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# RELATIVE RANK OF THE FINITE FULL TRANSFORMATION SEMIGROUP WITH RESTRICTED RANGE

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ABSTRACT. In this paper, we determine the relative rank of the semigroup $T(X,Y)$ of all transformations on a finite set $X$ with restricted range $Y$ modulo the semigroup of all extensions of the bijections on $Y$, modulo the idempotent order-preserving transformations in $T(X,Y)$, and modulo the semigroup of all order-preserving transformations in $T(X,Y)$.

## 1. INTRODUCTION AND PRELIMINARIES

The rank of a semigroup $S$, denoted rank$(S)$, is the minimum size of a generating set for $S$ [**8**]. The ranks of certain finite semigroups were studied in [**4**]. This concept was generalized in [**9**]. The authors introduced a 'new' rank property, the relative rank of $S$ modulo a subset $A$ of $S$. For a semigroup $S$, if $A \subseteq S$, then we call the minimum size of a set $B$ such that $\langle A \cup B \rangle = S$ the relative rank of $S$ modulo $A$, denoted rank$(S : A)$. In [**5**], the authors considered the relative rank of $T(X)$ modulo the semigroup $O(X)$ of all order-preserving maps on a finite linearly ordered set $X$, i.e., $\text{rank}(T(X) : O(X)) = 2$. The relative rank of $T(X)$ modulo the symmetric group $S(X)$ on a finite set $X$ is 1 [**10**].

Recall that for a finite linearly ordered set $(X; \leq)$, a map $\alpha \in T(X)$ is order preserving if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X$. For a finite set $X$ of size $n$, the semigroup $O(X)$ has been studied extensively. Its order is $\binom{2n-1}{n-1}$, its rank is $n$, it is idempotent generated, and the minimum size of a generating set of $O(X)$ consisting of idempotents, the idempotent rank, is $2n-2$, see [**4**] or [**7**].

The present paper deals with only finite transformation semigroups, i.e., $X$ is finite. The rank of the semigroup $T(X,Y)$ was determined in [**2**]. It is a 'large size'. Hence, we consider the relative rank of $T(X,Y)$ modulo the semigroup $O(X,Y)$ of all order-preserving maps from $X$ in $Y$ on a linearly ordered set $X$ as well as the semigroup $S(X,Y)$ of all maps $\alpha$ from $X$ in $Y$ whose restriction to $Y$ (denoted $\alpha|_Y$) is a bijection on $Y$, i.e., $\alpha|_Y \in S(Y)$. The semigroup $T(X,Y)$ was introduced by J. S. V. Symons in 1975 and called semigroup with restricted range [**12**].

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Let $X$ be a finite set of size $n$ and consider a subset $Y \subseteq X$ of size $m$. The semigroup $T(X,Y)$ was studied in [**1, 12, 6, 10, 11, 12**]. When $X$ is linearly ordered, $O(X,Y)$ can be written as

$$O(X,Y) := O(X) \cap T(X,Y)$$

and has order $\binom{n+m-1}{m-1}$ by the same calculations as in the proof of the order of $O(X)$ in [**7**]. The semigroup $O(X,Y)$ is not idempotent generated. It has rank $\binom{n-1}{m-1}+|Y^{\#}|$, where $Y^{\#}$ is the set of so-called captive elements [**1**].

The semigroup $S(X,Y)$ was firstly mentioned in [**11**]. The authors of this paper consider it as $\mathcal{J}$-class of the semigroup $F(X,Y) := \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\}$, namely $J(F,m) := \{\alpha \in F(X,Y) : |X\alpha| = m\}$. Notice that $J(F,m) = S(X,Y)$, $\text{rank}(J(F,m)) = m^{n-m}$ if $Y$ is a proper subset of $X$, and $\text{rank}(J(F,m)) = 2$ if $X = Y$.

The cardinality of the image of $\alpha$, $\text{im}\alpha := X\alpha$ is called rank of $\alpha$, denoted $\text{rank}\alpha = |\text{im}\alpha|$. The kernel of $\alpha$, denoted $\ker\alpha := \{(x,y) \in X \times X : x\alpha = y\alpha\}$, is an equivalence relation on $X$, which corresponds uniquely to a decomposition of $X$ into blocks, called $\ker\alpha$-classes. This justifies the notation $B \in \ker\alpha$ in case $B$ is $\ker\alpha$-class. Moreover, a set $T \subseteq X$ with $|B \cap T| = 1$ for all $B \in \ker\alpha$, is called transversal of $\ker\alpha$. If we restrict $\alpha$ to $T$, we obtain a map $\alpha|_T$ from $T$ in $Y$ defined by $x(\alpha|_T) := x\alpha$ for all $x \in T$. In particular, it is easy to verify that $\alpha \in S(X,Y)$ if and only if $Y$ is a transversal of $\ker\alpha$.

The rest of this paper is organized in three sections. In the next section, we determine the relative rank of $T(X,Y)$ modulo $S(X,Y)$. As a consequence, we obtain the already known rank of $T(X,Y)$, but here as the sum of the rank of $S(X,Y)$ and the relative rank $T(X,Y)$ modulo $S(X,Y)$. In the second section, we determine the number of idempotents in $O(X,Y)$ and the relative rank of $O(X,Y)$ modulo the set of its idempotents. In the last section, we give the relative rank of $T(X,Y)$ modulo $O(X,Y)$. If $Y = \{y\}$ is a singleton set, then $T(X,Y)$ contains only one element, namely the constant map $c_Y$ mapping all elements of $X$ to $y$. Hence, we drop the case $m = 1$ and assume without loss of generality that

$$X = \{x_1, \ldots, x_n\} \qquad \text{and} \qquad Y = \{x_{i_1}, \ldots, x_{i_m}\} \text{ with } m \geq 2.$$

## 2. The relative rank of $T(X,Y)$ modulo $S(X,Y)$

In this section, we determine the relative rank of $T(X,Y)$ modulo $S(X,Y)$. Since $T(X,Y) \setminus S(X,Y)$ is an ideal [**10**], the rank of $T(X,Y)$ is the sum of the relative rank of $T(X,Y)$ modulo $S(X,Y)$ and the rank of $S(X,Y)$.

Recall that for $1 \leq k \leq n$, the Stirling number of the second kind (or Sterling partition number) is the number of ways to decompose an $n$-element set into $k$ non-empty subsets, denoted $S(n,k)$, where

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

**Proposition 2.1.** *Let $X$ and $Y$ be subsets as in Section 1 with cardinality $n$ and $m$, respectively. The relative rank of $T(X,Y)$ modulo $S(X,Y)$ is $S(n,m)-m^{n-m}$.*

*Proof.* Let $\mathcal{D} := \{\ker\beta : \beta \in T(X,Y)\setminus S(X,Y), \operatorname{rank}\beta = m\}$. For each $D \in \mathcal{D}$, we choose a transformation $\alpha_D \in T(X,Y)$ with $\operatorname{im}\alpha_D = Y$ and $\ker\alpha_D = D$. It is easy to see that $\mathcal{D}$ is the set of all decompositions $D$ of $X$ into $m$ non-empty sets such that $Y$ is not a transversal of $D$. In order to calculate the cardinality of $D$, we need to count the ways to decompose the $n$-element set $X$ into $m$ non-empty subsets having $Y$ as transversal. This is a simple combinatorial problem: We have to distribute the elements of the set $X \setminus Y$ to the elements of $Y$. There are exactly $m^{n-m}$ ways to do this. Hence, $|\mathcal{D}| = S(n,m) - m^{n-m}$.

We will show that $S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}$ generates $T(X,Y)$. If this is the case, then $\operatorname{rank}(T(X,Y) : S(X,Y)) \leq S(n,m) - m^{n-m}$.

Let $\gamma \in T(X,Y) \setminus S(X,Y)$ with $\operatorname{rank}\gamma = m$. Then there is $D \in \mathcal{D}$ with $\ker\gamma = \ker\alpha_D = D$. Let $x_D \in x\alpha_D^{-1}$ for $x \in Y$. Then define $\theta$ from $X$ to $Y$ by

$$x\theta := \begin{cases} x_D\gamma & \text{if } x \in Y, \\ x_{i_1} & \text{otherwise.} \end{cases}$$

It is easy to see that $Y$ is a transversal of $\ker\theta$, i.e., $\theta \in S(X,Y)$. For $x \in X$. we have $x\alpha_D\theta = (x\alpha_D)_D\gamma = x\gamma$ since $(x\alpha_D)_D \in x\alpha_D\alpha_D^{-1}$ is in the $\ker\alpha_D$-class of $x$, which is also a $\ker\gamma$-class (since $\ker\gamma = \ker\alpha_D$). Therefore, $\gamma = \alpha_D\theta \in \langle S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}\rangle$.

Suppose now that $\{\beta \in T(X,Y) : \operatorname{rank}\beta = p\} \subseteq \langle S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}\rangle$ for some $p \leq m$ and take $\gamma \in T(X,Y)$ with $\operatorname{rank}\gamma = p-1$. We can assume without loss of generality that

$$\operatorname{im}\gamma = \{x_{i_1}, \dots, x_{i_{p-1}}\}.$$

Then there is $k \in \{1,\dots,p-1\}$ with $\left|x_{i_k}\gamma^{-1}\right| \geq 2$ and decompose $x_{i_k}\gamma^{-1}$ into two non-empty sets ($C_1$ and $C_2$, say). Define $\alpha$ from $X$ to $Y$ by

$$x\alpha := \begin{cases} x\gamma & \text{if } x \notin x_{i_k}\gamma^{-1}, \\ x_{i_k} & \text{if } x \in C_1, \\ x_{i_p} & \text{if } x \in C_2. \end{cases}$$

Clearly, $\operatorname{rank}\alpha = p$ and thus $\alpha \in \langle S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}\rangle$ by assumption. Now define $\delta$ from $X$ to $Y$ by

$$x\delta := \begin{cases} x & \text{if } x \in \operatorname{im}\gamma \setminus \{x_{i_k}\}, \\ x_{i_k} & \text{if } x \in \{x_{i_k}, x_{i_p}\}, \\ x_{i_p} & \text{if } x \notin (\operatorname{im}\gamma \cup \{x_{i_p}\}). \end{cases}$$

Clearly, $\operatorname{rank}\delta = p$ and thus $\delta \in \langle S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}\rangle$ by assumption. For $x \in X \setminus x_{i_k}\gamma^{-1}$, we have $x\alpha = x\gamma \in \operatorname{im}\gamma \setminus \{x_{i_k}\}$ and thus $x\alpha\delta = x\gamma$. For $x \in x_{i_k}\gamma^{-1}$, we have $x\alpha \in \{x_{i_k}, x_{i_p}\}$ and thus $x\alpha\delta = x_{i_k} = x\gamma$. This shows that $\gamma = \alpha\delta \in \langle S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}\rangle$. Thus, we have $T(X,Y) \subseteq \langle S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}\rangle$.

Next, we will show if $B \subseteq T(X,Y)$ with $\langle S(X,Y) \cup B\rangle = T(X,Y)$, then $\mathcal{D} \subseteq \{\ker\beta : \beta \in B\}$. If it is the case, then $\operatorname{rank}(T(X,Y) : S(X,Y)) \geq |\mathcal{D}| = S(m,n) - m^{n-m}$ and altogether, we have proved the equality as required. Assume that $\mathcal{D} \not\subseteq \{\ker \beta : \beta \in B\}$. Then there is $D \in \mathcal{D}$ such that $\ker \beta \neq D$ for all $\beta \in B$. From $\alpha_D \in \langle S(X,Y) \cup B \rangle$, it follows the existence of $\theta_1 \in S(X,Y) \cup B$ and $\theta_2 \in T(X,Y)$ with $\alpha_D = \theta_1\theta_2$. Thus $D = \ker \alpha_D = \ker \theta_1\theta_2 = \ker \theta_1$ since $\operatorname{im} \alpha_D = Y$. But $\ker \theta_1 = D$ implies $\theta_1 \notin B$. Hence, $\theta_1 \in S(X,Y)$ and $Y$ is a transversal of $\ker \theta_1$. This contradicts $\ker \theta_1 = D \in \mathcal{D}$. □

**Corollary 2.2.** $\operatorname{rank}(T(X,Y)) = S(n,m)$.

*Proof.* It follows from Proposition 2.1 and the fact that the rank of $S(X,Y)$ equals $m^{n-m}$. □

## 3. The relative rank of $O(X,Y)$ modulo its idempotents

In this section, we assume that $X$ is linearly ordered. Set

$$X = \{x_1 < \cdots < x_n\}$$

and $Y$ a non-empty subchain with $m$ elements, say

$$Y = \{x_{i_1} < \cdots < x_{i_m}\}.$$

Let $EO(X,Y)$ be the set of all idempotent order-preserving transformations in $T(X,Y)$. For a subchain $P = \{x_{p_1} < x_{p_2} < \cdots < x_{p_k}\}$ of $Y$ of size $k$, we define $g_P$ from $\{0, 1, \ldots, k-1\}$ in $\{1, \ldots, n\}$ by $g_P(l) := p_{l+1} - p_l$ for $l \in \{1, \ldots, k-1\}$ and $g_P(0) := 1$.

Next proposition determines how many idempotents with rank $m$ are in the semigroup $O(X,Y)$.

**Proposition 3.1.** $|EO(X,Y)| = \displaystyle\sum_{\emptyset \neq P \subseteq Y} \prod_{l=0}^{|P|-1} g_P(l)$.

*Proof.* Let $P$ be a non-empty subchain of $Y$. If $|P| = 1$, then there is exactly one idempotent with image $P$, where $\prod_{l=0}^{0} g_P(l) = g_P(0) = 1$. Admit $|P| \geq 2$ and let $l \in \{1, 2, \ldots, |P|-1\}$. Then there are $p_{l+1} - p_l$ ways to decompose $\{x_{p_l}, x_{p_l+1}, \ldots, x_{p_{l+1}}\}$ into two non-empty sets $C_1 < C_2$ ($x_1 \in C_1$ and $x_2 \in C_2$ implies $x_1 < x_2$). From the definition of $g_P$, we have $|\{\beta \in EO(X,Y) : \operatorname{rank} \beta = |P|\}| = g_P(0)g_P(1)\ldots g_P(|P|-1) = \prod_{l=0}^{|P|-1} g_P(l)$. Considering all non-empty subsets $P \subseteq Y$, we obtain the assertion. □

We consider the set

$$A(X,Y) := \{\beta \in O(X,Y) : \beta \notin EO(X,Y),\ \operatorname{rank} \beta = m\}.$$

**Lemma 3.2.** $\langle EO(X,Y) \cup A(X,Y) \rangle = O(X,Y)$.

*Proof.* We reason by induction on the rank of any $\gamma \in O(X,Y)$. Let $\operatorname{rank} \gamma = m$. Then $\gamma \in EO(X,Y) \cup A(X,Y)$, i.e., $\gamma \in \langle EO(X,Y) \cup A(X,Y) \rangle$.

Suppose that $\gamma \in \langle EO(X,Y) \cup A(X,Y) \rangle$ whenever $\gamma \in O(X,Y)$ with $\operatorname{rank} \gamma = k$ for some $k \leq m$, and let $\gamma \in O(X,Y)$ with $\operatorname{rank} \gamma = k-1$. We put $P := \operatorname{im} \gamma = \{x_{p_1} < x_{p_2} < \cdots < x_{p_{k-1}}\}$. Then there is $i \in \{1, \ldots, k-1\}$ with $\left|x_{p_i}\gamma^{-1}\right| \geq 2$. Moreover, there is $j \in \{1, \ldots, k-1\}$ with $x_{p_j+1} \notin P$ or $.x_{p_j-1} \notin P$. Suppose without loss of generality $x_{p_j+1} \notin P$. We decompose $x_{p_i}\gamma^{-1}$ into two non-empty sets $C_1 < C_2$ and put

$$\begin{aligned} D_l &:= x_{p_l}\gamma^{-1} && \text{for } 1 \leq l < i, \\ D_i &:= C_1 \\ D_{i+1} &:= C_2 \\ D_l &:= x_{p_{l-1}}\gamma^{-1} && \text{for } i+2 \leq l \leq k, \text{ as well as} \\ y_l &:= x_{p_l} && \text{for } 1 \leq l \leq j, \\ y_{j+1} &:= x_{p_j+1} \\ y_l &:= x_{p_{l-1}} && \text{for } j+1 < l \leq k. \end{aligned}$$

We define $\theta$ from $X$ to $Y$ by

$$x\theta := y_l \text{ if } x \in D_l,\ 1 \leq l \leq k.$$

It is easy to verify that $\theta$ is order-preserving with rank $k$. Hence, $\theta \in \langle EO(X,Y) \cup A(X,Y)\rangle$.
Suppose that $i = j$. Then we define $\varepsilon$ from $X$ to $Y$ by

$$x\varepsilon := \begin{cases} y_1 & \text{if } x < y_1, \\ y_l & \text{if } y_l \leq x < y_{l+1};\ 1 \leq l < i, \\ y_i & \text{if } y_i \leq x \leq y_{i+1}, \\ y_l & \text{if } y_{l-1} < x \leq y_l;\ i+1 < l \leq k, \\ y_k & \text{if } x > y_k. \end{cases}$$

It is easy to see that $\varepsilon$ is order-preserving as well as idempotent and thus $\varepsilon \in \langle EO(X,Y) \cup A(X,Y)\rangle$. Let $x \in D_l$ for some $l \in \{1, \ldots, k\}$. If $l < i$, then $x\theta\varepsilon = y_l = x_l = x\gamma$. If $l \in \{i, i+1\}$, then $x\theta\varepsilon = y_l\varepsilon = y_i = x_i = x\gamma$. And if $l > i+1$, then $x\theta\varepsilon = y_l\varepsilon = y_l = x_{l-1} = x\gamma$. This shows that $\gamma = \theta\varepsilon \in \langle EO(X,Y) \cup A(X,Y)\rangle$.

We consider now $j < i$. The case $j > i$ works analogously. For $0 \leq r \leq i-j-1$, we define $\varepsilon_r$ from $X$ in $Y$ by

$$x\varepsilon_r := \begin{cases} y_1 & \text{if } x < y_1, \\ y_l & \text{if } y_l \leq x < y_{l+1}; 1 \leq l < i-r, \\ y_{i-r+1} & \text{if } y_{i-r} \leq x \leq y_{i-r+1}, \\ y_l & \text{if } y_{l-1} < x \leq y_l; i-r+1 < l \leq k, \\ y_k & \text{if } x > y_k. \end{cases}$$

It is easy to see that $\varepsilon_r$ is order-preserving as well as idempotent and thus $\varepsilon_r \in \langle EO(X,Y) \cup A(X,Y)\rangle$. Let $x \in X$. If $x \in D_l$, $1 \leq l \leq j$, then $x\theta\varepsilon_0\varepsilon_1 \ldots \varepsilon_{i-j-1} = y_l\varepsilon_0\varepsilon_1 \ldots \varepsilon_{i-j-1} = y_l = x_l = x\gamma$. If $x \in D_l$, $i+2 \leq l \leq k$, then $x\theta\varepsilon_0\varepsilon_1 \ldots \varepsilon_{i-j-1} = y_l\varepsilon_0\varepsilon_1 \ldots \varepsilon_{i-j-1} = y_l = x_{l-1} = x\gamma$. If $x \in D_l$, $j+1 \leq l \leq i$, then

$$\begin{aligned} x\theta\varepsilon_0\varepsilon_1 \ldots \varepsilon_{i-j-1} &= y_l(\varepsilon_0\varepsilon_1 \ldots \varepsilon_{i-l})\varepsilon_{i-l+1} \ldots \varepsilon_{i-j-1} = y_l\varepsilon_{i-l}(\varepsilon_{i-l+1} \ldots \varepsilon_{i-j-1}) \\ &= y_{l+1}(\varepsilon_{i-l+1} \ldots \varepsilon_{i-j-1}) = y_{l+1} = x_l = x\gamma. \end{aligned}$$

If $x \in D_{l+1}$, then $x\theta\varepsilon_0\varepsilon_1 \dots \varepsilon_{i-j-1} = y_{i+1}\varepsilon_0\varepsilon_1 \dots \varepsilon_{i-j-1} = y_{i+1} = x_i = x\gamma$. As a consequence of the above reasoning, we get that $\gamma = \theta\varepsilon_0\varepsilon_1 \dots \varepsilon_{i-j-1} \in \langle EO(X,Y) \cup A(X,Y)\rangle$. □

Next lemma shows the size of $A(X,Y)$.

**Lemma 3.3.** $|A(X,Y)| = \binom{n-1}{m-1} - \prod_{l=0}^{m-1} g_P(l)$.

*Proof.* Any $\alpha \in O(X,Y)$ with $\operatorname{rank}\alpha = m$ has image $Y$ and it is uniquely determined by its kernel $\ker\alpha$. But this kernel is uniquely determined by the least elements in each $\ker\alpha$-class. Note that $x_1$ is mapped to $x_{i_1}$. Hence, there are $\binom{n-1}{m-1}$ ways to create this kernel $\ker\alpha$. Any $\alpha \in O(X,Y)$ with $\operatorname{rank}\alpha = m$ is idempotent if $Y$ is a transversal of $\ker\alpha$, then there are $\prod_{l=0}^{m-1} g_P(l)$ ways to create this $\ker\alpha$ as already shown in the proof of Proposition 3.1. Hence, $|A(X,Y)| = |\{\beta \in O(X,Y) : \beta \notin EO(X,Y),\ \operatorname{rank}\beta = m\}| = \binom{n-1}{m-1} - \prod_{l=0}^{m-1} g_P(l)$. □

We are now in the position to present the main result of this section.

**Theorem 3.4.** *With the above notations, the relative rank of the semigroup of order-preserving transformations $O(X,Y)$ modulo its idempotent elements satisfies*

$$\operatorname{rank}(O(X,Y) : EO(X,Y)) = \binom{n-1}{m-1} - \prod_{l=0}^{m-1} g_P(l).$$

*Proof.* By the Lemma 3.2 and 3.3, we have

$$\operatorname{rank}(O(X,Y) : EO(X,Y)) \leq |A(X,Y)| = \binom{n-1}{m-1} - \prod_{l=0}^{m-1} g_P(l).$$

For the equality, we show that each set $B \subseteq O(X,Y)$ with $\langle EO(X,Y) \cup B\rangle = O(X,Y)$ contains $A(X,Y)$.

Assume there is a set $B$ as above but $A(X,Y) \not\subseteq B$, i.e., there is $\gamma \in A(X,Y) \setminus B$. Since $\gamma \in O(X,Y) = \langle EO(X,Y) \cup B\rangle$, there are $\theta_1 \in EO(X,Y) \cup B$ and $\theta_2 \in O(X,Y)$ with $\gamma = \theta_1\theta_2$. We have $\ker\gamma = \ker\theta_1$ because $\operatorname{rank}\gamma = m$. Note that the elements in $A(X,Y)$ are uniquely, determined by their kernels. So we can conclude $\theta_1 = \gamma \in A(X,Y)$, i.e., $\theta_1 = \gamma \notin B$ and $\gamma = \theta_1 \in EO(X,Y)$, a contradiction. □

## 4. Relative rank of $T(X,Y)$ modulo $O(X,Y)$

This section is devoted to state and prove the main result of this paper: the computation the relative rank of $T(X,Y)$ modulo the semigroup of all order-preserving transformations in $T(X,Y)$. In the case $X = Y$, it is two. Set $X$ and $Y$ as in Section 3. We define

$$\mathcal{M} := \{\ker\beta : \beta \in T(X,Y),\ \operatorname{rank}\beta = m\} \setminus \{\ker\beta : \beta \in O(X,Y),\ \operatorname{rank}\beta = m\}.$$

**Lemma 4.1.** $|\mathcal{M}| = S(n,m) - \binom{n-1}{m-1}$.

*Proof.* It follows from $|\{\ker\beta : \beta \in O(X,Y),\ \operatorname{rank}\beta = m\}| = \binom{n-1}{m-1}$ (see proof of Lemma 3.3) and the fact that $|\{\ker\beta : \beta \in T(X,Y),\ \operatorname{rank}\beta = m\}| = S(n,m)$. □

For each $M \in \mathcal{M}$, we choose $\alpha_M \in T(X,Y)$ with $\operatorname{im}\alpha_M = Y$ and $\ker\alpha_M = M$. Clearly, $|\{\alpha_M : M \in \mathcal{M}\}| = S(n,m) - \binom{n-1}{m-1}$. Note for all $s \in S(Y)$ there is $\mu_S \in T(X,Y) \setminus O(X,Y)$ with $\mu_S|_Y = s$ whenever $s$ is not the identity map on $Y$.

**Lemma 4.2.** *If* $S \subseteq S(Y)$ *with* $\langle S\rangle = S(Y)$, *then*

$$T(X,Y) = \langle O(X,Y) \cup \{\mu_S : s \in S\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle .$$

*Proof.* Let $\gamma \in T(X,Y)$. Suppose that $\operatorname{rank}\gamma = m$. Then there is $\delta \in O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}$ with $\ker\delta = \ker\gamma$. For $x \in Y$, we choose $x_\delta \in x\delta^{-1}$ and define map $\theta$ from $X$ to $Y$ by

$$x\theta := \begin{cases} x_\delta\gamma & \text{if } x \in Y \\ x_{i_1} & \text{otherwise.} \end{cases}$$

If $a, b \in Y$ with $a\theta = b\theta$, then $a_\delta\gamma = b_\delta\gamma$, $a\delta^{-1}\gamma = b\delta^{-1}\gamma$, and $a = b$ since $\ker\delta = \ker\gamma$. Hence, $Y$ is a transversal of $\ker\theta$, i.e., $\theta|_Y \in S(Y) = \langle S\rangle$. That means that there is $\mu \in \langle\{\mu_S : s \in S\}\rangle$ with $\theta|_Y = \mu|_Y$.

Let $x \in X$. Then we have $x\delta\mu = x\delta\theta = (x\delta)_\delta\gamma = x\gamma$. This shows $\gamma = \delta\mu \in \langle O(X,Y) \cup \{\mu_S : s \in S\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle$.

Suppose now that $\gamma \in \langle O(X,Y) \cup \{\mu_S : s \in S\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle$ whenever $\operatorname{rank}\gamma = p$ for some $p \leq m$ and let $\operatorname{rank}\gamma = p-1$ with

$$\operatorname{im}\gamma = \{x_{j_1} < \cdots < x_{j_{p-1}}\} \subseteq Y.$$

Then there is $i \in \{1, 2, \ldots, p-1\}$ with $\left|x_{j_i}\gamma^{-1}\right| \geq 2$ and we decompose $x_{j_i}\gamma^{-1}$ into two sets $M_i < M_{i+1}$. So, there is $z \in Y \setminus \operatorname{im}\gamma$ and set

$$\begin{aligned} M_l &:= x_{j_l}\gamma^{-1} && \text{and } y_l := x_{j_l} \text{ for } 1 \leq l < i, \\ M_l &:= x_{j_{(l-1)}}\gamma^{-1} && \text{and } y_l := x_{j_{(l-1)}} \text{ for } i+1 < l \leq p, \\ y_i &:= x_{j_i} && \text{and } y_{i+1} := z. \end{aligned}$$

Now we define map $\alpha$ from $X$ to $Y$ by

$$x\alpha := y_l \qquad \text{if } x \in M_l,\ 1 \leq l \leq p.$$

For $x \in \operatorname{im}\gamma \cup \{z\}$, we choose $x^\alpha \in x\alpha^{-1}$ and define $\eta$ from $X$ to $Y$ by

$$x\eta := \begin{cases} x^\alpha\gamma & \text{if } x \in \{y_1, \ldots, y_{i-1}, y_{i+2}, \ldots, y_p\} \\ y_i & \text{if } x \in \{y_i, y_{i+1}\} \\ z & \text{if } x \in X \setminus \{y_1, \ldots y_p\}. \end{cases}$$

Notice that $X \setminus \{y_1, \ldots y_p\} \neq \emptyset$ since $p \leq m < n$. Hence, it is easy to verify that both $\alpha$ and $\eta$ have rank $p$ and thus

$$\alpha, \eta \in \langle O(X,Y) \cup \{\mu_S : s \in S\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle .$$

Let $x \in M_l$ for some $1 \leq l \leq p$. If $l \neq i$ and $l \neq i+1$, then $x\alpha\eta = (x\alpha)^\alpha\gamma = x\gamma$. If $l = i$ or $l = i+1$, then $x\alpha\eta = y_i = x_{j_i} = x\gamma$. This shows that

$\gamma = \alpha\eta \in \langle O(X,Y) \cup \{\mu_S : s \in S\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle$. The above reasoning proves the assertion $T(X,Y) \subseteq \langle O(X,Y) \cup \{\mu_S : s \in S\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle$. □

**Lemma 4.3.** *If $A \subseteq T(X,Y) \setminus O(X,Y)$ with $\langle O(X,Y) \cup A\rangle = T(X,Y)$ then $\mathcal{M} \subseteq \{\ker\beta : \beta \in A\}$.*

*Proof.* Let $A \subseteq T(X,Y) \setminus O(X,Y)$ with $\langle O(X,Y) \cup A\rangle = T(X,Y)$. Assume that there is $M \in \mathcal{M}$ with $M \notin \{\ker\beta : \beta \in A\}$. Since $\alpha_M \in T(X,Y) = \langle O(X,Y) \cup A\rangle$, there is an element $\theta_1 \in O(X,Y) \cup A$ and $\theta_2 \in T(X,Y)$ such that $\alpha_M = \theta_1\theta_2$. Because $\operatorname{rank}\alpha_M = m$, we obtain $\ker\alpha_M = \ker\theta_1$, i.e., $\ker\theta_1 = M$. Hence, $\theta_1 \notin A$ (by assumption) and $\theta_1 \notin O(X,Y)$ (since $M \notin \{\ker\beta : \beta \in O(X,Y)\}$), a contradiction. □

We define the following subset $P^*(X)$ of the power set $P(X)$ of $X$:
If $|X| \geq 5$, then $P^*(X) := P(X) \setminus (\{\emptyset, X\} \cup \{\{x\} : x \in X\})$,
if $|X| = 4$, then $P^*(X) := \{Y \subseteq X : |Y| \geq 2, |X \setminus Y| = 2$ or $\{x_2, x_3\} \subseteq Y\}$ and
if $|X| = 3$, then $P^*(X) := \{Y \subseteq X : |Y| = 2, x_2 \in Y\}$.
We call two elements $a, b \in X$ to be neighbors if $a$ is immediate successor or predecessor of $b$.

**Theorem 4.4.** *With the previous notations, assume that $Y \in P^*(X)$. Then*

$$\operatorname{rank}(T(X,Y) : O(X,Y)) = S(n,m) - \binom{n-1}{m-1}.$$

*Proof.* If $|X| \geq 5$ or $|X| = 4$ and $\{x_2, x_3\} \subseteq Y$, then there are $x \in X \setminus Y$ and $y_1, y_2 \in Y$ such that neither $y_1$ nor $y_2$ is neighbor of $x$. So we can put $M_i := \{\{r\} : r \in Y \setminus \{y_i\}\} \cup \{X \setminus (Y \setminus \{y_i\})\}$, $i = 1, 2$. It is easy to verify that $Y$ is transversal of $M_1$ as well as of $M_2$. Moreover, $M_1, M_2 \notin \{\ker\beta : \beta \in O(X,Y)\}$, i.e., $M_1, M_2 \in \mathcal{M}$. It is well known that the symmetric group $S(Y)$ on $Y$ is generated by two bijections ($s_1$ and $s_2$, say). We can assume without loss of generality that $\alpha_{M_1}|_Y = s_1$ and $\alpha_{M_2}|_Y = s_2$, i.e., $\mu_{s_1} = \alpha_{M_1}$ and $\mu_{s_2} = \alpha_{M_2}$.

If $|X| = 4$ and $|X \setminus Y| = 2$ or $|X| = 3$ and $x_2 \in Y_2$ then $|Y| = 2$. Here there are $x \in X \setminus Y$ and $y \in Y$ such that $x$ is not neighbor of $y$. Then we put $M_3 := \{\{r\} : r \in Y \setminus \{y\}\} \cup \{X \setminus (Y \setminus \{y\})\}$. It is easy to verify that $Y$ is transversal of $M_3$ and $M_3 \notin \{\ker\beta : \beta \in O(X,Y)\}$. Thus $M_3 \in \mathcal{M}$. The symmetric group $S(Y)$ on the two-element set $Y$ is generated by one bijection, say $s$. We can assume without loss of generality that $\alpha_{M_3}|_Y = s$, i.e., $\mu_s = \alpha_{M_3}$.

The above fact shows that there is $S \subseteq S(Y)$ with $\langle S\rangle = S(Y)$ such that $\{\mu_S : s \in S\} \subseteq \{\alpha_M : M \in \mathcal{M}\}$. Now we can use Lemma 4.2. It provides that $T(X,Y) = \langle O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}\rangle$ and thus $\operatorname{rank}(T(X,Y) : O(X,Y)) \leq |\{\alpha_M : M \in \mathcal{M}\}| = S(n,m) - \binom{n-1}{m-1}$ by Lemma 4.1. On the other hand, Lemma 4.3 shows that the minimum size of a relative generating set modulo $O(X,Y)$ is $|\mathcal{M}| = S(n,m) - \binom{n-1}{m-1}$ and altogether, we obtain the assertion $\operatorname{rank}(T(X,Y) : O(X,Y)) = S(n,m) - \binom{n-1}{m-1}$. □

**Theorem 4.5.** *If $Y \notin P^*(X)$, then*

$$\operatorname{rank}(T(X,Y) : O(X,Y)) = S(n,m) - \binom{n-1}{m-1} + 1.$$

*Proof.* If $|X| = 4$, $|X \setminus Y| = 1$, and $\{x_2, x_3\} \nsubseteq Y$, then there is exactly one pair $(x, y) \in (X \setminus Y) \times Y$ such that $x$ and $y$ are not neighbors. Then $M_1 := \{\{r\} : r \in Y \setminus \{y\}\} \cup \{\{x, y\}\}$ is the only element $M \in \mathcal{M}$ with $Y$ is transversal of $M$ and $M \notin \{\ker \beta : \beta \in O(X,Y)\}$. Note that the symmetric group $S(Y)$ is generated by two bijections ($s_1$ and $s_2$, say). We can assume without loss of generality that $\alpha_{M_1}|_Y = s_1$, i.e., $\mu_{s_1} = \alpha_{M_1}$. Then $\mu_{s_2} \notin O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}$.

If $|X| = 3$ and $x_2 \notin Y$, then $|Y| = 2$ and there is no $M \in \mathcal{M}$ with $Y$ is transversal of $M$ and $M \notin \{\ker \beta : \beta \in O(X,Y)\}$. The 2-element symmetric group $S(Y)$ is generated by one $s \in S(Y)$, where $\mu_s \notin O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}$.

We put $\alpha := \mu_{s_2}$ and $\alpha := \mu_s$, respectively. We can apply Lemma 4.2 and obtain $T(X,Y) = \langle O(X,Y) \cup \{\alpha\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle$ and thus

$$\operatorname{rank}(T(X,Y) : O(X,Y)) \leq |\{\alpha_M : M \in \mathcal{M}\} \cup \{\alpha\}| = S(n,m) - \binom{n-1}{m-1} + 1.$$

Let $A \subseteq T(X,Y) \setminus O(X,Y)$ with $\langle O(X,Y) \cup A\rangle = T(X,Y)$, then $\mathcal{M} \subseteq \{\ker \beta : \beta \in A\}$ by Lemma 4.3. If $\gamma \in O(X,Y)$ then $Y$ is not a transversal of $\ker \gamma$ or $\gamma|_Y$ is the identity map on $Y$. Hence, there is $S \subseteq A$ such that $\{s|_Y : s \in S\} = S(Y)$.

If $|X| = 4$ and $|X \setminus Y| = 1$ and $\{x_2, x_3\} \nsubseteq Y$ then $S$ contains at least two elements ($\mu_1$ and $\mu_2$, say). Note that $Y$ is a transversal of $\ker \mu_1$ as well as of $\ker \mu_2$ and $\ker \mu_1, \ker \mu_2 \notin \{\ker \beta : \beta \in O(X,Y)\}$. But we have only one $M \in \mathcal{M}$ with $Y$ is a transversal of $M$ and $M \notin \{\ker \beta : \beta \in O(X,Y)\}$. Hence, we need one additional element in $A$ which is not in $O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}$. Hence,

$$|A| \geq |\mathcal{D}| + 1 = S(n,m) - \binom{n-1}{m-1} + 1.$$

If $|X| = 3$ and $x_2 \notin Y$, then $|Y| = 2$ and $S(Y)$ is a cyclic group with one generator, i.e., $S$ has to contain one element, say $s$. But in this case $\mu_S \notin O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}$, we need one additional element in $A$ which is not in $O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}$. Hence,

$$|A| \geq |\mathcal{M}| + 1 = S(n,m) - \binom{n-1}{m-1} + 1.$$

□

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