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Relative nullity foliations and lightlike hypersurfaces in indefinite Kenmotsu manifolds

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Abstract

This paper deals with the relative nullity distributions of lightlike hypersurfaces of indefinite Kenmotsu space forms, tangent to the structure vector field. Theorems on parallel vector fields are obtained. We give characterization theorems for the relative nullity distributions as well as for Einstein, totally contact umbilical and flat lightlike hypersurfaces. We show that, under a certain condition, Einstein lightlike hypersurfaces in indefinite Kenmotsu space forms have parallel screen distributions. We prove that on a parallel (or totally umbilical) lightlike hypersurface, the relative nullity space coincides with the tangent vector space.

Key Words: η-Einstein lightlike hypersurfaces; Indefinite Kenmotsu manifold; Relative nullity foliation; Screen distribution.

1. Introduction

Nullity spaces in Riemannian geometry have been studied by many authors, see references [1], [5] and references therein. Abe and Magid in [1], for instance, extended the study of the relative nullity to isometric immersion between manifolds with indefinite metric. The present paper aims to investigate a similar concept, namely, relative nullity foliations of lightlike hypersurfaces of indefinite Kenmotsu manifolds. Many differences from the Riemannian case are due to the fact that the metric in consideration is degenerate. Further advancements in this topic are recent (see [3], for instance).

As is well known, contrary to timelike and spacelike hypersurfaces, the geometry of a lightlike hypersurface is different and rather difficult since the normal bundle and the tangent bundle have non-zero intersection. To overcome this difficulty, a theory on the differential geometry of lightlike hypersurfaces developed by Duggal and Bejancu [6] introduces a non-degenerate screen distribution and constructs the corresponding lightlike transversal vector bundle. This one enables to define an induced linear connection (depending on the screen distribution, and hence is not unique in general).

The paper is organized as follows. In Section 2, we recall some basic definitions and formulas for indefinite Kenmotsu manifolds and lightlike hypersurfaces of semi-Riemannian manifolds. In Section 3, for those lightlike hypersurfaces of indefinite Kenmotsu manifolds which are tangential to the structure vector field, we give the decomposition of almost contact metric, supported by an example. Theorems on parallel vector field are

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obtained. In Section 4, we study relative nullity distributions of lightlike hypersurfaces of indefinite Kenmotsu space forms. By Theorems 4.4, 4.5 and 4.9, we establish the characterization of the relative nullity distributions of an η-Einstein lightlike hypersurface and a parallel (or totally umbilical) lightlike hypersurface, tangent to the structure vector field, in an indefinite Kenmotsu space form. By Theorem 4.8, we show that lightlike hypersurfaces in indefinite Kenmotsu space forms are Einstein in the direction of a relative nullity space and the latter is an isotropic distribution under a certain condition. In the same theorem, we also show that, under a certain condition, an Einstein lightlike hypersurfaces have parallel screen distributions. We prove, under some conditions, that the geometry of the relative nullity distributions of lightlike hypersurfaces M is closely related with the geometry of M, the distributions $TM^{\perp}$ and $\overline{\phi}(TM^{\perp})$.

2. Preliminaries

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi},\xi,\eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -\mathbb{I} + \eta\otimes\xi,\ \ \eta(\xi)=1,\ \ \eta\circ\overline{\phi}=0 \text{ and } \overline{\phi}\xi = 0. \tag{2.1}$$

Then $(\overline{\phi},\xi,\eta,\overline{g})$ is called an almost contact metric structure on $\overline{M}$ if $(\overline{\phi},\xi,\eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that, for any vector field $\overline{X}$, $\overline{Y}$ on $\overline{M}$ [4]

$$\eta(\overline{X}) = \overline{g}(\xi,\overline{X}),\ \ \overline{g}(\overline{\phi}\,\overline{X},\overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X},\overline{Y}) - \eta(\overline{X})\,\eta(\overline{Y}). \tag{2.2}$$

If, moreover, $(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{\phi}\,\overline{X},\overline{Y})\xi - \eta(\overline{Y})\overline{\phi}\,\overline{X}$ and $\overline{\nabla}_{\overline{X}}\xi = \overline{X} - \eta(\overline{X})\xi$, where $\overline{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\overline{g}$, we call $\overline{M}$ an indefinite Kenmotsu manifold [9].

A plane section σ in $T_p\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by $\overline{X}$ and $\overline{\phi}\,\overline{X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to ξ. The sectional curvature of a $\overline{\phi}$-section σ is called a $\overline{\phi}$-sectional curvature. If a Kenmotsu manifold $\overline{M}$ has constant $\overline{\phi}$-sectional curvature c, then, by virtue of the Proposition 12 in [9], the curvature tensor $\overline{R}$ of $\overline{M}$ is given by

$$\begin{aligned} \overline{R}(\overline{X},\overline{Y})\overline{Z} &= \frac{c-3}{4}\left\{\overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y}\right\} + \frac{c+1}{4}\left\{\eta(\overline{X})\eta(\overline{Z})\overline{Y}\right. \\ &-\eta(\overline{Y})\eta(\overline{Z})\overline{X} + \overline{g}(\overline{X},\overline{Z})\eta(\overline{Y})\xi - \overline{g}(\overline{Y},\overline{Z})\eta(\overline{X})\xi + \overline{g}(\overline{\phi}\,\overline{Y},\overline{Z})\overline{\phi}\,\overline{X} \\ &\left. - \overline{g}(\overline{\phi}\,\overline{X},\overline{Z})\overline{\phi}\,\overline{Y} - 2\overline{g}(\overline{\phi}\,\overline{X},\overline{Y})\overline{\phi}\,\overline{Z}\right\},\ \ \overline{X},\ \overline{Y},\ \overline{Z}\in\Gamma(T\overline{M}). \end{aligned} \tag{2.3}$$

A Kenmotsu manifold $\overline{M}$ of constant $\overline{\phi}$-sectional curvature c will be called *Kenmotsu space form* and denoted by $\overline{M}(c)$.

Let $(\overline{M},\overline{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index s, $0 < s < 2n+1$ and let (M,g) be a hypersurface of $\overline{M}$, with $g = \overline{g}_{|M}$. M is a lightlike hypersurface of $\overline{M}$ if g is of constant rank $2n-1$ and the normal bundle $TM^{\perp}$ is a distribution of rank 1 on M [6]. A complementary bundle of $TM^{\perp}$ in TM is a rank $2n-1$ non-degenerate distribution over M. It is called a *screen distribution* and is often denoted by $S(TM)$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^{\perp}$ lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface.

Theorem 2.1 **[6]** *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$. Then, there exists a unique vector bundle $N(TM)$ of rank 1 over M such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exist a unique section N of $N(TM)$ on $\mathcal{U}$ satisfying*

$$\overline{g}(N, E) = 1 \text{ and } \overline{g}(N, N) = \overline{g}(N, W) = 0, \ \forall W \in \Gamma(S(TM)|_{\mathcal{U}}). \tag{2.4}$$

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote by Γ(E) the smooth sections of the vector bundle E. Also by $\perp$ and $\oplus$ we denote the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 2.1 we may write down the following decompositions:

$$\begin{aligned} TM &= S(TM) \perp TM^{\perp}, \\ T\overline{M} &= TM \oplus N(TM) = S(TM) \perp (TM^{\perp} \oplus N(TM)). \end{aligned} \tag{2.5}$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $(\overline{M}, \overline{g})$. Then by using the second decomposition of (2.5), we have Gauss and Weingarten formulae in the form

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \text{ and } \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \nabla X, Y \in \Gamma(TM_{|\mathcal{U}}), \, V \in \Gamma(N(TM)), \tag{2.6}$$

where $\nabla_X Y$, $A_V X \in \Gamma(TM)$. and $h(X, Y), \nabla^{\perp}_X V \in \Gamma(N(TM)).\nabla$ is an induced a symmetric linear connection on M, $\nabla^{\perp}$ is a linear connection on the vector bundle $N(TM)$, h is a $\Gamma(N(TM))$-valued symmetric bilinear form and A_V is the shape operator of M concerning V.

Equivalently, consider a normalizing pair $\{E, N\}$ as in Theorem 2.1. Then (2.6) takes the form, for any X, $Y \in \Gamma(TM|_{\mathcal{U}})$,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)\, N \text{ and } \overline{\nabla}_X N = -A_N X + \tau(X) N, \tag{2.7}$$

where B, A_N, τ and ∇ are called the local second fundamental form, the local shape operator, the transversal differential 1-form and the induced linear torsion free connection, respectively, on $TM_{|U}$.

It is important to mention that the second fundamental form B is independent of the choice of screen distribution. From (2.7), we obtain

$$B(X, Y) = \overline{g}(\overline{\nabla}_X Y, E), \ \ \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{2.8}$$

$$\tau(X) = \overline{g}(\nabla^{\perp}_X N, E), \ \ \forall X \in \Gamma(TM|_{\mathcal{U}}). \tag{2.9}$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition of TM. We have, for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$,

$$\nabla_X PY = \nabla^*_X PY + C(X, PY) E \text{ and } \nabla_X E = -A^*_E X - \tau(X) E, \tag{2.10}$$

where $\nabla^*_X PY$ and $A^*_E X$ belong to $\Gamma(S(TM))$. C, A^*_E and ∇^* are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$. The induced linear connection ∇ is not a metric connection and we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\theta(Z) + B(X,Z)\theta(Y), \forall X,\ Y \in \Gamma(TM|_{\mathcal{U}}), \tag{2.11}$$

where θ is a differential 1-form locally defined on M by $\theta(\cdot) := \overline{g}(N,\cdot)$.

Also, we have, $g(A^*_E X, PY) = B(X,PY)$, $g(A^*_E X, N) = 0$, $B(X,E) = 0$.

Finally, using (2.7), the curvature tensor fields $\overline{R}$ and R of $\overline{M}$ and M are related as

$$\begin{aligned} &\overline{R}(X,Y)Z = R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\} N, \\ &\text{where } (\nabla_X B)(Y,Z) = X.B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \end{aligned} \tag{2.12}$$

3. Lightlike hypersurfaces of indefinite Kenmotsu manifolds

Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite Kenmotsu manifold and (M,g) be a lightlike hypersurface of $(\overline{M},\overline{g})$, *tangent to the structure vector field* ξ *(*$\xi \in TM$*).* If E is a local section of $TM^\perp$, then $\overline{g}(\overline{\phi}E, E) = 0$, and $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^\perp)$ is a distribution on M of rank 1 such that $\overline{\phi}(TM^\perp) \cap TM^\perp = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^\perp)$ as a vector subbundle. If we consider a local section N of $N(TM)$, since $\overline{g}(\overline{\phi}\,N, E) = -\overline{g}(N, \overline{\phi}\,E) = 0$, we deduce that $\overline{\phi}\,E$ is also tangent to M and belongs to $S(TM)$. On the other hand, since $\overline{g}(\overline{\phi}\,N, N) = 0$, we see that the component of $\overline{\phi}\,N$ with respect to E vanishes. Thus $\overline{\phi}\,N \in \Gamma(S(TM))$. From (2.1), we have $\overline{g}(\overline{\phi}\,N, \overline{\phi}E) = 1$. Therefore, $\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))$ (direct sum but not orthogonal) is a nondegenerate vector subbundle of $S(TM)$ of rank 2. If M is tangent to the structure vector field ξ, then, we may choose $S(TM)$ so that ξ *belongs to* $S(TM)$. Using this, and since $\overline{g}(\overline{\phi}E, \xi) = \overline{g}(\overline{\phi}N, \xi) = 0$, there exists a nondegenerate distribution D_0 of rank $2n-4$ on M such that

$$S(TM) = \left\{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp <\xi>, \tag{3.1}$$

where $\langle\xi\rangle$ is the distribution spanned by ξ, that is, $\langle\xi\rangle = Span\{\xi\}$. It is easy to check that the distribution D_0 is invariant under $\overline{\phi}$, i.e. $\overline{\phi}(D_0) = D_0$.

Example 3.1 We consider the 7-dimensional manifold $\overline{M}^7 = \left\{(x_1, ..., x_7) \in \mathbb{R}^7 : x_7 > 0\right\}$, where $x = (x_1, x_2, ..., x_7$ are the standard coordinates in $\mathbb{R}^7$. The vector fields

$$\begin{aligned} &e_1 = x_7\frac{\partial}{\partial x_1},\ \ e_2 = x_7\frac{\partial}{\partial x_2},\ \ e_3 = x_7\frac{\partial}{\partial x_3},\ \ e_4 = x_7\frac{\partial}{\partial x_4},\ \ e_5 = -x_7\frac{\partial}{\partial x_5}, \\ &e_6 = -x_7\frac{\partial}{\partial x_6},\ \ e_7 = -x_7\frac{\partial}{\partial x_7} \end{aligned}$$

are linearly independent at each point of $\overline{M}$. Let $\overline{g}$ be the semi-Riemannian metric defined by $\overline{g}(e_i, e_j) = 0$, $\forall i \neq j$, $i, j = 1, 2, ..., 7$ and $\overline{g}(e_k, e_k) = 1$, $\forall k = 1, 2, 3, 4, 7$, $\overline{g}(e_m, e_m) = -1$, $\forall m = 5, 6$. Let η be the 1-form defined by $\eta(\overline{X}) = \overline{g}(\overline{X}, e_7)$, for any $\overline{X} \in \Gamma(T\overline{M})$.

Let $\overline{\phi}$ be the $(1, 1)$ tensor field defined by $\overline{\phi}e_1 = -e_2$, $\overline{\phi}e_2 = e_1$, $\overline{\phi}e_3 = -e_4$, $\overline{\phi}e_4 = e_3$, $\overline{\phi}e_5 = -e_6$, $\overline{\phi}e_6 = e_5$, $\overline{\phi}e_7 = 0$. Then using the linearity of $\overline{\phi}$ and $\overline{g}$, we have $\eta(e_7) = 1$, $\overline{\phi}^2\overline{X} = -\overline{X} + \eta(\overline{X})e_7$, $\overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y})$, for any $\overline{X}$, $\overline{Y} \in \Gamma(T\overline{M})$. Thus, for $e_7 = \xi$, $(\overline{\phi}, \xi, \eta, \overline{g})$ defines an almost contact metric structure on $\overline{M}$. Let $\overline{\nabla}$ be the Levi-Civita connection with respect to the metric $\overline{g}$. Then, we have $[e_i, e_7] = e_i$, $\forall i = 1, 2, ..., 6$ and $[e_i, e_j] = 0$, $\forall i \neq j$, $i, j = 1, 2, ..., 6$. The metric connection $\overline{\nabla}$ of the metric $\overline{g}$ is given by

$$\begin{aligned} 2\overline{g}(\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}) &= \overline{X}.\overline{g}(\overline{Y}, \overline{Z}) + \overline{Y}.\overline{g}(\overline{Z}, \overline{X}) - \overline{Z}.\overline{g}(\overline{X}, \overline{Y}) - \overline{g}(\overline{X}, [\overline{Y}, \overline{Z}]) \\ &\quad -\overline{g}(\overline{Y}, [\overline{X}, \overline{Z}]) + \overline{g}(\overline{Z}, [\overline{X}, \overline{Y}]), \end{aligned}$$

which is known as Koszul's formula. Using this formula, the non-vanishing covariant derivatives are given by $\overline{\nabla}_{e_1}e_1 = -e_7$, $\overline{\nabla}_{e_2}e_2 = -e_7$, $\overline{\nabla}_{e_3}e_3 = -e_7$, $\overline{\nabla}_{e_4}e_4 = -e_7$, $\overline{\nabla}_{e_5}e_5 = e_7$, $\overline{\nabla}_{e_6}e_6 = e_7$, $\overline{\nabla}_{e_1}e_7 = e_1$, $\overline{\nabla}_{e_2}e_7 = e_2$, $\overline{\nabla}_{e_3}e_7 = e_3$, $\overline{\nabla}_{e_4}e_7 = e_4$, $\overline{\nabla}_{e_5}e_7 = e_5$, $\overline{\nabla}_{e_6}e_7 = e_6$. From these relations, it follows that the manifold $\overline{M}$ satisfies $\overline{\nabla}_{\overline{X}}\xi = \overline{X} - \eta(\overline{X})\xi$. Hence, $\overline{M}^7$ is indefinite Kenmotsu manifold. We now define a hypersurface M of $(\overline{M}^7, \overline{\phi}, \xi, \eta, \overline{g})$ as $M = \left\{x \in \overline{M}^7 : x_5 = x_2\right\}$. Thus, the tangent space TM is spanned by $\{U_i\}_{1 \leq i \leq 6}$, where $U_1 = e_1$, $U_2 = e_2 - e_5$, $U_3 = e_3$, $U_4 = e_4$, $U_5 = e_6$, $U_6 = \xi$ and the 1-dimensional distribution $TM^{\perp}$ of rank 1 is spanned by E, where $E = e_2 - e_5$. It follows that $TM^{\perp} \subset TM$. Then M is a 6-dimensional lightlike hypersurface of $\overline{M}^7$. Also, the transversal bundle $N(TM)$ is spanned by $N = \frac{1}{2}(e_2 + e_5)$. On the other hand, by using the almost contact structure of $\overline{M}^7$ and also by taking into account the decomposition (3.1), the distribution D_0 is spanned by $\{F, \overline{\phi}F\}$, where $F = U_3$, $\overline{\phi}F = -U_4$ and the distributions $\langle\xi\rangle$, $\overline{\phi}(TM^{\perp})$ and $\overline{\phi}(N(TM))$ are spanned, respectively, by ξ, $\overline{\phi}E = U_1 + U_5$ and $\overline{\phi}N = \frac{1}{2}(U_1 - U_5)$. Hence, M is a lightlike hypersurface of $\overline{M}^7$.

Moreover, from (2.5) and (3.1) we obtain the decompositions

$$TM = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp <\xi> \perp TM^{\perp}, \tag{3.2}$$

$$T\overline{M} = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp <\xi> \perp (TM^{\perp} \oplus N(TM)). \tag{3.3}$$

Now, we consider the distributions on M, $D := TM^{\perp} \perp \overline{\phi}(TM^{\perp}) \perp D_0$, $D' := \overline{\phi}(N(TM))$. Then D is invariant under $\overline{\phi}$ and

$$TM = D \oplus D' \perp \langle\xi\rangle. \tag{3.4}$$

Let us consider the local lightlike vector fields $U := -\overline{\phi}N$, $V := -\overline{\phi}E$. Then, from (3.4), any $X \in \Gamma(TM)$ is written as $X = RX + QX + \eta(X)\xi$, $QX = u(X)U$, where R and Q are the projection morphisms of TM into D and D', respectively, and u is a differential 1-form locally defined on M by $u(\cdot) := g(V, \cdot)$. Applying

$\overline{\phi}$ to X and (2.1), one obtains $\overline{\phi}\,X = \phi\,X + u(X)\,N$, where ϕ is a tensor field of type $(1,1)$ defined on M by $\phi\,X := \overline{\phi}\,RX$. Also, we obtain, for any $X \in \Gamma(TM)$,

$$B(X,\xi) = 0, \tag{3.5}$$

$$\phi^2\,X = -\,X + \eta(X)\xi + u(X)\,U, \tag{3.6}$$

$$\text{and} \quad \nabla_X\xi = X - \eta(X)\xi. \tag{3.7}$$

For the sake of future use, we have the following identities: for any X, $Y \in \Gamma(TM)$,

$$C(X,\xi) = \theta(X), \tag{3.8}$$

$$B(X,U) = C(X,V) \tag{3.9}$$

$$(\nabla_X u)Y = -B(X,\phi Y) - u(Y)\tau(X) - \eta(Y)u(X), \tag{3.10}$$

$$(\nabla_X\phi)Y = \overline{g}(\overline{\phi}X,Y)\xi - \eta(Y)\phi X - B(X,Y)U + u(Y)A_N X. \tag{3.11}$$

Proposition 3.2 *Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$. The Lie derivative of g with respect to the vector field V is given by, for any X, $Y \in \Gamma(TM)$,*

$$(L_V g)(X,Y) = X.u(Y) + Y.u(X) + u([X,Y]) - 2u(\nabla_X Y). \tag{3.12}$$

Proof. The proof follows by direct calculation. □

The relation (3.12) can be written in terms of B using the following relation

$$u(\nabla_X Y) = B(X,\phi Y) + u(X)\eta(Y), \;\; \forall X,\; Y \in \Gamma(TM). \tag{3.13}$$

As the geometry of a lightlike hypersurface depends on the chosen screen distribution, it is important to investigate the relationship between geometrical objects induced by two screen distributions.

We ask the following question: Is the Lie derivative L_V (3.12) independent of the choice of a screen distribution $S(TM)$? The answer is negative. Indeed, we prove the following with respect to a change in $S(TM)$. Suppose a screen $S(TM)$ changes to another screen $S(TM)'$. Following are the local transformation equations due to this change (see details in [6], p. 87).

$$\begin{aligned} W_i' &= \sum_{j=1}^{2n-1} W_i^j (W_j - \epsilon_j c_j E), \\ N' &= N - \frac{1}{2}\{\sum_{i=1}^{2n-1} \epsilon_i (c_i)^2\}E + W, \\ \tau'(X) &= \tau(X) + B(X,W), \\ \nabla'_X Y &= \nabla_X Y + B(X,Y)\{\frac{1}{2}(\sum_{i=1}^{2n-1} \epsilon_i (c_i)^2)E - W\}, \end{aligned} \tag{3.14}$$

where $W = \sum_{i=1}^{2n-1} c_i W_i$, $\{W_i\}$ and $\{W_i'\}$ are the local orthonormal bases of $S(TM)$ and $S(TM)'$ with respective transversal sections N and N' for the same null section E. Here c_i and W_i^j are smooth functions on $\mathcal{U}$ and $\{\epsilon_1, ..., \epsilon_{2n-1}\}$ is the signature of the basis $\{W_1, ..., W_{2n-1}\}$. The Lie derivatives L_V and L'_V of the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related through the relation (see [10]):

$$(L'_V g)(X,Y) = (L_V g)(X,Y) - u(X)B(Y,W) - u(Y)B(X,W).$$

The Lie derivative L_V is unique, that is, L_V is independent of $S(TM)$, if and only if, the second fundamental form h (or equivalently B) of M vanishes identically on M.

If a $(2n+1)$-dimensional Kenmotsu manifold $\overline{M}$ has a constant $\overline{\phi}$-sectional curvature c, then the Ricci tensor $\overline{Ric}$ and the scalar curvature $\overline{r}$ are given by [9]

$$\overline{Ric} = \frac{1}{2}\left(n(c-3)+c+1\right)\overline{g} - \frac{1}{2}(n+1)(c+1)\eta\otimes\eta, \tag{3.15}$$

$$\overline{r} = \frac{1}{2}\left(n(2n+1)(c-3) - n(c+1)\right). \tag{3.16}$$

This means that $\overline{M}$ is η-Einstein. Since $\overline{M}$ is Kenmotsu space form and Einstein, by Corollary 9 in [9], $\overline{M}$ is an Einstein one and consequently, $c+1=0$, that is, $c=-1$. So, the Ricci tensor (3.15) becomes $\overline{Ric} = -2n\overline{g}$ and the scalar curvature is given by $\overline{r} = -2n(2n+1)$.

Thus, *if a Kenmotsu manifold $\overline{M}$ is a space form, then it is Einstein and $c=-1$.*

Let $\overline{M}(c)$ be an indefinite Kenmotsu space form and M be a lightlike hypersurface of $\overline{M}(c)$. Let us consider the pair $\{E, N\}$ on $\mathcal{U}\subset M$ (see Theorem 2.1) and by using (2.12), we obtain

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \tau(Y)B(X,Z) - \tau(X)B(Y,Z). \tag{3.17}$$

Theorem 3.3 *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}$, with $\xi\in TM$. Then, the Lie derivative of the local second fundamental form B with respect to ξ is given by*

$$(L_\xi B)(X,Y) = (1-\tau(\xi))B(X,Y),\ \forall\, X,\, Y\in\Gamma(TM). \tag{3.18}$$

Proof. Using (2.12), we obtain

$$(\nabla_\xi B)(X,Y) = (L_\xi B)(X,Y) - 2B(X,Y). \tag{3.19}$$

Likewise, using again (2.12), we have

$$(\nabla_X B)(\xi,Y) = -B(X,Y). \tag{3.20}$$

Subtracting (3.19) and (3.20), we obtain

$$(\nabla_\xi B)(X,Y) - (\nabla_X B)(\xi,Y) = (L_\xi B)(X,Y) - B(X,Y). \tag{3.21}$$

From (3.17) and after calculations, the left hand side of (3.21) becomes

$$(\nabla_\xi B)(X,Y) - (\nabla_X B)(\xi,Y) = -\tau(\xi)B(X,Y). \tag{3.22}$$

The expressions (3.21) and (3.22) implies $(L_\xi B)(X,Y) = (1-\tau(\xi))B(X,Y)$. □

Next, we give characterization on parallel lightlike hypersuface of an indefinite Kenmotsu manifold. In fact, it shows that there do not exist non-totally geodesic totally umbilical lightlike hypersurfaces of indefinite Kenmotsu manifolds, tangent to the structure vector field ξ.

The second fundamental form h of M is said to be parallel if $(\nabla_X h)(Y, Z) = 0$, $\forall X, Y, Z \in \Gamma(TM)$. That is,

$$(\nabla_X B)(Y, Z) = -\tau(X)B(Y, Z). \tag{3.23}$$

In [13], Sahin characterizes lightlike hypersurfaces with parallel second fundamental form in Lorentzian manifold. He showed that there do not exist non-totally geodesic parallel lightlike hypersurfaces in a Lorentzian manifold.

Theorem 3.4 *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in TM$. If the second fundamental form h of M is parallel, then M is totally geodesic.*

Proof. Suppose that the second fundamental form h of M is parallel. Then (3.23) is satisfied. Using (3.23), we obtain

$$(\nabla_\xi B)(X, Y) = -\tau(\xi)B(X, Y). \tag{3.24}$$

From (2.12) and using (3.18), the left hand side of (3.24) becomes

$$(\nabla_\xi B)(X, Y) = (L_\xi B)(X, Y) - 2B(X, Y) = -(1 + \tau(\xi))B(X, Y). \tag{3.25}$$

From expressions (3.24) and (3.25) we complete the proof. □

The covariant derivative of the second fundamental form h depends on ∇, N and τ which depend on the choice of the screen vector bundle. The covariant derivatives ∇ of $h = B \otimes N$ and ∇' of $h' = B \otimes N'$ in the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related as follows: for any X, Y, $Z \in \Gamma(TM)$,

$$\overline{g}((\nabla'_X h')(Y, Z), E) = \overline{g}((\nabla_X h)(Y, Z), E) + \mathcal{L}_{(X,Y)}Z,$$

with $\mathcal{L}_{(X,Y)}Z = B(X, Y)B(Z, W) + B(X, Z)B(Y, W) + B(Y, Z)B(X, W)$.

It is easy to check that the parallelism of h is independent of the screen distribution $S(TM)$ ($\nabla' h' \equiv \nabla h$) if and only the second fundamental form B of M vanishes identically on M.

We note that the Theorem 3.4 arises when the local second fundamental form B of M is also parallel. So, the Theorem 3.4 generates some lightlike geometric aspects on any parallel lightlike hypersurface of an indefinite Kenmotsu manifold by using the Theorem 2.2 in ([6], p. 88).

Note that the 1-form τ in (2.9) depends on the vector field E and it is easy to see that if $\overline{E} = \lambda E$ with λ a positive smooth function on M, the associated 1-form $\overline{\tau}$ is related to τ by

$$\tau(X) = \overline{\tau}(X) + X(\ln \lambda), \quad \forall X \in \Gamma(TM|_{\mathcal{U}}). \tag{3.26}$$

The induced connection ∇ on the lightlike hypersurface M is not metric in general and the Ricci tensor associated is not symmetric, contrary to the case of semi-Riemannian manifolds. However, for η-Einstein lightlike hypersurfaces, that is, the Ricci tensor Ric tensor satisfies $Ric(X, Y) = k_1 g(X, Y) + k_2\eta(X)\eta(Y)$, due to the symmetry of the induced degenerate metric g and $\eta \otimes \eta$, the Ricci tensor is symmetric, and the notion of η-Einstein manifold does not depend on the choice of the screen distribution $S(TM)$. Consequently

Proposition 3.5 *On a lightlike η-Einstein hypersurface of an indefinite Kenmotsu manifold, tangent to the structure vector field, the 1-form τ in (2.9) is closed.*

Proof. Define the Ricci tensor Ric as $Ric(X,Y) = \text{trace}(Z \longrightarrow R(Z,X)Y)$, for any X, $Y \in \Gamma(TM)$ where R is the curvature tensor of the induced connection ∇.

Consider a local quasi-orthogonal frame field $\{X_0, N, X_i\}_{i=1,\dots,2n-1}$ on $\overline{M}$ where $\{X_0, X_i\}$ is a local frame field on M with respect to the decomposition (3.3) with N, the unique section of transversal bundle $N(TM)$ satisfying (2.4), and $E = X_0$. Put $R_{ls} := Ric(X_s, X_l)$ and $R_{0k} := Ric(X_k, X_0)$. Using the frame field $\{X_0, N, X_i\}$, a direct calculation gives locally $R_{ls} - R_{sl} = 2d\tau(X_l, X_s)$ and $R_{0k} - R_{k0} = 2d\tau(X_0, X_k)$. Since the Ricci tensor is symmetric on M which is η-Einstein, we have $d\tau = 0$. □

From Proposition 3.5, τ is closed. Poincaré Lemma implies locally on $\mathcal{U}$, $\tau = df$ for some function $f \in \mathcal{F}(\mathcal{U})$, that is

$$\tau(X) = X.f. \tag{3.27}$$

Using (3.26), the relation (3.27), for $\lambda = \exp(f)$, yields $\tau(X) = \overline{\tau}(X) + X(\ln\lambda) = \overline{\tau}(X) + X.f = \overline{\tau}(X) + \tau(X)$, therefore $\overline{\tau}(X) = 0$, for any $X \in \Gamma(TM|_{\mathcal{U}})$. Then, by taking $\overline{E} = \exp(f)E$, one obtains $\overline{\tau} = 0$ on $\mathcal{U}$. The corresponding $\overline{N}$ is $\overline{N} = (1/\exp(f))N$. Therefore, we have the following proposition.

Proposition 3.6 *Let $(M, g, S(TM))$ be a lightlike η-Einstein hypersurface of an indefinite Kenmotsu manifold, with $\xi \in TM$. There exists on all coordinate neighbourhood $\mathcal{U}$, a pair $\{E, N\}$ such that the 1-form τ in (2.9) vanishes identically.*

This result is also true in case of any Einstein lightlike hypersurface of a semi-Riemannian manifold [3].

Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. By definition $Ric(X,Y) = \text{trace}(Z \longrightarrow R(Z,X)Y)$, for any X, $Y \in \Gamma(TM)$ and we have

$$\begin{aligned} Ric(X,Y) &= \sum_{i=1}^{2n-4} \varepsilon_i \overline{g}(R(F_i,X)Y,F_i) + \overline{g}(R(\xi,X)Y,\xi) + \overline{g}(R(E,X)Y,N) \\ &+ \overline{g}(R(\overline{\phi}E,X)Y,\overline{\phi}N) + \overline{g}(R(\overline{\phi}N,X)Y,\overline{\phi}E), \end{aligned} \tag{3.28}$$

where $\{F_i\}_{1\leq i\leq 2n-4}$ is an orthogonal basis of D_0 and $\varepsilon_i = g(F_i, F_i) \neq 0$, since the distribution D_0 is non-degenerate. Since $c = -1$, so from Gauss and Codazzi equations, we obtain

$$\begin{aligned}
\overline{g}(R(F_i, X)Y, F_i) &= \overline{g}(X, F_i)\overline{g}(Y, F_i) - \varepsilon_i\overline{g}(X, Y) + B(X, Y)C(F_i, F_i) \\
&\quad - B(F_i, Y)C(X, F_i), \qquad (3.29)\\
\overline{g}(R(\xi, X)Y, \xi) &= \eta(Y)\eta(X) - \overline{g}(X, Y) - B(\xi, Y)C(X, \xi), \qquad (3.30)\\
\overline{g}(R(E, X)Y, N) &= -\overline{g}(X, Y), \qquad (3.31)\\
\overline{g}(R(\overline{\phi}E, X)Y, \overline{\phi}N) &= u(Y)v(X) - \overline{g}(X, Y) + B(X, Y)C(\overline{\phi}E, \overline{\phi}N)\\
&\quad - B(\overline{\phi}E, Y)C(X, \overline{\phi}N), \qquad (3.32)\\
\overline{g}(R(\overline{\phi}N, X)Y, \overline{\phi}E) &= u(X)v(Y) - \overline{g}(X, Y) + B(X, Y)C(\overline{\phi}N, \overline{\phi}E)\\
&\quad - B(\overline{\phi}N, Y)C(X, \overline{\phi}E). \qquad (3.33)
\end{aligned}$$

So substituting (3.29), (3.30), (3.31), (3.32) and (3.33) in (3.28) and by regrouping like terms, we have the result.

Proposition 3.7 *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in TM$. Then the Ricci tensor Ric is given by, for any X, $Y \in \Gamma(TM)$,*

$$Ric(X, Y) = ag(X, Y) + B(X, Y)trA_N - B(A_N X, Y), \qquad (3.34)$$

where $a = -(2n-1)$ and trace, tr, is written with respect to g restricted to $S(TM)$.

Note that the Ricci tensor does not depend on the choice of the vector field E of the distribution $TM^{\perp}$. From (3.34), we have

$$Ric(X, Y) - Ric(Y, X) = B(A_N X, Y) - B(A_N Y, X). \qquad (3.35)$$

This means that the Ricci tensor of a lightlike hypersurface M of an indefinite Kenmotsu space form $\overline{M}(c)$ is not symmetric in general. So, only some privileged conditions on the local second fundamental form of M may enable the Ricci tensor to be symmetric. It is easy to check that the Ricci tensor (3.34) of M is symmetric if and only if the shape operator of M is symmetric with respect to the second fundamental form B of M (see [7] for details). Also by Theorem 3.4, the Ricci tensor of the induced connection ∇ of any parallel lightlike hypersurface, which becomes totally geodesic and consequently Einstein lightlike hypersurface, is symmetric.

Are there any others, with symmetric induced Ricci tensors, but not necessarily totally geodesic or shape operator symmetric with respect to the second fundamental form? Here is one such class.

First, we recall the definition of screen conformal lightlike hypersurfaces of a semi-Riemannian manifold $\overline{M}$. A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold is screen locally conformal if the shape operator A_N and A^*_E of M and its screen distribution $S(TM)$, respectively, are related by [2]

$$A_N = \varphi\, A^*_E, \qquad (3.36)$$

where φ is a non-vanishing smooth function on $\mathcal{U}$ in M. In case $\mathcal{U} = M$ the screen conformality is said to be global. Such a submanifold has some important and desirable properties, for instance, the integrability of its screen distribution (see [2] for details).

We have the following result proved in [2].

Theorem 3.8 *Let $(M, g, S(TM))$ be a locally (or globally) screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in TM$. Then the Ricci tensor of the induced connection ∇ is symmetric.*

A submanifold M is said to be totally contact umbilical lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$ if the second fundamental form h of M satisfies [12]:

$$h(X,Y) \quad = \quad \{g(X,Y) - \eta(X)\eta(Y)\}\, H + \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi), \tag{3.37}$$

for any X, $Y \in \Gamma(TM)$, where H is a normal vector field on M (that is $H = \lambda N$, λ is a smooth function on $\mathcal{U} \subset M$). In case when M is a lightlike hypersurface of a Kenmotsu manifold $\overline{M}$, it becomes η-totally umbilical, that is, $h(X,Y) = \lambda\,\{g(X,Y) - \eta(X)\eta(Y)\}\, N$, since $h(\cdot,\,\xi) = B(\cdot,\,\xi)N = 0$. We have

Theorem 3.9 *Let $(M, g, S(TM))$ be a locally (or globally) screen conformal lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in TM$. If M is totally contact umbilical lightlike, then M is η-Einstein.*

Proof. The proof follows directly from (3.34), (3.36) and (3.37). □

Theorem 3.9 gives the existence of an η-Einstein lightlike hypersurface in indefinite Kenmotsu space form, tangent to the structure vector field ξ.

4. Relative nullity distributions of lightlike hypersurfaces of indefinite Kenmotsu space forms

Let M be a lightlike hypersurface of indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. The relative nullity space at a point x is defined by

$$T^{*0}(x) = \{X \in T_xM : A^*_E X = 0, \;\; \forall\, E \in T_xM^{\perp}\}. \tag{4.1}$$

The relative nullity space is characterized as

$$T^{*0}(x) = \{X \in T_xM : h(X, PY) = 0, \;\; \forall\, Y \in T_xM\}. \tag{4.2}$$

The dimension $\nu(x)$ of $T^{*0}(x)$ is called the index of relative nullity at x. The value $\nu_0 = \min_{x \in M} \nu(x)$ is called the index of minimum relative nullity [5].

Writing A^*_E as, for any $X \in \Gamma(TM)$,

$$A^*_E X = \sum_{i=1}^{2n-4} \frac{B(X, F_i)}{g(F_i, F_i)} F_i + B(X, V)U + B(X, U)V, \tag{4.3}$$

with $g(F_i, F_i) \neq 0$ and using $B(., \xi) = 0$, it is easy to check that $A^*_E\xi = A^*_E E = 0$. Therefore, $\dim T^{*0}(x) \geq 2, \ \forall x \in M$. Moreover

$$T_xM^{\perp} \perp \langle\xi\rangle_x \subset T^{*0}(x). \tag{4.4}$$

Hence, $T^{*0}(x)$ is a degenerate distribution along M and $\nu_0 = 2$.

Now, say that the screen distribution $S(TM)$ is totally umbilical if on any coordinates neighborhood $\mathcal{U} \subset M$, there exists a smooth function φ such that [6]

$$C(X, PY) = \varphi g(X, PY), \forall X, Y \in \Gamma(TM_{|\mathcal{U}}). \tag{4.5}$$

If we assume that the screen distribution $S(TM)$ of the lightlike hypersurface M with $\xi \in TM$ is totally umbilical, then it follows that C is symmetric on $\Gamma(S(TM)_{|\mathcal{U}})$ and hence according to Theorem 2.3 in [6], the distribution $S(TM)$ is integrable. Also, we have $A_N X = \varphi PX$ and $C(E, PX) = 0$. Since $\overline{\phi}\xi = 0$ and by using $\eta(A_N X) = -\theta(X)$, we have $\eta(A_N\xi) = \varphi\overline{g}(\xi, \xi) = -\theta(\xi) = 0$ which implies that $\varphi = 0$, so the screen distribution $S(TM)$ is totally geodesic. This is equivalent, by using the Proposition 2.7 in [6] page 89, to the parallelism of $S(TM)$ and the vanishing of the shape operator A_N. Therefore, we have the following result.

Lemma 4.1 *Let $(M, g, S(TM)$ be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$, with $\xi \in TM$ such that $S(TM)$ is totally umbilical. Then*

$$\dim T^{*0}(x) \geq 3, \ \ \forall x \in M.$$

Proof. Taking $X = U$ in (4.3), using $B(., \xi) = 0$ and (3.9), one obtains

$$A^*_E U = \sum_{i=1}^{2n-4} \frac{C(F_i, V)}{g(F_i, F_i)} F_i + C(V, V)U + C(U, V)V. \tag{4.6}$$

If $S(TM)$ is totally umbilical, then $A^*_E U = 0$, so $U \in T^{*0}(x)$. With the aid of (4.4), one obtains

$$T_xM^{\perp} \perp \langle\xi\rangle_x \perp D'_x \subset T^{*0}(x), \tag{4.7}$$

which completes the proof. □

The orthogonal complement $(T^{*0}(x))^{\perp}$ of $T^{*0}(x)$ in T_xM is denoted by $T^{*1}(x)$.

Proposition 4.2

$$T^{*1}(x) = \text{span}\{A^*_E Y, \ \ Y \in T_xM, \ E \in T_xM^{\perp}\} \perp T_xM^{\perp}.$$

Proof. It is obvious to check that $T_xM^{\perp} \subset T^{*1}(x)$. Then, there exists a set $\Delta(x)$ such that $T^{*1}(x) = \Delta(x) \perp T_xM^{\perp}$. Now we want to show that $\Delta(x) = \text{span}\{A^*_E Y\}$. Given any $E \in T_xM^{\perp}$, $Y \in T_xM$ and $X \in T^{*0}(x)$,

$$g(X, A^*_E Y) = g(A^*_E X, Y) = 0,$$

so, $A_E^* Y \in \Delta(x)$. On the other hand, let $Z \in \text{span}\{A_E^* Y\}^{\perp_S}$ and $Y \in T_x M$, where $\perp_S$ denotes the orthogonality symbol in the screen distribution $S(TM)$. We have

$$0 = g(Z, A_E^* Y) = g(A_E^* Z, Y), \forall\, Y \in T_x M.$$

Then, $A_E^* Z \in S(TM) \cap T_x M^\perp = \{0\}$, that is, $A_E^* Z = 0$ and $Z \in T^{*0}(x)$.

Thus $\text{span}\{A_E^* Y\}^{\perp_S} \subset T^{*0}(x)$ and $T^{*1}(x) \subset \text{span}\{A_E^* Y\}$. Since $A_E^* Y \notin T_x M^\perp$, then $\Delta(x) \subset \text{span}\{A_E^* Y\}$ which completes the proof. □

A submanifold M is said to be η-Einstein if its induced Ricci tensor Ric tensor satisfies

$$Ric(X, Y) = k_1 g(X, Y) + k_2 \eta(X)\eta(Y), \tag{4.8}$$

where the nonzero functions k_1 and k_2 are not necessarily constant on M. If M is an η-Einstein lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^{2n+1} (n > 1)$ with $\xi \in TM$, the functions k_1 and k_2 satisfy

$$k_1 + k_2 = -(2n - 1). \tag{4.9}$$

Lemma 4.3 *Let M be an η-Einstein lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$ such that the Ricci tensor Ric of M is parallel. The nonzero functions k_1 and k_2 satisfy*

$$dk_i = 0, \ \ \forall\, i = 1, 2. \tag{4.10}$$

Moreover, if we choose, at each point $p \in M$, a connected open set G such that $T_p G = T_p M$, then, the nonzero functions k_1 and k_2 are constants on M.

Proof. Let M be lightlike η-Einstein hypersurface. Then, the induced Ricci tensor Ric tensor satisfies $Ric(X, Y) = k_1 g(X, Y) + k_2 \eta(X)\eta(Y)$. Using (2.11), the covariant derivative of the induced Ricci tensor Ric gives

$$\begin{aligned}
&(\nabla_X Ric)(Y, Z) = (X.k_1) g(Y, Z) + k_1 (\nabla_X g)(Y, Z) + (X.k_2)\eta(Y)\eta(Z) \\
&+ k_2 \eta(Z)(\nabla_X \eta)Y + k_2 \eta(Y)(\nabla_X \eta)Z \\
&= (X.k_1)\{g(Y, Z) - \eta(Y)\eta(Z)\} + k_1 \{B(X, Y)\theta(Z) + B(X, Z)\theta(Y)\} \\
&+ k_2 \eta(Z)\{g(X, Y) - \eta(X)\eta(Y)\} + k_2 \eta(Y)\{g(X, Z - \eta(X)\eta(Z)\}.
\end{aligned} \tag{4.11}$$

Taking $Y = V$ and $Z = U$ into (4.11), one obtains

$$(\nabla_X Ric)(V, U) = X.k_1 = dk_1(X). \tag{4.12}$$

On the other hand, the Ricci tensor is parallel, that is, $(\nabla_X Ric)(V, U) = 0$ which completes (4.10). The last assertion is obvious. □

Note that a hypersurface of a 3-dimensional indefinite Kenmotsu manifold, tangent to the structure vector field ξ is of dimension 1 and its tangent space is reduced to the distribution spanned by ξ which is *nondegenerate*. This means that the dimension 3 is too low to develop the theory and this agrees with the decomposition (3.3) which requires $2n-4\geq 0$, that is $n\geq 2$.

By virtue of Theorem 3.4, a parallel lightlike hypersurface M of an indefinite Kenmotsu space form $\overline{M}^{2n+1}(c)\,(n>1)$ with $\xi\in TM$ is also an Einstein one.

Theorem 4.4 *Let $(M,g,S(TM))$ be an η-Einstein lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}^{2n+1}(c)\,(n>1)$, with $\xi\in TM$. Let G be the set of points in M where $\nu(x)=\nu_0$. Then,*

1. *The relative nullity distribution T^{*0} of the screen is smooth on G,*
2. *G is an open set in M,*
3. *If $k_1\neq a$, then, on G, we have*
   - (a) *The foliation T^{*0} contains an isotropic subspace along M which does not contain the distribution $\langle\xi\rangle$,*
   - (b) *$T^{*0}=TM^{\perp}\perp\langle\xi\rangle$,*
   - (c) *The relative nullity distribution T^{*0} is integrable and the leaves are totally geodesic in M and $\overline{M}$,*
   - (d) *Moreover, if $S(TM))$ is totally umbilical, then $T^{*0}=TM^{\perp}\perp\langle\xi\rangle\perp D'$.*

Proof. (1) Let x_0 be an element of G. From (4.4), we have

$$T^{*0}(x_0)=P(T^{*0}(x_0))\perp T_{x_0}M^{\perp}\perp\langle\xi\rangle_{x_0}. \tag{4.13}$$

Let $\perp_S$ denotes the orthogonality symbol in the screen distribution $S(TM)$. For $Y\in T_{x_0}M$, $E\in T_{x_0}M^{\perp}$ and $X\in P(T^{*0}(x_0))$, we have $g(A^*_EY,X)=g(Y,A^*_EX)=0$, so,

$$\text{span}\{A^*_EY\}\subset P(T^{*0}(x_0))^{\perp_S}.$$

Let $Z\in\text{span}\{A^*_EY\}^{\perp_S}$ and $Y\in T_{x_0}M$. We have $0=g(Z,A^*_EY)=g(A^*_EZ,Y)$, $\forall\ Y\in T_xM$. Then $A^*_EZ\in S(TM)\cap T_{x_0}M^{\perp}=\{0\}$, that is, $A^*_EZ=0$ and $Z\in P(T^{*0}(x_0))$. Thus

$$\text{span}\{A^*_EY\}^{\perp_S}\subset P(T^{*0}(x_0))\ \text{ and }\ P(T^{*0}(x_0))^{\perp_S}\subset\text{span}\{A^*_EY\}.$$

Consequently $P(T^{*0}(x_0))^{\perp_S}=\text{span}\{A^*_EY\}$ and $T^{*1}(x_0)=\text{span}\{A^*_EY\}\perp T_{x_0}M^{\perp}$. There exist vector fields $Y_1,...,Y_{2n-\nu+1}\in T_{x_0}M$ such that $\{E(x_0),A^*_{E(x_0)}Y_1,...,A^*_{E(x_0)}Y_{2n-\nu+1}\}$, represent a basis of $T^{*1}(x)$.

Take smooth local extensions of $E(x_0)$ and $Y_1,...,Y_{2n-\nu+1}\in T_{x_0}M$ in $TM^{\perp}$ and TM respectively. By continuity, the vector fields $\{E(x_0),Y_1,...,Y_{2n-\nu+1}\}$ remain linearly independent in a neighborhood $\mathcal{V}\subset G$ of x_0 and then T^{*1} is a smooth distribution. Consequently, T^{*0} is smooth distribution.

(2) follows immediately from the arguments developed in (1).

(3) Suppose that $k_1\neq a$. (a) From (3.34) and the fact that M is η-Einstein (4.8), we have, for any X, $Y\in\Gamma(TM)$,

$$(a-k_1)g(X,Y)-k_2\eta(X)\eta(Y)+g(A^*_EX,Y)trA_N-g(A_NX,A^*_EY)=0, \tag{4.14}$$

But $X, Y \in T^{*0}(x) - \langle\xi\rangle_x$ implies $A^*_E X = A^*_E Y = 0$ and since $a - k_1 \neq 0$, (4.14) is reduced to $g(X,Y) = 0$, $\forall\, X, Y \in T^{*0}(x) - \langle\xi\rangle_x$. That is, the distribution $T^{*0} - \langle\xi\rangle \subset T^{*0}$ is isotropic along M.

(b) Take $X \in T^{*0}(x) - \langle\xi\rangle_x$ and $Y \in \Gamma(T_x M)$ with $x \in G$, using (3.34) and the fact that M is η-Einstein, again we have

$$(a - k_1)g(X,Y) - k_2\eta(X)\eta(Y) + B(X,Y)trA_N - B(A_N X, Y) = 0, \tag{4.15}$$

Since M is η-Einstein, so the induced Ricci tensor is symmetric and from the relation (3.34) the shape operator of M is symmetric with respect to the second fundamental form B of M. That is $B(A_N X, Y) = B(X, A_N Y)$. Therefore, relation (4.15) becomes

$$(a - k_1)g(X,Y) - k_2\eta(X)\eta(Y) + g(A^*_E X, Y)trA_N - g(A^*_E X, A_N Y) = 0.$$

Since $X \in T^{*0}(x) - \langle\xi\rangle_x$, we have

$$(a - k_1)g(X,Y) = 0, \text{ that is } g(X,Y) = 0, \text{ for } A^*_E X = 0 \text{ and } k_1 \neq a.$$

So $X \in T^{*0}(x) - \langle\xi\rangle_x$ implies $g(X,Y) = 0$, $\forall\, Y \in T_x M$, so $X \in T_x M^\perp$ and we deduce that $T^{*0}(x) - \langle\xi\rangle_x \subset T_x M^\perp$. Therefore $T^{*0}(x) \subset T_x M^\perp \perp \langle\xi\rangle_x$. From (4.4), we conclude that

$$T^{*0}(x) = T_x M^\perp \perp \langle\xi\rangle_x.$$

(c) From Gauss and Codazzi equations, for all $E \in \Gamma(TM^\perp)$ and X, Y, $Z \in \Gamma(TM)$, we have

$$\overline{g}(\overline{R}(X,Y)Z, E) = \overline{g}((\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), E). \tag{4.16}$$

Take $X \in \Gamma(TM)$ and Y, $Z \in T^{*0}(x)$, $x \in G$. Since $(\nabla_X h)(Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$, Then

$$\begin{aligned}&\overline{g}((\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), E) = X.B(Y,Z) - Y.B(X,Z) - \tau(X)B(Y,Z)\\ &+\tau(Y)B(X,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z) + B(\nabla_Y X, Z) + B(X, \nabla_Y Z)\end{aligned} \tag{4.17}$$

Using (2.3) and the fact that the indefinite Kenmotsu space form is of constant curvature $c = -1$, the left hand side of (4.16) vanishes and by Proposition 3.6, the relation (4.17) becomes

$$\begin{aligned}0 &= X.B(Y,Z) - Y.B(X,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z) + B(\nabla_Y X, Z)\\ &+ B(X, \nabla_Y Z).\end{aligned} \tag{4.18}$$

From Proposition 4.2, we have $B(Y,Z) = \overline{g}(h(Z, PY), E) + \theta(Y)B(Z,E) = 0$, for Y, $Z \in T^{*0}(x)$. Similarly, $B(X,Z) = 0$.

On the other hand, we have

$$\begin{aligned}B(\nabla_X Y, Z) &= B(\nabla_X PY, Z) + X.\theta(Y)B(Z,E) + \theta(Y)B(\nabla_X E, Z)\\ &= B(\nabla^*_X PY, Z) = 0, \text{ for } Z \in T^{*0}(x) \text{ and } \nabla^*_X PY \in \Gamma(S(TM)).\end{aligned}$$

Also $B(\nabla_Y X, Z) = 0$.

Relation (4.18) becomes $B(X, \nabla_Y Z) - B(Y, \nabla_X Z) = 0$. But

$$\begin{aligned} B(Y, \nabla_X Z) &= B(Y, \nabla_X PZ) + \theta(X) B(Y, \nabla_X E) \\ &= B(Y, \nabla^*_X PZ) + \theta(X) B(Y, -A^*_E X - \tau(X) E) = 0. \end{aligned}$$

Consequently $h(\nabla_Y Z, PX) = 0$, for any $X \in \Gamma(TM)$. From (4.2), we deduce that $\nabla_Y X \in T^{*0}(x)$. This implies that $T^{*0}(x)$ is involutive with totally geodesic leaves in both M and $\overline{M}$.

(d) Take $X \in T^{*0}(x) - (\langle\xi\rangle_x \perp D'_x))$ with $x \in G$, using (3.34) and the fact that M is η-Einstein and if $S(TM)$ is totally umbilical, the relation (4.14) is reduced to

$$(a - k_1) g(X, Y) - k_2 \eta(X) \eta(Y) = 0.$$

Since $X \in T^{*0}(x) - (\langle\xi\rangle_x \perp D'_x)$, that is $A^*_E X = 0$, we have $(a - k_1) g(X, Y) = 0$ which gives $g(X, Y) = 0$, for $k_1 \neq a$. So $X \in T^{*0}(x) - (\langle\xi\rangle_x \perp D'_x)$ implies $g(X, Y) = 0$, $\forall\ Y \in T_x M$ and we deduce that $T^{*0}(x) - (\langle\xi\rangle_x \perp D'_x) \subset T_x M^\perp$. Therefore $T^{*0}(x) \subset T_x M^\perp \perp \langle\xi\rangle_x \perp D'_x$. From (4.7), we conclude that $T^{*0}(x) = T_x M^\perp \perp \langle\xi\rangle_x \perp D'_x$ which completes the proof. □

Note that items (1) and (2) of Theorem 4.4 do not depend on the η-Einstein condition on the lightlike hypersurface M. Item (c) coincides with the one given in the main Theorem of Atindogbe et al in [3]. This is due to the fact that, in both cases, Einstein lightlike hypersurface of Lorentzian space $\mathbb{R}^{n+2}_1 (n \geq 3, c = 0)$ [3] and η-Einstein lightlike hypersurface of an indefinite Kenmotsu space form $(c = -1)$, the differential 1-form τ vanishes. This leads, in both cases, to the same equation (4.18).

Theorem 4.5 *Let $(M, g, S(TM))$ be an η-Einstein lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}^{2n+1}(c)\,(n > 1)$, with $\xi \in TM$ such that the Ricci tensor Ric of M is parallel. Let G be a connected open set where the index of relative nullity $\nu(x) = \nu_0$. Then, on G we have:*

(a) *The foliation T^{*0} contains an isotropic subspace along M which does not contain the distribution $\langle\xi\rangle$,*

(b) $T^{*0} = TM^\perp \perp \langle\xi\rangle$,

(c) *The relative nullity distribution T^{*0} is integrable and the leaves are totally geodesic in M and $\overline{M}$,*

(d) *If $S(TM))$ is totally umbilical, then on G, $T^{*0} = TM^\perp \perp \langle\xi\rangle \perp D'$.*

Proof. Since the open set G is connected, then by Lemma 4.3, the nonzero functions k_1 and k_2 are constants. As an indefinite Kenmotsu space form $\overline{M}^{2n+1}(c)\,(n > 1)$ is of constant curvature $c = -1$, then $k_1 + k_2 = a = -(2n - 1)$, that is, $a - k_1 = k_2 \neq 0$ by (4.8) and the rest follows from Theorem 4.4. □

Next, we prove, under some conditions, that the geometry of the relative nullity distributions of lightlike hypersurfaces M is closely related with the geometry of M, the distributions $TM^\perp$ and $\overline{\phi}(TM^\perp)$.

Lemma 4.6 *Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$. Then, M is $D \perp \langle\xi\rangle$-totally geodesic if and only if, for any $X \in \Gamma(D \perp \langle\xi\rangle)$,*

$$A_E^* X = u(A_N X)V. \tag{4.19}$$

Proof. The proof follows by direct calculation using (4.3). □

It is well known that if the lightlike hypersurface (M, g) is totally geodesic, the induced connection ∇ on M is torsion-free and g-metric. Also, the shape operator A_E^* vanishes identically on M (see Theorem 2.2. [6] p. 88). This vanishing property failed when the lightlike hypersurface M, with $\xi \in TM$, is $D \perp \langle\xi\rangle$-totally geodesic. That is, only some privileged conditions on the screen distribution of M may enable to get the $D \perp \langle\xi\rangle$-version of the Theorem 2.2 in [6].

Also, it is known that lightlike submanifolds whose screen distribution is integrable have interesting properties. Therefore, we investigate the effect of integrability of the screen distributions. It is now easy to see that the distribution $D \perp \langle\xi\rangle$ is integrable if and only if $B(X, \phi Y) = B(\phi X, Y)$, $\forall X,\ Y \in \Gamma(D \perp \langle\xi\rangle)$.

Theorem 4.7 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}^{2n+1}(c)$, with $\xi \in TM$. Let Ω be an open set where the index of relative nullity ν is equal to some constant. Suppose the distribution $D \perp \langle\xi\rangle$ is integrable. Then, on Ω, the following assertions are equivalent:*

- (i) *M is $D \perp \langle\xi\rangle$-totally geodesic,*
- (ii) *$TM^\perp$ is a $D \perp \langle\xi\rangle$-parallel on M,*
- (iii) *$\overline{\phi}(TM^\perp)$ is a $D \perp \langle\xi\rangle$-Killing distribution on M,*
- (iv) *$T^{*0} = D \perp \langle\xi\rangle$.*

Proof. The equivalence of (i) and (iv) follows from Proposition 4.2. By using the second equation of (2.10), we obtain the equivalence of (i) and (ii). Next, we prove the equivalence of (i) and (iii). Using the fact $D \perp \langle\xi\rangle$ is integrable, we have, for any X, $Y \in \Gamma(D_x \perp \langle\xi\rangle_x)$ with $x \in \Omega$, $(L_V g)(X, Y) = -B(X, \phi Y) - B(Y, \phi X) = -2B(X, \phi Y)$. The equivalence follows from this equation, since $B(., \xi) = 0$. □

Theorem 4.8 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}^{2n+1}(c)$ $(n > 1)$, with $\xi \in TM$. Let Ω be an open set where the index of relative nullity ν is equal to some constant. Then, on Ω, the following assertions hold*

- (i) *M is Einstein on the direction of T^{*0},*
- (ii) *If M is flat, then T^{*0} is an isotropic distribution along M,*
- (iii) *If M is Einstein and $(trA_N)X - A_N X \notin \langle\xi\rangle$, for any $X \in \Gamma(TM)$, then the screen distribution $S(TM)$ is parallel.*

Proof. (i) For any X, $Y \in T_xM$ with $x \in \Omega$,

$$Ric(X,Y) = -(2n-1)g(X,Y) + g(A^*_E X, Y)trA_N - g(A_N X, A^*_E Y). \tag{4.20}$$

So, if X, $Y \in T^{*0}(x)$, then $A^*_E X = A^*_E Y = 0$ and the induced Ricci tensor above becomes $Ric(X,Y) = -(2n-1)g(X,Y)$, that is, M is Einstein in T^{*0}.

(ii) If M is flat, then the induced Ricci tensor Ric on M vanishes and (4.20) becomes $0 = -(2n-1)g(X,Y) + g(A^*_E X, Y)trA_N - g(A_N X, A^*_E Y)$. Taking X, $Y \in T^{*0}(x)$, we get $g(X,Y) = 0$, for $n > 1$, that is the distribution is isotropic along M.

(iii) Suppose that M is Einstein. Since $B(.,\xi) = 0$, the induced Ricci tensor on M satisfies $Ric(X,Y) = -(2n-1)g(X,Y)$, for any X, $Y \in T_xM$, with $x \in \Omega$. Using (4.20), we obtain $B(X,Y)trA_N - B(A_N X, Y) = 0$ which implies that $B((trA_N)X - A_N X, Y) = 0$, that is, $g((trA_N)X - A_N X, A^*_E Y) = 0$. Since $(trA_N)X - A_N X \notin \langle\xi\rangle_x$, we have $(trA_N)X - A_N X = \theta(X)(trA_N)E$. So, $A_N X = (X - \theta(X)E)\, trA_N = (trA_N)PX$ and the screen distribution $S(TM)$ is totally umbilical. Consequently, $trA_N = g(A_N\xi, \xi) = 0$, that is, the shape operator A_N is trace-free. Using this, we get $A_N X = 0$, $\forall\, X \in T_xM$. By Proposition 2.7 in [6] page 89, the screen distribution $S(TM)$ is parallel. □

Let M be a lightlike hypersurface of an indefinite Kenmotsu manifolds $\overline{M}$ with $\xi \in TM$. It is easy to check that M is (D, D')-mixed totally geodesic if and only if, $A_N X \in \Gamma(\overline{\phi}(TM^\perp) \perp D_0)$, $\forall\, X \in \Gamma(D)$ [11]. Using this, we have $\eta((trA_N)X - A_N X) = 0$, $\forall\, X \in \Gamma(D)$. That is, $(trA_N)X - A_N X \notin \langle\xi\rangle$, $\forall\, X \in \Gamma(D)$. Therefore, there exist vector fields on M which satisfy the extra condition of the third assertion of the Theorem 4.8.

A submanifold M is said to be totally umbilical lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$ if the local second fundamental form B of M satisfies $B(X,Y) = \rho g(X,Y)$, for any X, $Y \in \Gamma(TM)$, where ρ is a smooth function on $\mathcal{U} \subset M$ [6].

If M is a totally umbilical lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$, then we have $B(X,Y) = \rho g(X,Y)$, for any X, $Y \in \Gamma(TM)$, which implies that $0 = B(\xi,\xi) = \rho$. Hence M is totally geodesic. Therefore we have the following theorem.

Theorem 4.9 *Let $(M, g, S(TM))$ be a parallel (or totally umbilical) lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in TM$. Let Ω be an open set where the index of relative nullity ν is equal to some constant. Then, on Ω,*

$$T^{*0} = TM. \tag{4.21}$$

Proof. The proof follows from Theorem 3.4 and (4.2). □

The Theorems 4.7 and 4.9 can be extended by using Theorem 4.10 in [10] and Theorem 2.2 in [6] page 88 in order to know more about the geometry of parallel lightlike hypersurface M.

Example 4.10 Let M be a lightlike hypersurface of $\overline{M}^7$ defined in Example 3.1. The tangent space TM is spanned by $\{U_i\}_{1\le i\le 6}$, where $U_1 = e_1$, $U_2 = e_2 - e_5$, $U_3 = e_3$, $U_4 = e_4$, $U_5 = e_6$, $U_6 = \xi$ and the 1-dimensional distribution $TM^\perp$ of rank 1 is spanned by E, where $E = e_2 - e_5$. Also, the transversal bundle

$N(TM)$ is spanned by $N = \frac{1}{2}(e_2 + e_5)$. It follows that $TM^{\perp} \subset TM$. Then M is a 6-dimensional lightlike hypersurface of $\overline{M}^7$ having a local quasi-orthogonal field of frames $\{U_1, U_2 = E, U_3, U_4, U_5, U_6 = \xi, N\}$ along M. Denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}^7$. Then, by straightforward calculations, we obtain $\overline{\nabla}_X N = 0$, $\forall X \in \Gamma(TM)$. Using these equations above, the differential 1-form τ vanishes i.e. $\tau(X) = 0$, for any $X \in \Gamma(TM)$. So, from the Gauss and Weingarten formulae, we have $A_N X = 0$, $A^*_E X = 0$ and $\nabla_X E = 0$, $\forall X \in \Gamma(TM)$. Therefore, by Duggal-Bejancu theorems (Theorem 2.2 and Proposition 2.7) in [6] the lightlike hypersurface M of $\overline{M}^7$ is totally geodesic and its distribution is parallel. Using (3.34), M is Einstein and the foliation T^{*0} is given by $T^{*0} = TM$ on an open set where the index of relative nullity ν is equal to some constant.

Theorem 4.11 *Let $(M, g, S(TM))$ be a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$, with $\xi \in TM$, such that $S(TM)$ is totally umbilical. Let Ω be an open set where the index of relative nullity ν is equal to some constant. Then, on Ω, the following assertions are equivalent:*

(i) *M is $D \perp \langle\xi\rangle$-totally geodesic,*

(ii) *$TM^{\perp}$ is a $D \perp \langle\xi\rangle$-parallel on M,*

(iii) *The distribution $\overline{\phi}(TM^{\perp})$ is $D \perp \langle\xi\rangle$-parallel on M,*

(iv) *$T^{*0} = D \perp \langle\xi\rangle$.*

Proof. The equivalence of (i) and (iv) follows from (4.2). Now, we prove the equivalence of (i) and (ii). Since the screen distribution $S(TM)$ is totally umbilical, $S(TM)$ is totally geodesic, that is, $C(X,Y) = 0$, for any $X, Y \in \Gamma(S(T_xM))$ with $x \in \Omega$. In particular, for any $X \in \Gamma(\overline{\phi}(T_xM^{\perp}) \perp D_{0x} \perp \langle\xi\rangle_x)$, $C(X, V) = u(A_N X) = 0$. Since $C(E, V) = 0$, we have, for any $X_0 \in \Gamma(D_x \perp \langle\xi\rangle_x)$, $u(A_N X_0) = 0$. From the Lemma 4.6, M is $D \perp \langle\xi\rangle$-totally geodesic if and only if, for any $X_0 \in \Gamma(D_x \perp \langle\xi\rangle_x)$, $A^*_E X_0 = 0$ and using the second equation of (2.10), we obtain the required equivalence. Next, we prove the equivalence of (ii) and (iii). Suppose $TM^{\perp}$ is a $D \perp \langle\xi\rangle$-parallel on M. Then, for any $X_0 \in \Gamma(D_x \perp \langle\xi\rangle_x)$, $A^*_E X_0 = 0$. Since the normal bundle $\overline{\phi}(TM^{\perp})$ is a distribution on M of rank 1 and spanned by $\overline{\phi}E$, then, for any $Y_0 \in \Gamma(\overline{\phi}(TM^{\perp}))$,

$$\nabla_{X_0} Y_0 = (X_0.v(Y_0) - v(Y_0)\tau(X_0))\overline{\phi}E \in \Gamma(\overline{\phi}(TM^{\perp})),$$

since $Y_0 = v(Y_0)\overline{\phi}E$. So, the distribution $\overline{\phi}(TM^{\perp})$ is $D \perp \langle\xi\rangle$-parallel. Conversely, suppose the distribution $\overline{\phi}(TM^{\perp})$ is $D \perp \langle\xi\rangle$-parallel. Then, for any $X_0 \in \Gamma(D_x \perp \langle\xi\rangle_x)$ and $Y_0 = v(Y_0)\overline{\phi}E \in \Gamma(\overline{\phi}(T_xM^{\perp}))$, $\nabla_{X_0} Y_0 \in \Gamma(\overline{\phi}(T_xM^{\perp}))$. Since $\overline{\phi}(TM^{\perp})$ is spanned by $\overline{\phi}E$, there exist a smooth function $\lambda \neq 0$ on M such that $\nabla_{X_0} Y_0 = \lambda\overline{\phi}E$. We have

$$\lambda = \overline{g}(X_0.v(Y_0)\overline{\phi}E + v(Y_0)\nabla_{X_0}\overline{\phi}E, \overline{\phi}N) = X_0.v(Y_0) - v(Y_0)\tau(X_0).$$

On the other hand, $\nabla_{X_0} Y_0 = (X_0.v(Y_0) - v(Y_0)\tau(X_0))\overline{\phi}E - v(Y_0)\phi(A^*_E X_0)$. So, we have

$$\begin{aligned}
\nabla_{X_0} Y_0 &= (X_0.v(Y_0) - v(Y_0)\tau(X_0))\overline{\phi}E - v(Y_0)\phi(A^*_E X_0) \\
&= (X_0.v(Y_0) - v(Y_0)\tau(X_0))\overline{\phi}E,
\end{aligned}$$

that is, $v(Y_0)\phi(A^*_E X_0) = 0$. Taking $Y_0 = V$, we have $\phi(A^*_E X_0) = 0$. Applying ϕ to this and using (3.6), $A^*_E X_0 = \eta(A^*_E X_0)\xi + u(A^*_E X_0)U = -u(X_0)\xi + B(X_0, V)U = \lambda u(X_0)U = 0$, since M is totally contact umbilical and $u(X_0) = 0$, $\forall\, X_0 \in \Gamma(D_x \perp \langle\xi\rangle_x)$. This completes the proof. □

It is known that the local second fundamental form B of a lightlike hypersurface M on $\mathcal{U}$ is independent of the choice of the screen distribution [6]. Thus, the Theorems above which depend exclusively on B are stable with respect to any change of the screen distribution.

Let P and P' be projections of TM on $S(TM)$ and $S(TM)'$, respectively with respect to the orthogonal decomposition of TM. So, any vector field X on M can be written as $X = PX + \theta(X)E = P'X + \theta'(X)E$, where $\theta(X) = \overline{g}(X, N)$ and $\theta'(X) = \overline{g}(X, N')$. Then, using equations (3.14) we have

$$P'X = PX - \omega(X)E \quad \text{and} \quad C'(X, P'Y) = C'(X, PY), \quad \forall\, X, Y \in \Gamma(TM). \tag{4.22}$$

Here ω is the dual 1-form of $W = \sum_{i=1}^{2n-1} c_i W_i$, characteristic vector field of the screen change, with respect to the induced metric g of M defined by $\omega(\cdot) = g(\cdot, W)$.

The relationship between the local second fundamental forms C and C' of the screen distribution $S(TM)$ and $S(TM)'$, respectively, is given by (using equations (3.14))

$$C'(X, PY) = C(X, PY) - \frac{1}{2}\omega(\nabla_X PY + B(X, Y)W). \tag{4.23}$$

All equations or conditions above depending only on the local fundamental form C (making equations non unique) are independent of the screen distribution $S(TM)$ if and only if $\omega(\nabla_X PY + B(X, Y)W) = 0$, for any $X,\ Y \in \Gamma(TM)$.

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