda^{\Phi,\Psi}\Phi) \tag{4.10}$$

$$=2\int_0^\infty \mathrm{d}r(\Phi, \{(1+r)^{-1}-[r+(\Delta_{\Phi,\Psi})^{1/2}]^{-1}\}\Phi),$$

where the change of the order of $r$- and $\lambda$- integrations is allowed because the integral is definite in the Lebesgue sense (finite or $+\infty$) due to (3.2).

By taking the limit $\varepsilon\to+0$ and $L\to+\infty$ and by substituting (4.10) and the same formula for the pair $\psi_\alpha, \phi_\alpha$, we obtain Theorem 3.7 (1) for this case.

(b) *The case where $\phi_\alpha$ is independent of $\alpha$*: By (3.3) and by the same computation as (4.10), we obtain

$$S(\psi_\alpha/\phi)=-2\int_0^\infty \mathrm{d}r(\Phi,\{(1+r)^{-1}-[r+(\Delta_{\Psi_\alpha,\Phi})^{1/2}]^{-1}\}\Phi) \tag{4.11}$$

where the boundedness

$$\int_0^\infty \lambda \mathrm{d}(\Phi, E_\lambda^{\Psi_\alpha,\Phi}\Phi)=\|(\Delta_{\Psi_\alpha,\Phi})^{1/2}\Phi\|^2=\psi_\alpha(s(\phi))<\infty \tag{4.12}$$

guarantees the definiteness of the integral in (4.11). (Note that $s(\psi_\alpha)\geqq s(\phi_\alpha)=s(\phi)$.)

By Lemma 4.1 and by the same argument as the Case (a), we obtain

$$\text{(4.13)} \qquad \liminf S(\psi_\alpha/\phi) \geqq -2\int_0^\infty dr(\Phi, \{(1+r)^{-1} - [r + (\Delta_{\Psi,\Phi})^{1/2}]^{-1}\}\Phi).$$

Since $(\Delta_{\Psi,\Phi})^{1/2}$ commutes with $s^M(\Psi) = Js^{M'}(\Psi)J$, the inner product in (4.13) is the sum of contributions from the expectation values in $(\mathbf{1} - s^M(\Psi))\Phi$ and $s^M(\Psi)\Phi$. The first one is given by

$$-2\int_0^\infty dr(\Phi, \{(1+r)^{-1} - r^{-1}\}(\mathbf{1} - s^M(\Psi))\Phi) = +\infty$$

if

$$(\Phi, (\mathbf{1} - s^M(\Psi))\Phi) = \phi(\{\mathbf{1} - s(\psi)\}) > 0,$$

i.e. if $s(\psi) \geqq s(\phi)$ does not hold. The second one is either finite or $+\infty$ by (4.12). Hence if $s(\psi) \geqq s(\phi)$ does not hold, then

$$\text{(4.14)} \qquad \lim S(\psi_\alpha/\phi) = +\infty = S(\psi/\phi).$$

If $s(\psi) \geqq s(\phi)$ holds, then (4.13) already proves Theorem 3.7 (1) for the present case.

(c) *General case*: Let $\omega$ be a normal faithful state. For $\varepsilon > 0$, we obtain

$$\liminf S(\psi_\alpha + \varepsilon\omega/\phi_\alpha) \geqq S(\psi + \varepsilon\omega/\phi)$$

by the Case (a). By Theorem 3.6 (4),

$$S(\psi_\alpha + \varepsilon\omega/\phi_\alpha) \leqq S(\psi_\alpha/\phi_\alpha).$$

Hence

$$\liminf S(\psi_\alpha/\phi_\alpha) \geqq S(\psi + \varepsilon\omega/\phi).$$

By taking the limit $\varepsilon \to +0$ and using the Case (b), we obtain Theorem 3.7 (1) for the general case.

*Proof of Theorem* 3.7 (2). If $\omega' \geqq \lambda^{-1}\omega$ for $\lambda > 0$, then (3.7) implies

$$(\Delta_{\Omega',\Omega} + r)^{-1} \leqq (\lambda^{-1}\Delta_{\Omega,\Omega} + r)^{-1}.$$

Due to the identity

$$(r+\rho^{1/2})^{-1}=\pi^{-1}\int_0^\infty(\rho+x)^{-1}(x+r^2)^{-1}x^{1/2}dx\,,\quad r>0\,,$$

for a positive self-adjoint $\rho$, this implies

$$(r+(\Delta_{\Omega',\Omega})^{1/2})^{-1}\leqq(r+\lambda^{-1/2}(\Delta_{\Omega,\Omega})^{1/2})^{-1}.$$

Hence

$$\begin{aligned}\omega(\mathbf{1})&\geqq(\Omega,\{(1+r)^{-1}-[r+(\Delta_{\Omega',\Omega})^{1/2}]^{-1}\}\Omega)\\&\geqq\omega(\mathbf{1})\{(1+r)^{-1}-(\lambda^{-1/2}+r)^{-1}\}\\&=\omega(\mathbf{1})(1+r)^{-1}(1+\lambda^{1/2}r)^{-1}(1-\lambda^{1/2}).\end{aligned}$$

Therefore

$$\begin{aligned}(4.15)\qquad -\varepsilon\omega(\mathbf{1})&\leqq-\int_0^\varepsilon dr(\Omega,\{(1+r)^{-1}-[r+(\Delta_{\Omega',\Omega})^{1/2}]^{-1}\}\Omega)\\&\leqq\omega(\mathbf{1})\log\{(1+\varepsilon)(1+\lambda^{1/2}\varepsilon)^{-1}\}\end{aligned}$$

for $\varepsilon>0$. We also have for $L>0$

$$\begin{aligned}(4.16)\qquad \omega(\mathbf{1})\log(1+L^{-1})&=-\int_L^\infty(\Omega,\{(1+r)^{-1}-r^{-1}\}\Omega)\\&\geqq-\int_L^\infty dr(\Omega,\{(1+r)^{-1}-[r+(\Delta_{\Omega',\Omega})^{1/2}]^{-1}\}\Omega)\\&=-\int_L^\infty dr(\Omega,(1+r)^{-1}[r+(\Delta_{\Omega',\Omega})^{1/2}]^{-1}\{(\Delta_{\Omega',\Omega})^{1/2}-\mathbf{1}\}\Omega)\\&\geqq-\int_L^\infty(1+r)^{-1}r^{-1}dr\|(\Delta_{\Omega',\Omega})^{1/2}\Omega\|^2\\&=-\omega'(\mathbf{1})\log(1+L^{-1}).\end{aligned}$$

where the last inequality is obtained by using the spectral decomposition $\Delta_{\Omega',\Omega}=\int\lambda dE_\lambda$ and majorizing $(r+\lambda^{1/2})^{-1}(\lambda^{1/2}-1)$ by $r^{-1}\lambda$ for $0\leqq\lambda$. Since

$$\begin{aligned}\lim\int_\varepsilon^L dr(\Phi_\alpha,\{(1+r)^{-1}-[r+(\Delta_{\Psi_\alpha,\Phi_\alpha})^{1/2}]^{-1}\}\Phi_\alpha)\\=\int_\varepsilon^L dr(\Phi,\{(1+r)^{-1}-[r+(\Delta_{\Psi,\Phi})^{1/2}]^{-1}\}\Phi),\end{aligned}$$

the estimates (4.15) and (4.16) for $(\omega',\omega)=(\psi_\alpha,\phi_\alpha)$ and for $(\omega',\omega)=(\psi,\phi)$ yield

$$(4.17)\quad \lim \int_0^\infty dr(\Phi_\alpha, \{(1+r)^{-1} - [r+(\Delta_{\Psi_\alpha,\Phi_\alpha})^{1/2}]^{-1}\}\Phi_\alpha)$$
$$= \int_0^\infty dr(\Phi, \{(1+r)^{-1}[r+(\Delta_{\Psi,\Phi})^{1/2}]^{-1}\}\Phi).$$

Since $\lambda\psi_\alpha \geqq \phi_\alpha$ and its consequence $\lambda\psi \geqq \phi$ imply $s(\psi_\alpha) \geqq s(\phi_\alpha)$ and $s(\psi) \geqq s(\phi)$, the equations (4.17) and an expression of the form (4.11) for $s(\psi_\alpha/\phi_\alpha)$ and $s(\psi/\phi)$ imply Theorem 3.7 (2).

*Proof of Theorem* 3.7 (3). This follows from Theorem 3.7 (1) and Theorem 3.6 (4). Q.E.D.

*Remark* 4.2. The argument leading to (4.14) implies that the formula

$$(4.18)\quad S(\psi/\phi) = -2\int_0^\infty dr(\Phi, \{(1+r)^{-1} - [r+(\Delta_{\Psi,\Phi})^{1/2}]^{-1}\}\Phi),$$

which is used in (4.11) for the case $s(\psi) \geqq s(\phi)$, holds for a general pair $\psi$ and $\phi$ (even if $s(\psi) \geqq s(\phi)$ does not hold). this is not the case for the formula of the form (4.10).

*Proof of Theorem* 3.9 (1). Let $\omega_0$ be a faithful state, $\omega = \omega_0 + \psi + \phi$, and $1 > \varepsilon > \mu > 0$. The proof of Lemma 3 in [1] (without the assumption $\psi \leqq k\phi$ there) implies

$$(4.19)\quad \liminf S(E_{N_\alpha}\psi_\varepsilon/E_{N_\alpha}\phi_\mu) \geqq S(\psi_\varepsilon/\phi_\mu)$$

where

$$\psi_\varepsilon = (1-\varepsilon)\psi + \varepsilon\omega, \quad \phi_\mu = (1-\mu)\phi + \mu\omega.$$

By the convexity (Theorem 3.8 (1)),

$$(4.20)\quad S(E_{N_\alpha}\psi_\varepsilon/E_{N_\alpha}\phi_\mu)$$
$$\leqq (1-\mu)S(E_{N_\alpha}\psi_\varepsilon/E_{N_\alpha}\phi) + \mu S(E_{N_\alpha}\psi_\varepsilon/E_{N_\alpha}\omega).$$

By Theorem 3.6 (4) and (3), we have

$$(4.21)\quad S(E_{N_\alpha}\psi_\varepsilon/E_{N_\alpha}\omega) \leqq S(E_{N_\alpha}(\varepsilon\omega)/E_{N_\alpha}\omega)$$
$$= -\omega(\mathbf{1})\log\varepsilon < \infty.$$

By Theorem 3.7 (2)

(4. 22) $$\lim_{\mu\to 0} S(\psi_\varepsilon/\phi_\mu)=S(\psi_\varepsilon/\phi).$$

The formulas (4. 20), (4. 21) and (4. 22) imply in the limit $\mu\to+0$

(4. 23) $$\liminf S(E_{N_\alpha}\psi_\varepsilon/E_{N_\alpha}\phi)\geqq S(\psi_\varepsilon/\phi).$$

By Theorem 3. 7 (3),

(4. 24) $$\lim_{\varepsilon\to 0} S(\psi_\varepsilon/\phi)=S(\psi/\phi).$$

By Theorem 3. 6 (4),

(4. 25) $$S(E_{N_\alpha}\psi_\varepsilon/E_{N_\alpha}\phi)\leqq S(E_{N_\alpha}(1-\varepsilon)\psi/E_{N_\alpha}\phi)$$
$$=S(E_{N_\alpha}\psi/E_{N_\alpha}\phi)-\phi(\mathbf{1})\log(1-\varepsilon).$$

The formulas (4. 23), (4. 24) and (4. 25) imply in the limit $\varepsilon\to+0$ Theorem 3. 9 (1).

*Proof of Theorem* 3. 9 (2). This follows from Theorem 3. 9 (1) and Theorem 3. 8 (2) ($\gamma$). Q.E.D.

*Proof of Theorem* 3. 10. First consider the case where $\phi$ is faithful. Then $\Omega$ given by (2. 8) is cyclic and separating for $M$. From the definition of the perturbed state and the expression (2. 10), we obtain

(4. 26) $$\Psi(h)\otimes e_{11}+\Phi\otimes e_{22}=\Omega(h\otimes u_{11}).$$

By (4. 13) of [8], we have

(4. 27) $$\log\Delta_{\Omega(h\otimes u_{11})}=\log\Delta_\Omega+h\otimes u_{11}-j(h)\otimes J_K u_{11}J_k.$$

Here $j(h)$ denotes $JhJ$. By (2. 10), we obtain

(4. 28) $$\log\Delta_{\Psi(h),\Phi}=\log\Delta_{\Psi,\Phi}+h,$$

(4. 29) $$\log\Delta_{\Phi,\Psi(h)}=\log\Delta_{\Phi,\Psi}-j(h).$$

By (3. 3), for example, we obtain Theorem 3. 10 for the present case of a faithful $\phi$.

For the general case, we apply the result just proved to

$$\phi_\varepsilon=(1-\varepsilon)\phi+\varepsilon\psi,\quad \varepsilon>0,$$

which is faithful:

(4. 30) $$S(\psi^h/\phi_\varepsilon) = -(1-\varepsilon)\phi(h) - \varepsilon\psi(h) + S(\psi/\phi_\varepsilon).$$

From the convexity of the relative entropy, we obtain

$$S(\psi^h/\phi_\varepsilon) \leqq (1-\varepsilon) S(\psi^h/\phi) - \varepsilon\psi(h).$$

Combining the limit $\varepsilon \to +0$ of this relation with Theorem 3. 7 (1), we obtain

(4. 31) $$\lim_{\varepsilon \to +0} S(\psi^h/\phi_\varepsilon) = S(\psi^h/\phi).$$

For $h=0$, we have the same equation for $\psi$. Hence the first equation of Theorem 3. 10 follows from (4. 30). The second equation of Theorem 3. 10 is trivially true for a non-faithful $\phi$ because both sides of the equation is then $+\infty$.

## § 5. Relative Entropy of States of $C^*$-Algebras

For two positive linear functionals $\psi$ and $\phi$ of a $C^*$-algebra $\mathfrak{A}$, we define the relative entropy $S(\psi/\phi)$ by

(5. 1) $$S(\psi/\phi) \equiv S(\tilde{\psi}/\tilde{\phi})$$

where $\tilde{\psi}$ and $\tilde{\phi}$ are the unique normal extension of $\psi$ and $\phi$ to the enveloping von Neumann algebra $\mathfrak{A}''$.

If the cyclic representation $\pi_\psi$ associated with $\psi$ does not quasi-contain the cyclic representation $\pi_\phi$ associated with $\phi$, then the central support of $\tilde{\psi}$ does not majorize that of $\tilde{\phi}$, hence $s(\tilde{\psi}) \geq s(\tilde{\phi})$ does not hold. Therefore

(5. 2) $$S(\psi/\phi) = +\infty$$

if $\pi_\psi$ does not quasi-contain $\pi_\phi$.

From the definition (5. 1), it follows that

(5. 3) $$S(\psi/\phi) = S(\hat{\psi}/\hat{\phi})$$

where $\hat{\psi}$ and $\hat{\phi}$ are the unique normal extension of $\psi$ and $\phi$ to $M = \pi(\mathfrak{A})''$ where $\pi = \pi_\psi \oplus \pi_\phi$. If $\mathfrak{A}$ is separable, then $M = \pi(\mathfrak{A})''$ for this $\pi$ has a separable predual and hence all results in previous sections apply. In particular, if $\mathfrak{A}_\alpha$ is a monotone increasing net of nuclear $C^*$-subalgebras of $\mathfrak{A}$ generating $\mathfrak{A}$, then

$$\lim S(E_{\mathfrak{A}_\alpha}\psi/E_{\mathfrak{A}_\alpha}\phi)=S(\psi/\phi). \tag{5.4}$$

This implies the result in [2] that if

$$\sup_\alpha S(E_{\mathfrak{A}_\alpha}\psi/E_{\mathfrak{A}_\alpha}\phi)<\infty,$$

then $\pi_\psi$ quasi-contains $\pi_\phi$.

If $\mathfrak{A}$ is separable, then the restriction of the envelopping von Neumann algebra $\mathfrak{A}''$ to a direct sum of a denumerable number of cyclic representations of $\mathfrak{A}$ has a faithful normal state. Hence Theorems 3.6, 3.8, and 3.9 as well as Theorem 3.7 for sequences are valid for positive linear functionals of $C^*$-algebras.

If $\psi$ is a positive linear functional of a $C^*$-algebra $\mathfrak{A}$ such that the corresponding cyclic vector $\Psi$ for the associated cyclic representation $\pi_\psi$ of $\mathfrak{A}$ is separating for the weak closure $\pi_\psi(\mathfrak{A})''$, then the perturbed state $\psi^h$ for $h=h^*\in\mathfrak{A}$ is defined by

$$\psi^h(a)=(\Psi[\pi_\psi(h)],\pi_\psi(a)\Psi[\pi_\psi(h)]),\quad a\in\mathfrak{A}. \tag{5.5}$$

For such $\psi$, Theorem 3.10 holds for $C^*$-algebras.

## § 6. Conditional Entropy

Let $\mathfrak{A}$ be a UHF algebra with an increasing suquence of finite dimensional factors $\mathfrak{A}_n$ generating $\mathfrak{A}$. Let $\mathfrak{A}^c_{m,n}$ be the relative commutant of $\mathfrak{A}_n$ in $\mathfrak{A}_m$. The conditional entropy $\tilde{S}_n(\phi)$ of a positive linear functional $\phi$ of $\mathfrak{A}$ is defined by

$$\tilde{S}_n(\phi)=\lim_{m\to\infty}(S(E_{\mathfrak{A}_m}\phi)-S(E_{\mathfrak{A}^c_{m,n}}\phi)) \tag{6.1}$$

where

$$S(\psi)=-\psi(\log\rho_\psi)$$

for a positive linear functional $\psi$ of a finite dimensional factor $\mathfrak{M}_k$ and $\rho_\psi$ is the density matrix of $\psi$ defined by

$$\psi(a)=\tau(\rho_\psi a),\quad a\in\mathfrak{M}_k$$

with the unique trace state $\tau$ of $\mathfrak{M}_k$. ([3])

Let $\mathfrak{A}^c{}_{,n}$ be the relative commutant of $\mathfrak{A}_n$ in $\mathfrak{A}$, $\omega$ be the restriction of $\phi$ to $\mathfrak{A}^c{}_{,n}$ and $\omega'$ be any positive linear functional on $\mathfrak{A}^c{}_{,n}$. Then

(6. 2) $S(E_{\mathfrak{A}_n}\phi)-S(E_{\mathfrak{A}^c_{m,n}}\phi)=-S(E_{\mathfrak{A}_m}(\tau_n\otimes\omega')/E_{\mathfrak{A}_m}\phi)+S(E_{\mathfrak{A}^c_{m,n}}\omega'/E_{\mathfrak{A}^c_{m,n}}\omega)$

where $\tau_n$ is the unique trace state on $\mathfrak{A}_n$, because the density matrices for $E_{\mathfrak{A}_m}(\tau_n\otimes\omega')$ and for $E_{\mathfrak{A}^c_{m,n}}\omega'$ are the same element of $\mathfrak{A}$.

By taking the limit $m\to\infty$ and using (5. 4), we obtain

$$\tilde{S}_n(\phi)=S(\omega'/\omega)-S(\tau_n\otimes\omega'/\phi). \tag{6. 3}$$

Since the left hand side is finite, it follows that if either $S(\omega'/\omega)$ or $S(\tau_n\otimes\omega'/\phi)$ is finite, then both quantities are finite and (6. 3) holds. This formula has been used in [3].

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