Relative Pricing of Options with Stochastic Volatility[*]

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Abstract

This paper offers a new approach for pricing options on assets with stochastic volatility. We start by taking as given the prices of a few simple, liquid European options. More specifically, we take as given the "surface" of Black-Scholes implied volatilities for European options with varying strike prices and maturities. We show that the Black-Scholes implied volatilities of at-the-money options converge to the underlying asset's instantaneous (stochastic) volatility as the time to maturity goes to zero. We model the stochastic processes followed by the implied volatilities of options of all maturities and strike prices as a joint diffusion with the stock price. In order for no arbitrage opportunities to exist in trading the stock and these options, the drift of the processes followed by the implied volatilities is constrained in such a way that it is fully characterized by the volatilities of the implied volatilities. Finally, we suggest how to use the arbitrage-free joint process for the stock price and its volatility to price other derivatives, such as standard but illiquid options as well as exotic options, using numerical methods. Our approach simply requires as inputs the stock price and the implied volatilities at the time the exotic option is to be priced, as well as estimates of the volatilities of the implied volatilities.

KEY WORDS: stochastic volatility, Black-Scholes, implied volatility, options

1. Introduction

Extensive empirical evidence suggests that the volatility of many assets' prices is stochastic. This affects the pricing and hedging of options written on these assets, and therefore creates the need for simple and efficient models to price and hedge these derivatives. We provide such a model.

The traditional approach to pricing options on stocks with stochastic volatility starts by specifying the joint process for the stock price and its volatility and makes some assumption about the market price of volatility risk. Then, it uses the risk-adjusted joint process followed by the stock price and its volatility to price options, in closed-form if possible, but more likely with numerical methods. This is the approach followed by Bakshi, Cao and Chen (1997), Heston (1993), Hull and White (1987), and Stein and Stein (1991), among others. These models are typically calibrated to the prices of a few options or estimated from the time series of stock prices. Unfortunately, they cannot fit the prices of more than a few options at a time.

In contrast, we start by taking as given the prices of a few simple, liquid European options. More specifically, we take as given the "surface" of Black-Scholes implied volatilities for European call options with varying strike prices and maturities. We show that the Black-Scholes implied volatilities of at-the-money options converge to the underlying asset's instantaneous (stochastic) volatility as the time to maturity goes to zero.[1] Intuitively, one instant before the option expires, the effect of stochastic volatility on the option price is negligible. Then, the Black-Scholes formula

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[1]There is nothing special about using the Black-Scholes formula in our approach. It is just a convenient, and well known, mapping from prices of options (that are actually traded) to volatilities (that are typically quoted). In particular, the mapping is continuous in the strike price and time to maturity, and is such that the volatility of the stock price is equal to the limit as time to maturity goes to zero of at-the-money implied volatilities. Other formulas with these properties would work just as well.

accurately prices the option and, as a result, its implied volatility corresponds to the instantaneous volatility of the underlying asset.[2]

We then model the stochastic processes followed by the implied volatilities of options of all maturities and strike prices as a joint diffusion with the stock price. In order for no arbitrage opportunities to exist in trading the stock and these options, the drift of the processes followed by the implied volatilities is constrained in such a way that it is fully characterized by the volatilities of the implied volatilities. Finally, we suggest how to use the arbitrage-free joint process for the stock price and its volatility to price other derivatives, such as standard but illiquid options as well as exotic options, using numerical methods. Our approach simply requires as inputs the stock price and the implied volatilities at the time the exotic option is to be priced, as well as estimates of the volatilities of the implied volatilities.

This approach allows us to exactly fit the prices of the simple options at the time of pricing the exotic option. This is particularly important as these options might be used to hedge the position in the exotic and accurate hedging is only possible if the model used to price the exotic exactly fits the prices of the hedging instruments.

Our approach relates to existing models of option pricing with stochastic volatility in a way similar to the relation between the Heath, Jarrow and Morton (HJM, 1992) model and the traditional models of the term structure of interest rates, such as Cox, Ingersoll and Ross (CIR, 1985) and Vasicek (1977). Models such as CIR take as input the process for the short term interest rate and its market price of risk to obtain the risk-adjusted dynamics of the short interest rate and then

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[2]Actually, this result is not as trivial as this intuitive argument suggests. Only the implied volatilities of at-the-money options converge to the instantaneous volatility. The implied volatilities of out-of-the-money or in-the-money options diverge as the options approach expiration.

price bonds and other interest rate derivatives. HJM take as given the current term structure of (forward) interest rates and their volatilities, to then obtain the risk-adjusted dynamics of the short term interest rate and price interest rate derivatives (other than bonds).

Our approach should not be confused with the option pricing models based on "implied binomial trees", of Derman and Kani (1994), Dupire (1994), and Rubinstein (1994). These models assume that the stock price volatility is a deterministic function of the stock price itself and time, so that the stock price is still the only source of uncertainty. Trading in the stock and a bond is therefore enough to replicate any stock derivatives. These papers provide a method of calibrating the volatility of the stock as a function of the stock and time in a way that matches a number of simple option prices computed from a binomial or trinomial tree. Exotics can then be priced off of the same tree. However, since volatilities are predicted to vary deterministically with the stock and time, it is unlikely that the relationship will still hold exactly in all dates posterior to the fit date. In fact Aït-Sahalia, Wang and Yared (2001), Jackwerth and Rubinstein (1996) and Dumas, Fleming and Whaley (1998) have found evidence against this class of models.

In recent work, Derman and Kani (1997) offer a model that is similar in spirit to ours. They model the dynamics of what they term the "local volatility surface" and find a no-arbitrage condition that it must satisfy. The problem with their approach is that local volatilities are not readily measurable and that there is no explicit relationship between option prices and local volatilities.[3] Furthermore,

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[3]The local volatilities $\varsigma(t, s, K)$ are defined at time t as the stock volatility at future time s and stock level $S(s) = K$ that would price all observed options correctly. Formally

$$\varsigma(t, s, K)^2 = 2 \frac{\frac{\partial C}{\partial s}(t, s, K) + (r - q)K \frac{\partial C}{\partial K}(t, s, K) + qC(t, s, K)}{K^2 \frac{\partial^2 C}{\partial K^2}(t, s, K)}, \tag{1}$$

where $C(t, s, K)$ denotes the time t price of a call option with maturity at time s and strike price K.

the resulting arbitrage-free process for the stock price volatility is rather complicated.[4] It is virtually impossible to use the model in continuous time, and only a binomial approximation is possible.

The rest of the paper is organized as follows. Section 2 states the Black-Scholes formula and the "Greeks". Section 3 presents the stochastic process followed by implied volatilities as a joint diffusion process with the stock price. We show that the Black-Scholes implied volatilities of at-the-money options converge to the stock's instantaneous volatility as the time to maturity goes to zero. Section 4 derives the no arbitrage constraint that the drift of the implied volatilities has to satisfy. In Section 5 we suggest how to use our approach to price illiquid and exotic options. Section 6 concludes the paper.

2. The Black-Scholes Formula and the Greeks

We look at a simple approach to price options on an asset with stochastic volatility, that we term a *stock*, although it could as well be a stock index, an exchange rate, or the price of a commodity. The element that seems to be most important in pricing these options is that the volatility of the stock is stochastic, and correlated with the stock price itself. Although there is ample evidence that dividend yields and interest rates are also stochastic, we take them to be constant as a first approximation.

Consider the price at time t of a call option on a stock worth S, paying a constant (continuous) dividend yield q, with maturity $s \geq t$, and strike price K, as given by the Black-Scholes formula:

$$C(t, S, V; s, K) = Se^{-q(s-t)}N(d_1(t, S, V; s, K)) - Ke^{-r(s-t)}N(d_2(t, S, V; s, K)), \qquad (2)$$

[4]The drift of the local volatilities is obtained as an integral of the forward transition density of the stock price, computed at time t with the volatility surface as if it were non-random.

$$d_1(t, S, V; s, K) = \frac{\log(S/K) + \left(r - q + \frac{1}{2}V^2\right)(s - t)}{V\sqrt{s - t}},$$

$$d_2(t, S, V; s, K) = d_1(t, S, V; s, K) - V\sqrt{s - t}, \tag{3}$$

where V denotes the implied volatility and r is the continuously compounded interest rate. The function $N(x)$ is the standard normal cumulative distribution function evaluated at x, and $n(x)$ is the corresponding density.

For later use, we state the "Greeks":[5]

$$\Theta(t, S, V; s, K) \equiv \frac{\partial C}{\partial t}(t, S, V; s, K) = -\frac{Se^{-q(s-t)}n(d_1(t, S, V; s, K))V}{2\sqrt{s - t}}$$

$$+qSe^{-q(s-t)}N(d_1(t, S, V; s, K))$$

$$-rKe^{-r(s-t)}N(d_2(t, S, V; s, K)),$$

$$\Delta(t, S, V; s, K) \equiv \frac{\partial C}{\partial S}(t, S, V; s, K) = e^{-q(s-t)}N(d_1(t, S, V; s, K)),$$

$$\Lambda(t, S, V; s, K) \equiv \frac{\partial C}{\partial V}(t, S, V; s, K) = Se^{-q(s-t)}n(d_1(t, S, V; s, K))\sqrt{s - t},$$

$$\Gamma(t, S, V; s, K) \equiv \frac{\partial^2 C}{\partial S^2}(t, S, V; s, K) = \frac{e^{-q(s-t)}n(d_1(t, S, V; s, K))}{SV\sqrt{s - t}},$$

$$\Omega(t, S, V; s, K) \equiv \frac{\partial^2 C}{\partial V^2}(t, S, V; s, K) = Se^{-q(s-t)}n(d_1(t, S, V; s, K))$$

$$\times \frac{d_1(t, S, V; s, K)d_2(t, S, V; s, K)\sqrt{s - t}}{V},$$

$$\Pi(t, S, V; s, K) \equiv \frac{\partial^2 C}{\partial S \partial V}(t, S, V; s, K) = -e^{-q(s-t)}n(d_1(t, S, V; s, K))$$

$$\times \frac{d_2(t, S, V; s, K)}{V}. \tag{4}$$

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[5]We make use of the relation $Se^{-q(s-t)}n(d_1(t, S, V; s, K)) = Ke^{-r(s-t)}n(d_2(t, S, V; s, K))$.

3. Stochastic Dynamics

Assume that the stock price follows:

$$\frac{dS(t)}{S(t)} = \mu_S(t)dt + \sigma_{S1}(t)dW_1(t), \tag{5}$$

with stochastic volatility σ_{S1} that we leave as yet unspecified, following a joint diffusion with the stock price. The stock also pays a dividend at a constant rate q.

We allow the implied volatilities of each time to maturity $T \equiv s-t$ and moneyness $X \equiv S(t)/K$ to be stochastic.[6] [7] Assume that the implied volatilities of any fixed time to maturity and moneyness have dynamics given by:

$$dV(t,T,X) = \mu_V(t,T,X)dt + \sigma_{V1}(t,T,X)dW_1(t) + \sigma_{V2}(t,T,X)dW_2(t), \tag{6}$$

where W_2 is a Brownian motion orthogonal to W_1. We emphasize that $V(t,T,X)$ is the volatility at time t of an option with time to maturity T and moneyness X. The notation does *not* stand for the volatility at time t being a function of T and X. Similarly, $\mu_V(t,T,X)$, $\sigma_{V1}(t,T,X)$, and $\sigma_{V2}(t,T,X)$ are arbitrary processes[8] indexed by T and X, that evolve through time t. In very simple applications we may want to make $\mu_V(t,T,X)$, $\sigma_{V1}(t,T,X)$, and $\sigma_{V2}(t,T,X)$ deterministic function of T and X. In other applications, we may make these coefficient processes functions of the value of some state variables at time t, possibly including the stock price $S(t)$ or the stock's instantaneous volatility $\sigma_{S1}(t)$. In even more complicated applications, we may take $\mu_V(t,T,X)$, $\sigma_{V1}(t,T,X)$, and $\sigma_{V2}(t,T,X)$ to depend on $V(t,T,X)$ itself, for each T and X.

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[6]This is, of course, contrary to the assumptions of the Black-Scholes model, which assumes that the stock's volatility, and therefore all option implied volatilities are constant and equal.

[7]Note that the volatility surface is only well defined when $0 < T < \infty$ and $0 < X < \infty$, or $(T,X) = (0,1)$.

[8]That must, of course, satisfy conditions for the integrals defining the dynamics of V to be well defined.

We choose to model the process followed by the implied volatilities with fixed *time to maturity* and fixed *moneyness*, rather than fixed *maturity date* and fixed *strike price* because we feel that the former are more stationary, and therefore easier to parameterize and estimate. We prove the following Lemma in Appendix.

LEMMA *The Black-Scholes implied volatilities of at-the-money options converge to the stock price volatility when the time to maturity goes to zero, that is,*

$$\sigma_{S1}(t) = \lim_{T \to 0} V(t, T, 1) \equiv V(t, 0, 1). \tag{7}$$

We can thus obtain the instantaneous volatility of the stock price as the "corner" ($T = 0$ and $X = 1$) of the observed implied volatility surface.[9]

4. No Arbitrage

We postulate the existence of a stochastic discount factor (SDF) with the following dynamics:[10]

$$\frac{dM(t)}{M(t)} = -rdt - \phi_1(t)dW_1(t) - \phi_2(t)dW_2(t). \tag{8}$$

Then the product of the price of any financial security with the SDF must be a martingale.

We define W_1^* and W_2^*, with dynamics given by:

$$dW_1^*(t) = dW_1(t) + \phi_1(t)dt, \tag{9}$$

and

$$dW_2^*(t) = dW_2(t) + \phi_2(t)dt, \tag{10}$$

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[9]Much in the same way that the instantaneous interest rate is identified with the intercept of the fitted term structure of interest rates.

[10]The use of this SDF is an artifact to simplify the derivation of the arbitrage free dynamics of the implied volatilities. We will see that we do not need to know the market prices of risk ϕ_1 and ϕ_2 to obtain the risk adjusted dynamics of the implied volatilities, so that we are indeed only pricing by no-arbitrage.

which are orthogonal Brownian motions under the risk-adjusted probability measure corresponding to the SDF.

The risk adjusted process for the stock price is then:

$$\frac{dS(t)}{S(t)} = (r - q)dt + \sigma_{S1}(t)dW_1^*(t), \tag{11}$$

since we have, by a standard argument, that:

$$\phi_1(t) = \frac{\mu_S(t) + q - r}{\sigma_{S1}(t)}. \tag{12}$$

Similarly, the risk adjusted returns to the options are equal to the short term interest rate. But we also know (by assumption) that option prices are functions of the stock price, the implied volatilities, and the options' time to maturity. Therefore, the dynamics of option prices are fully determined by the dynamics of the stock price and the dynamics of the implied volatilities by Itô's Lemma. We can equate the drift of the options that we obtain in this manner with the short term interest rate and obtain a constraint on the drift of the implied volatilities.

To impose the no-arbitrage condition on the call options, we need to look at the dynamics of the option price with fixed *maturity date* rather than fixed *time to maturity*, and with fixed *strike price* rather than fixed *moneyness*. For fixed maturity date s and strike K, we denote the implied volatility of this option at time $t \le s$ by $V(t, s - t, S(t)/K)$. In Appendix, we prove the following proposition using Itô's Lemma.

PROPOSITION *The stochastic dynamics of the implied volatility process with fixed maturity date*

and strike price follows:[11]

$$
\begin{aligned}
d_t V(t, s - t, S(t)/K) &= \left[\mu_V(t, s - t, S(t)/K) - \frac{\partial V}{\partial T}(t, s - t, S(t)/K) \right.\\
&\quad + \frac{\partial V}{\partial X}(t, s - t, S(t)/K) \frac{S(t)}{K} \mu_S(t)\\
&\quad + \frac{1}{2} \frac{\partial^2 V}{\partial X^2}(t, s - t, S(t)/K) \left(\frac{S(t)}{K} \right)^2 \sigma_{S1}(t)^2\\
&\quad \left. + \frac{\partial \sigma_{V1}}{\partial X}(t, s - t, S(t)/K) \frac{S(t)}{K} \sigma_{S1}(t) \right] dt\\
&\quad + \left[\sigma_{V1}(t, s - t, S(t)/K) + \frac{\partial V}{\partial X}(t, s - t, S(t)/K) \frac{S(t)}{K} \sigma_{S1}(t) \right] dW_1(t)\\
&\quad + \sigma_{V2}(t, s - t, S(t)/K) dW_2(t).
\end{aligned}
\tag{13}
$$

The only non-standard term in equation (13) is the last term of the drift. It comes from the covariation between the increment of order $dW_1(t)$ in the implied volatilities and the increment of order $dW_1(t)$ in the dynamics of the stock price.

We can now write the dynamics of the call option price as:

$$
\begin{aligned}
d_t C &= \Theta dt + \Delta dS + \Lambda d_t V + \frac{1}{2} \Gamma (dS)^2 + \frac{1}{2} \Omega (d_t V)^2 + \Pi (dS \times d_t V)\\
&= \left[\Theta + \Delta S \mu_S + \Lambda \mu_V - \Lambda \frac{\partial V}{\partial T} + \Lambda \frac{\partial V}{\partial X} \frac{S}{K} \mu_S + \frac{1}{2} \Lambda \frac{\partial^2 V}{\partial X^2} \left(\frac{S}{K} \right)^2 \sigma_{S1}^2 \right.\\
&\quad + \Lambda \frac{\partial \sigma_{V1}}{\partial X} \frac{S}{K} \sigma_{S1} + \frac{1}{2} \Gamma S^2 \sigma_{S1}^2 + \frac{1}{2} \Omega \left(\sigma_{V1}^2 + \sigma_{V2}^2 \right)\\
&\quad \left. + \Omega \frac{\partial V}{\partial X} \frac{S}{K} \sigma_{V1} \sigma_{S1} + \frac{1}{2} \Omega \left(\frac{\partial V}{\partial X} \right)^2 \left(\frac{S}{K} \right)^2 \sigma_{S1}^2 + \Pi S \sigma_{V1} \sigma_{S1} + \Pi \frac{\partial V}{\partial X} \frac{S^2}{K} \sigma_{S1}^2 \right] dt\\
&\quad + \left[\Delta S \sigma_{S1} + \Lambda \sigma_{V1} + \Lambda \frac{\partial V}{\partial X} \frac{S}{K} \sigma_{S1} \right] dW_1 + \Lambda \sigma_{V2} dW_2,
\end{aligned}
\tag{14}
$$

where the arguments of the processes follow the same logic as in equation (13), and have been omitted for brevity.

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[11]We denote the differential operator by d_t to emphasize that we take into account that, as time passes, the stock price changes and there is a corresponding shift from one fixed moneyness and time to maturity implied volatility process to another.

We now assume that there are no arbitrage opportunities in trading the options and the stock, so that the product of each of these securities with the SDF must be a martingale.

Then, the drifts of the implied volatilities will be such that:

$$
\begin{aligned}
\mu_V \;=\; & \frac{C}{\Lambda}r - \frac{\Theta}{\Lambda} - \frac{\Delta}{\Lambda}S\mu_S + \frac{\partial V}{\partial T} - \frac{\partial V}{\partial X}\frac{S}{K}\mu_S - \frac{1}{2}\frac{\partial^2 V}{\partial X^2}\left(\frac{S}{K}\right)^2 \sigma_{S1}^2 \\
& -\frac{\partial \sigma_{V1}}{\partial X}\frac{S}{K}\sigma_{S1} - \frac{1}{2}\frac{\Gamma}{\Lambda}S^2\sigma_{S1}^2 - \frac{1}{2}\frac{\Omega}{\Lambda}\left(\sigma_{V1}^2 + \sigma_{V2}^2\right) - \frac{\Omega}{\Lambda}\frac{\partial V}{\partial X}\frac{S}{K}\sigma_{V1}\sigma_{S1} \\
& -\frac{1}{2}\frac{\Omega}{\Lambda}\left(\frac{\partial V}{\partial X}\right)^2\left(\frac{S}{K}\right)^2\sigma_{S1}^2 - \frac{\Pi}{\Lambda}S\sigma_{V1}\sigma_{S1} - \frac{\Pi}{\Lambda}\frac{\partial V}{\partial X}\frac{S^2}{K}\sigma_{S1}^2 \\
& +\phi_1\left(\frac{\Delta}{\Lambda}S\sigma_{S1} + \sigma_{V1} + \frac{\partial V}{\partial X}\frac{S}{K}\sigma_{S1}\right) + \phi_2\sigma_{V2}.
\end{aligned}
\tag{15}
$$

Noting that this condition must hold for all $T \equiv s - t$ and $X \equiv S(t)/K$, we have proved the following theorem.

THEOREM *The risk-adjusted dynamics of the implied volatilities can finally be written as:*

$$
\begin{aligned}
dV(t,T,X) \;=\; & \left[\frac{d_2(t,T,X)}{V(t,T,X)\sqrt{T}}\sigma_{V1}(t,T,X)V(t,0,1) \right. \\
& -\frac{d_1(t,T,X)d_2(t,T,X)}{2V(t,T,X)}\left(\sigma_{V1}(t,T,X)^2 + \sigma_{V2}(t,T,X)^2\right) \\
& -\frac{\partial \sigma_{V1}}{\partial X}(t,T,X)XV(t,0,1) \\
& +\frac{V(t,T,X)^2 - V(t,0,1)^2}{2V(t,T,X)T} + \frac{\partial V}{\partial T}(t,T,X) - \frac{\partial V}{\partial X}(t,T,X)X(r-q) \\
& +\frac{d_2(t,T,X)}{V(t,T,X)\sqrt{T}}\frac{\partial V}{\partial X}(t,T,X)XV(t,0,1)^2 \\
& -\frac{d_1(t,T,X)d_2(t,T,X)}{V(t,T,X)}\frac{\partial V}{\partial X}(t,T,X)X\sigma_{V1}(t,T,X)V(t,0,1) \\
& -\frac{d_1(t,T,X)d_2(t,T,X)}{2V(t,T,X)}\left(\frac{\partial V}{\partial X}(t,T,X)\right)^2 X^2V(t,0,1)^2 \\
& \left. -\frac{1}{2}\frac{\partial^2 V}{\partial X^2}(t,T,X)X^2V(t,0,1)^2 \right] dt \\
& +\sigma_{V1}(t,T,X)dW_1^*(t) + \sigma_{V2}(t,T,X)dW_2^*(t),
\end{aligned}
\tag{16}
$$

where we have (re)defined:

$$
\begin{aligned}
d_1(t,T,X) &= \frac{\log(X) + \left[r - q + \frac{1}{2}V(t,T,X)^2\right]T}{V(t,T,X)\sqrt{T}}, \\
d_2(t,T,X) &= d_1(t,T,X) - V(t,T,X)\sqrt{T}.
\end{aligned}
\tag{17}
$$

5. Pricing Illiquid and Exotic Options

One main application of our result is to price exotic options. The price of a European stock derivative which pays $H(s)$ at future date s is equal to

$$
H(t) = e^{-r(s-t)} \mathrm{E}_t^* \left[H(s)\right],
\tag{18}
$$

where the expectation is taken with respect to the joint diffusion of the stock price and its volatility under the risk-adjusted probability measure

$$
\frac{dS(t)}{S(t)} = [r - q]dt + V(t,0,1)dW_1^*(t),
\tag{19}
$$

$$
\begin{aligned}
dV(t,0,1) &= \left[\frac{\sigma_{V1}(t,0,1)}{V(t,T,X)}\left(r - q - \frac{1}{2}V(t,0,1)^2\right) - \frac{\partial \sigma_{V1}}{\partial X}(t,0,1)V(t,0,1) + 2\frac{\partial V}{\partial T}(t,0,1)\right. \\
&\quad \left. - \frac{1}{2}\frac{\partial V}{\partial X}(t,0,1)V(t,0,1)^2 - \frac{1}{2}\frac{\partial^2 V}{\partial X^2}(t,0,1)V(t,0,1)^2\right]dt \\
&\quad + \sigma_{V1}(t,0,1)dW_1^*(t) + \sigma_{V2}(t,0,1)dW_2^*(t),
\end{aligned}
\tag{20}
$$

where we have used the result of the Theorem by letting $X = 0$ and $T \to 0$. This expectation can in general be approximated with Monte Carlo simulation methods.

The approach can easily be extended to price American derivatives, and derivatives with path-dependent payoffs, using the results of Longstaff and Schwartz (2001).

6. Future Work

Our approach can be generalized to allow for stochastic interest rates and a stochastic dividend yield, which may be particularly relevant to the pricing of currency and commodity options. To include stochastic interest rates, we need simply attach an HJM model of the term structure of forward rates to our model. Stochastic dividend yields can be treated in the framework of Miltersen and Schwartz (1997).

We can also extend our model to price bond options when the term structure of interest rates has stochastic volatility. In order to do this, we can model the implied volatilities of options of all maturities and strike prices on bonds of all maturities. Therefore, instead of modeling the dynamics of an implied volatility "surface", we will need to model the dynamics of an implied volatility "cube".

Following Santa-Clara and Sornette (2001), we can extend our model by using stochastic string shocks instead of Brownian motions to drive the uncertainty in the implied volatilities.[12]

Besides the relative pricing of options, an interesting area of application of our model of stochastic implied volatilities is in risk management. It is well known that one difficulty of value-at-risk approaches, whether based on simulation or delta methods, lies in dealing with portfolios that include options. Our approach allows the joint simulation of the underlying and its implied volatility surface, making it trivial to price any options for each realization of the simulated process.

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[12]Actually, since implied volatilities are two-dimensional, and keeping with the physical analogy, we might refer to the shocks to the implied volatility surface as "stochastic membranes".

Appendix

Proof of Lemma For simplicity (re)define:

$$C(t, T, X) \equiv C(t, S(t), V(t, T, X), r, q, t + T, S(t)/X), \tag{A.1}$$

$$d(t, T, X) \equiv d(t, S(t), V(t, T, X), r, q, t + T, S(t)/X). \tag{A.2}$$

For at-the-money-forward calls, $X = e^{-(r-q)T}$, we have in the Black-Scholes formula:

$$d(t, T, e^{-(r-q)T}) = \frac{1}{2} V(t, T, e^{-(r-q)T})\sqrt{T} \tag{A.3}$$

We take a first order Taylor approximation of the standard Normal distribution function to find:

$$N(d(t, T, e^{-(r-q)T})) = \frac{1}{2} + \frac{V(t, T, e^{-(r-q)T})\sqrt{T}}{\sqrt{8\pi}} + O(T) \tag{A.4}$$

for small T. Now, the call option price will be:

$$C(t, T, e^{-(r-q)T}) = \frac{1}{\sqrt{2\pi}} S(t)e^{-qT} V(t, T, e^{-(r-q)T})\sqrt{T} + O(T). \tag{A.5}$$

So that:

$$V(t, 0, 1) = \lim_{T \to 0} V(t, T, e^{-(r-q)T}) = \lim_{T \to 0} \frac{\sqrt{2\pi}}{\sqrt{T}} \frac{C(t, T, e^{-(r-q)T})}{S(t)e^{-qT}} \tag{A.6}$$

We now show that the r.h.s. of the above expression is equal to the stock volatility.

We use the pricing relation:

$$C(t, T, e^{-(r-q)T}) = \mathrm{E}_t^* \left[e^{-rT} \max \left(S(t+T) - S(t)e^{(r-q)T}, 0 \right) \right] \tag{A.7}$$

and the fact that, for small T:

$$S(t+T) = S(t)e^{(r-q)T} + S(t)\sigma_{S1}(t)(W^*(t+T) - W^*(t)) + O(T), \tag{A.8}$$

where the second term is the Brownian increment, which is normally distributed with mean zero and variance T. Then

$$\mathrm{E}_t^* \left[e^{-rT} \max(S(t+T) - S(t)e^{(r-q)T}, 0) \right] = \mathrm{E}_t^* \left[e^{-rT} \max(S(t)\sigma_{S1}(t)(W^*(t+T) - W^*(t)), 0) \right] + O(T),$$

$$(A.9)$$

and we can take the limit of the r.h.s. as:

$$\lim_{T \to 0} \frac{1}{\sqrt{T}} \mathrm{E}_t^* \left[e^{-rT} \max(S(t)\sigma_{S1}(t)(W^*(t+T) - W^*(t)), 0) \right] = \lim_{T \to 0} \frac{1}{\sqrt{2\pi}} S(t)e^{-rT}\sigma_{S1}(t) \quad (A.10)$$

where we have used the fact that for a normal random variable z with mean zero and variance v,

$$\mathrm{E}[\max(z, 0)] = \frac{\sqrt{v}}{\sqrt{2\pi}}. \qquad \qquad \Box$$

Proof of Proposition Making the second argument of the implied volatility processes change deterministically with time, $T = s - t$, adds a term to the drifts of the implied volatilities, equal to $-\frac{\partial V}{\partial T}(t, s - t, S(t)/K)$. In order to figure out the adjustment that needs to be made when the third index of the implied volatility processes varies (stochastically) with the stock price, we simplify the problem to ease notation, and consider a stochastic process S and a family V of stochastic processes indexed by X which follow a joint diffusion:[13]

$$\frac{dS(t)}{S(t)} = \mu_S(t)dt + \sigma_S(t)dW(t), \qquad (A.11)$$

$$dV(t, X) = \mu_V(t, X)dt + \sigma_V(t, X)dW(t), \qquad (A.12)$$

for all fixed X, where W is a standard Brownian motion. We assume that, for fixed t, $V(t, X)$ is sufficiently differentiable in X. μ_S, σ_S, μ_V and σ_V satisfy all the usual regularity conditions.

Now define a new process U by:

$$U(t) \equiv V(t, S(t)). \qquad (A.13)$$

---

[13] $W_2(t)$ is also ignored for simplicity.

At every point in time, t, we pick one element of the family V, and the element we pick is governed by the value of the process S at that time t.

Fix X and integrate V with respect to t:

$$V(t, X) = V(0, X) + \int_0^t \mu_V(u, X)du + \int_0^t \sigma_V(u, X)dW(u). \tag{A.14}$$

So, by definition:

$$U(t) \equiv V(t, S(t)) = V(0, S(t)) + \int_0^t \mu_V(u, S(t))du + \int_0^t \sigma_V(u, S(t))dW(u). \tag{A.15}$$

This is the stochastic process followed by U. To find its differential form, just apply Itô's Lemma:

$$
\begin{aligned}
dU(t) = {} & \frac{\partial V}{\partial X}(0, S(t))\,dS(t) + \frac{\partial^2 V}{\partial X^2}(0, S(t))\frac{1}{2}d\langle S(t)\rangle \\
& + \mu_V(t, S(t))\,dt + \left(\int_0^t \frac{\partial \mu_V}{\partial X}(u, S(t))du\right)dS(t) + \frac{1}{2}\left(\int_0^t \frac{\partial^2 \mu_V}{\partial X^2}(u, S(t))du\right)d\langle S(t)\rangle \\
& + \sigma_V(u, S(t))dW(t) + \left(\int_0^t \frac{\partial \sigma_V}{\partial X}(u, S(t))dW(u)\right)dS(t) \\
& + \frac{1}{2}\left(\int_0^t \frac{\partial^2 \sigma_V}{\partial X^2}(u, S(t))dW(u)\right)d\langle S(t)\rangle + d\left\langle S(t), \int_0^t \sigma_V(u, S(t))dW(u)\right\rangle, \tag{A.16}
\end{aligned}
$$

where

$$d\left\langle S(t), \int_0^t \sigma_V(u, S(t))dW(u)\right\rangle = \frac{\partial \sigma_V}{\partial X}(t, S(t))S(t)\sigma_S(t)dt. \tag{A.17}$$

Angles denote the quadratic variation of a process or the quadratic covariation of two processes. The last covariation term arises from applying Itô's Lemma to the product of a function of S and a stochastic integral. The particular aspect of this application is that the function of S is actually part of the stochastic integral.

Collecting terms:

$$
\begin{aligned}
dU(t) = {} & \left[\left(\frac{\partial V}{\partial X}(0, S(t)) + \int_0^t \frac{\partial \mu_V}{\partial X}(u, S(t))du + \int_0^t \frac{\partial \sigma_V}{\partial X}(u, S(t))dW(u)\right)S(t)\mu_S(t) \right. \\
& \left. + \frac{1}{2}\left(\frac{\partial^2 V}{\partial X^2}(0, S(t)) + \int_0^t \frac{\partial^2 \mu_V}{\partial X^2}(u, S(t))du + \int_0^t \frac{\partial^2 \sigma_V}{\partial X^2}(u, S(t))dW(u)\right)S(t)^2\sigma_S(t)^2\right.
\end{aligned}
$$

$$+\mu_V\left(t,S(t)\right)+\frac{\partial\sigma_V}{\partial X}(t,S(t))S(t)\sigma_S(t)\Bigg]dt$$

$$+\Bigg[\left(\frac{\partial V}{\partial X}\left(0,S(t)\right)+\int_0^t\frac{\partial\mu_V}{\partial X}(u,S(t))du+\int_0^t\frac{\partial\sigma_V}{\partial X}(u,S(t))dW\left(u\right)\right)S(t)\sigma_S(t)$$

$$+\sigma_V(u,S(t))\Bigg]dW(t). \tag{A.18}$$

or, using the integral form of V:

$$dU\left(t\right)=\Bigg[\frac{\partial V}{\partial X}\left(t,S(t)\right)S(t)\mu_S(t)+\frac{1}{2}\frac{\partial^2 V}{\partial X^2}\left(t,S(t)\right)S(t)^2\sigma_S(t)^2+\mu_V\left(t,S(t)\right)$$

$$+\frac{\partial\sigma_V}{\partial X}(t,S(t))S(t)\sigma_S(t)\Bigg]dt+\Bigg[\frac{\partial V}{\partial X}\left(t,S(t)\right)S(t)\sigma_S(t)+\sigma_V(u,S(t))\Bigg]dW(t). \tag{A.19}$$

The proof can be easily carried to the most general case. $\qquad\square$

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