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Relative Order of Entire Functions in Terms of Their Maximum Terms

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Abstract

In the paper we study different properties of relative order of entire functions defined on the basis of their maximum terms.

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1 Introduction, Definitions and Notations.

Let f and g be two entire functions and $F(r) = \max\{|f(z)| : |z| = r\}$, $G(r) = \max\{|g(z)| : |z| = r\}$. If f in non-constant then $F(r)$ is strictly increasing and continuous and its inverse

$$F^{-1} : (|f(0)|, \infty) \rightarrow (0, \infty)$$

exists and is such that

$$\lim_{s\rightarrow\infty} F^{-1}(s) = \infty.$$

Bernal([1]) introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows :

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : F(r) < G(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log\ G^{-1}F(r)}{\log\ r}.\end{aligned}$$

The definition coincides with the classical one if $g(z) = exp\ z$.

For an entire function f defined in the open complex plane $\mathbb{C}$ the maximum term $\mu(r,f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu(r,f) = \max_{n\geq 0}(|a_n|r^n)$. For sake of definiteness we denote $\mu(r,f)$ by $\mu_1(r)$. Similarly for entire g, $\mu_2(r)$ stands for $\mu(r,g)$.

In the paper we give an alternative definition of $\rho_g(f)$ i.e. the relative order of f with respect to g in terms of their maximum terms and find some applications of it. We do not explain the standard definitions and notations in the theory of entire functions as those are available in ([3]).

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *If f be entire and $\alpha > 1, 0 < \beta < \alpha$, then for all large r,*

$$\mu_1(\alpha r) \ \geq\ \beta\mu_1(r).$$

Proof. Since $f = \sum_{n=0}^{\infty} a_n z^n$ and $\mu_1(r) = \max_{n\geq 0}(|a_n|r^n)$
then

$$\begin{aligned}\mu_1(\alpha r) &= \max_{n\geq 0}|a_n|(\alpha r)^n \\ &= \max_{n\geq 0}|a_n|(\alpha)^n(r)^n\end{aligned}$$

and

$$\beta\mu_1(r) \ =\ \beta\max_{n\geq 0}|a_n|r^n.$$

As $\alpha > 1$ therefore $\alpha^n \geq 1$ for $n \geq 0$.
As we take maximum value for large r therefore $n = 0$, the maximum value does not occur and as $0 < \beta < \alpha$ so

$$0 < \beta < \alpha^n$$

holds.
Hence $\mu_1(\alpha r) \;\geq\; \beta\mu_1(r)$.
This proves the lemma. □

Lemma 2.

$$\rho_g(f) \;=\; \limsup_{r\to\infty} \frac{log\ \mu_2^{-1}\mu_1(r)}{log\ r}.$$

Proof. For $0 \;\leq\; r \;<\; R$, the following inequality for entire f and g is well known

$$\mu_1(r,f) \;\leq\; M(r,f) \;=\; F(r) \;\leq\; \frac{R}{R-r}\mu_1(R;f)$$

and

$$\mu_2(r,g) \;\leq\; M(r,g) \;=\; G(r) \;\leq\; \frac{R}{R-r}\mu_2(R;g).$$

Putting $R = 2r$ in above we get that

$$\mu_1(r,f) \;\leq\; F(r) \;\leq\; \frac{2r}{2r-r}\mu_1(2r,f)$$

and

$$\mu_2(r,g) \;\leq\; G(r) \;\leq\; \frac{2r}{2r-r}\mu_2(2r,g).$$

i.e.,

$$\begin{aligned} \mu_1(r) \;&\leq\; F(r) \;\leq\; 2\mu_1(2r) \\ \text{and}\;\; \mu_2(r) \;&\leq\; G(r) \;\leq\; 2\mu_2(2r). \end{aligned} \tag{1}$$

Now applying Lemma 1 by considering $2r$ in place of r we obtain that on taking $\alpha \;=\; 3,\; \beta \;=\; 2\; (0 < 2 = \beta < \alpha = 3)$,

$$\begin{aligned} \mu_1(3.2r) \;&>\; 2\mu_1(2r) \\ \text{and}\;\; \mu_2(3.2r) \;&>\; 2\mu_2(2r). \end{aligned}$$

Thus from (1) we get that

$$\begin{aligned} \mu_1(r) \;&\leq\; F(r) \;\leq\; 2\mu_1(2r) \;<\; \mu_1(6r) \\ \text{and}\;\; \mu_2(r) \;&\leq\; G(r) \;\leq\; 2\mu_2(2r) \;<\; \mu_2(6r). \end{aligned}$$

We consider from above only the following:

$$\begin{aligned} \mu_1 \;&\leq\; F(r) \;<\; \mu_1(6r) \\ \text{and}\;\; \mu_2 \;&\leq\; G(r) \;<\; \mu_2(6r). \end{aligned}$$

Now

$$G^{-1}(\mu_2(r)) \leq r.$$

Let us take $\mu_2(r) = k$ i.e., $r = \mu_2^{-1}(k)$.
So

$$G^{-1}(k) \leq \mu_2^{-1}(k).$$

For large r, $\mu_2(r)$ is also large.
Hence k is large.
Using $F(r)$ in place of k we get that

$$\begin{aligned} G^{-1}(F(r)) &\leq \mu_2^{-1}(F(r)) \\ \text{i.e., } G^{-1}F(r) &\leq \mu_2^{-1}F(r). \end{aligned}$$

But $F(r) < \mu_1(6r)$.
So we get from above that

$$G^{-1}F(r) < \mu_2^{-1}(\mu_1(6r)) = (\mu_2^{-1}\mu_1)(6r). \tag{2}$$

Now as $G(r) < \mu_2(6r)$ therefore

$$r < G^{-1}(\mu_2(6r)).$$

Let $\mu_2(6r) = k_0$.
So $6r = \mu_2^{-1}(k_0)$
i.e., $r = \frac{1}{6}\mu_2^{-1}(k_0)$.
Therefore we obtain that

$$\frac{1}{6}\mu_2^{-1}(k_0) < G^{-1}(k_0).$$

Now for large r, k_0 is large and we put $k_0 = F(r)$.
So from above it follows that

$$\frac{1}{6}\mu_2^{-1}(F(r)) < G^{-1}(F(r)).$$

But as $\mu_1(r) \leq F(r)$, therefore

$$\begin{aligned} \frac{1}{6}\mu_2^{-1}(\mu_1(r)) &\leq \frac{1}{6}\mu_2^{-1}(F(r)) < G^{-1}(F(r)) \\ \text{i.e., } \frac{1}{6}\mu_2^{-1}(\mu_1(r)) &< G^{-1}F(r). \end{aligned} \tag{3}$$

Now from (2) and (3)we obtain that

$$\begin{aligned}
\frac{1}{6}\mu_2^{-1}\mu_1(r) \; &< \; G^{-1}F(r) \; < \; \mu_2^{-1}\mu_1(6r) \\
i.e., \; \frac{\log\ \mu_2^{-1}\mu_1(r) + O(1)}{\log\ r} \; &< \; \frac{\log\ G^{-1}F(r)}{\log\ r} < \frac{\log\ \mu_2^{-1}\mu_1(6r)}{\log\ (6r) + O(1)} \\
i.e., \; \limsup_{r\to\infty}\frac{\log\ \mu_2^{-1}\mu_1(r) + O(1)}{\log\ r} \; &\leq \; \limsup_{r\to\infty}\frac{\log\ G^{-1}F(r)}{\log\ r} \leq \frac{\log\ \mu_2^{-1}\mu_1(6r)}{\log\ (6r) + O(1)} \\
i.e., \; \rho_g(f) \; &= \; \limsup_{r\to\infty}\frac{\log\ \mu_2^{-1}\mu_1(r)}{\log\ r}.
\end{aligned}$$

This proves the lemma. □

3 Theorems.

In this section we present the main results of our paper.

Definition 1. *Two entire functions g_1 and g_2 are said to be asymptotically equivalent if there exists $l,\ 0 < l < \infty$ such that*

$$\frac{G_1(r)}{G_2(r)} \to \; l \quad as \; r \to \infty$$

and in this case we write $g_1 \sim g_2$.
If $g_1 \sim g_2$ then clearly $g_2 \sim g_1$.

Theorem 1. *If $g_1 \sim g_2$ and f is entire, then $\rho_{g_1}(f) = \rho_{g_2}(f)$.*

Proof. Let $\epsilon > 0$. By Lemma 1 for all large r

$$\mu_2(r) < (1+\epsilon)\mu_2^{'}(r) < \mu_2^{'}(\alpha r) \tag{4}$$

where $\alpha > 1$ is such that $1 + \epsilon < \alpha$ and also $\mu_2(r), \mu_2^{'}(r)$ respectively denote the maximum terms of g_1 and g_2.
From (4),

$$\begin{aligned}
r \; &< \; \mu_2^{-1}(\mu_2^{'}(\alpha r)) \\
\text{i.e.,}\frac{1}{\alpha}\mu_2^{'\,-1}(t) \; &< \; \mu_2^{-1}(t), t = \mu_2^{'}(\alpha r) \\
\text{i.e.,}\mu_2^{'\,-1}(r) \; &< \; \alpha\mu_2^{-1}(r) \;\; \text{for all large } r.
\end{aligned} \tag{5}$$

Now by Lemma 2 we get that in view of (5)

$$\begin{aligned}
\rho_{g_2}(f) &= \limsup_{r\to\infty}\frac{\log\ \mu_2'^{-1}\mu_1(r)}{\log\ r}\\
&\le \limsup_{r\to\infty}\frac{\log\ \alpha\mu_2^{-1}\mu_1(r)}{\log\ r}\\
&= \limsup_{r\to\infty}\frac{\log\ \mu_2^{-1}\mu_1(r)}{\log\ r} = \rho_{g_1}(f).
\end{aligned}$$

The reverse inequality is clear because $g_2 \sim g_1$ and so $\rho_{g_1}(f) = \rho_{g_2}(f)$. This proves the theorem. □

Remark 1. *Theorem 1 has also been proved by Lahiri and Banerjee ([2]) using maximum modulus functions.*

Remark 2. *Under the same assumption, Theorem 1 can also be deduced for relative lower order.*

Remark 3. *The converse of Theorem 1 is not true as we see in the following example.*

Example 1. *Let us consider the functions $g_1(z) = e^z$ and $g_2(z) = e^{3z}$. Then*

$$G_1(r) = e^r \text{ and } G_2(r) = e^{3r}.$$

Now $\mu_2(r) \le G_1(r) \le \frac{R}{R-r}\mu_2(R)$ gives that for $R = 2r$,

$$\mu_2(r) \le G_1(r) \le 2\mu_2(2r).$$

Also for $R = 2r$, the inequality

$$\mu_2'(r) \le G_2(r) \le \frac{R}{R-r}\mu_2'(R)$$

gives that

$$\mu_2'(r) \le G_2(r) \le 2\mu_2'(2r),$$

so that

$$\begin{aligned}
&\frac{\mu_2(r)}{\mu_2'(r)} \le \frac{G_1(r)}{\frac{1}{2}G_2(\frac{r}{2})} = \frac{2G_1(r)}{G_2(\frac{r}{2})}\\
i.e.,\ &\frac{\mu_2(r)}{\mu_2'(r)} \le \frac{2e^r}{e^{\frac{3r}{2}}}\\
i.e.,\ &\frac{\mu_2(r)}{\mu_2'(r)} \to 0 \ \text{ as } \ r \to \infty.
\end{aligned}$$

Hence g_1 *not* $\sim g_2$.
But

$$\begin{aligned}
\rho_{g_2}(f) &= \limsup_{r\to\infty} \frac{\log\ \mu_2^{-1}\mu_1(r)}{\log\ r} \\
&= \limsup_{r\to\infty} \frac{\log\ G_2^{-1}F(r)}{\log\ r} \quad = \quad \limsup_{r\to\infty} \frac{\log\ \left(\frac{1}{3}\log\ F(r)\right)}{\log\ r} \\
&= \limsup_{r\to\infty} \frac{\log\ (\log\ F(r))}{\log\ r} \quad = \quad \limsup_{r\to\infty} \frac{\log\ G_1^{-1}F(r)}{\log\ r} \\
&= \limsup_{r\to\infty} \frac{\log\ \mu_2^{-1}\mu_1(r)}{\log\ r} \quad = \quad \rho_{g_1}(f).
\end{aligned}$$

Theorem 2. *Let f_1, f_2, g be three entire functions and $f_1 \sim f_2$. Then $\rho_g(f_1) = \rho_g(f_2)$.*

Proof. Since $f_1 \sim f_2$, for $\epsilon_1 > 0$, $\exists\ R_1 > 0$ such that

$$\begin{aligned}
\mu_1(r) &< (l+\epsilon_1)\mu_1^{'}(r),\ 0 < l < \infty,\ r \geq R_1 \\
&< \mu_1^{'}(\beta r) \qquad (6)
\end{aligned}$$

by Lemma 1, where $\beta > 1$ is such that $1 + \epsilon_1 < \beta$. Now

$$\begin{aligned}
\rho_g(f_1) &= \limsup_{r\to\infty} \frac{\log\ \mu_2^{-1}\mu_1(r)}{\log\ r} \\
&\leq \limsup_{r\to\infty} \frac{\log\ \mu_2^{-1}\mu_1^{'}(\beta r)}{\log\ r}, \quad \text{from(6)}.
\end{aligned}$$

For $0 < \epsilon_2 < 1$, there exists $R_2 > 0$ such that for $r \geq R_2$,

$$\log\ r > (1-\epsilon_2)\log\ \beta r.$$

So

$$\begin{aligned}
\rho_g(f_1) &\leq \limsup_{r\to\infty} \frac{\log\ \mu_2^{-1}\mu_1^{'}(\beta r)}{(1-\epsilon_2)\log\ \beta r} \\
&= \frac{1}{1-\epsilon_2}\rho_g(f_2).
\end{aligned}$$

Since $0 < \epsilon_2 < 1$ is arbitrary, $\rho_g(f_1) \leq \rho_g(f_2)$.
Since also $f_2 \sim f_1$, we obtain $\rho_g(f_2) \leq \rho_g(f_1)$.
This proves the theorem. □

Remark 4. *Theorem 2 has also been proved by Lahiri and Banerjee [2]using maximum modulus functions.*

Theorem 3. *Let f_1, f_2, g_1, g_2 be four entire functions. If $f_1 \sim f_2$ and $g_1 \sim g_2$, then*

$$\rho_{g_1}(f_1) = \rho_{g_2}(f_1) = \rho_{g_1}(f_2) = \rho_{g_2}(f_2).$$

Theorem 3 follows from Theorem 1 and Theorem 2.

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