

Relativistic classical integrable tops and quantum R-matrices

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ABSTRACT: We describe classical top-like integrable systems arising from the quantum exchange relations and corresponding Sklyanin algebras. The Lax operator is expressed in terms of the quantum non-dynamical R -matrix even at the classical level, where the Planck constant plays the role of the relativistic deformation parameter in the sense of Ruijsenaars and Schneider (RS). The integrable systems (relativistic tops) are described as multidimensional Euler tops, and the inertia tensors are written in terms of the quantum and classical R -matrices. A particular case of \mathfrak{gl}_N system is gauge equivalent to the N -particle RS model while a generic top is related to the spin generalization of the RS model. The simple relation between quantum R -matrices and classical Lax operators is exploited in two ways. In the elliptic case we use the Belavin's quantum R -matrix to describe the relativistic classical tops. Also by the passage to the noncommutative torus we study the large N limit corresponding to the relativistic version of the nonlocal 2d elliptic hydrodynamics. Conversely, in the rational case we obtain a new \mathfrak{gl}_N quantum rational non-dynamical R -matrix via the relativistic top, which we get in a different way — using the factorized form of the RS Lax operator and the classical Symplectic Hecke (gauge) transformation. In particular case of \mathfrak{gl}_2 the quantum rational R -matrix is 11-vertex. It was previously found by Cherednik. At last, we describe the integrable spin chains and Gaudin models related to the obtained R -matrix.

KEYWORDS: Integrable Equations in Physics, Quantum Groups, Gauge Symmetry

ARXIV EPRINT: [1405.7523](https://arxiv.org/abs/1405.7523)

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1 Introduction

We start with the quantum exchange relations [40–45] for the quantum \mathfrak{gl}_N -valued L -operators:

$$R_{12}^h(z-w) \hat{L}_1^\eta(z) \hat{L}_2^\eta(w) = \hat{L}_2^\eta(w) \hat{L}_1^\eta(z) R_{12}^h(z-w), \quad (1.1)$$

where the quantum non-dynamical R -matrix satisfies the quantum Yang-Baxter equation

$$R_{12}^h(z-w) R_{13}^h(z) R_{23}^h(w) = R_{23}^h(w) R_{13}^h(z) R_{12}^h(z-w) \quad (1.2)$$

and unitarity condition

$$R_{12}^h(z) R_{21}^h(-z) = f^h(z) 1 \otimes 1 \quad (1.3)$$

with some function $f^h(z)$.

In this paper we consider a class of solutions of (1.1) and (1.2) having simple pole at $z = 0$ and satisfying relation

$$\hat{L}^\eta(z) = \text{tr}_2 \left(R_{12}^\eta(z) \hat{S}_2 \right), \quad \hat{S} = \text{Res}_{z=0} \hat{L}^\eta(z), \quad (1.4)$$

where \hat{S} is \mathfrak{gl}_N -valued operator. Then (1.1) leads to (quadratic) Sklyanin algebra [72, 73] for \hat{S} which we denote as $\mathcal{A}_{\hbar,\eta}^{\text{skl}}$. Notice here that we use two parameters \hbar and η in (1.1) (it is customary to consider $\eta = \hbar$). In fact, one can even eliminate the η -dependence (see (1.20)) but we will see that it is useful to keep two free parameters from the very beginning.

In the classical limit $\hbar \rightarrow 0$ the matrix components of the residue \hat{S} become \mathbb{C} -valued coordinates on the phase space of an integrable system described by the Lax matrix $L^\eta(z)$ (it coincides with $\hat{L}^\eta(z)$, where \hat{S} is replaced with $\mathfrak{gl}(N, \mathbb{C})$ -valued S)

$$L^\eta(z) = \text{tr}_2 (R_{12}^\eta(z) S_2) , \quad S = \text{Res}_{z=0} L^\eta(z) \tag{1.5}$$

and the standard quadratic r -matrix structure:

$$\{L_1^\eta(z), L_2^\eta(w)\} = [L_1^\eta(z) L_2^\eta(w), r_{12}(z-w)] \tag{1.6}$$

with the classical r -matrix $r_{12}(z)$. We call this type of models *relativistic integrable tops* because η will be shown to play the role of the relativistic deformation parameter in the sense of Ruijsenaars and Schneider [69, 70]. The underlying Poincaré invariance is discussed in section 5.2.

Thus, when $\eta = \hbar$ we have simple relation (1.5) between the classical Lax operator and the quantum R -matrix, i.e. having quantum R -matrix we can define the classical integrable system. Write R -matrix in the standard \mathfrak{gl}_N basis $(E_{ij})_{ab} = \delta_{ia} \delta_{jb}$ as

$$R_{12}^\hbar(z) = \sum_{i,j,k,l=1}^N R_{ij,kl}^\hbar(z) E_{ij} \otimes E_{kl} . \tag{1.7}$$

Then it follows from (1.5) that

$$L^\eta(z) = L^\eta(z, S) = \sum_{i,j,k,l=1}^N R_{ij,kl}^\eta(z) E_{ij} S_{lk} . \tag{1.8}$$

The latter leads to the converse statement, i.e. having the classical Lax matrix $L^\eta(z)$ we can find the quantum R -matrix as

$$R_{12}^\hbar(z) = \frac{\partial L_1^\hbar(z)}{\partial S_2} = \sum_{k,l=1}^N \frac{\partial L^\hbar(z)}{\partial S_{lk}} \otimes E_{kl} \tag{1.9}$$

The purpose of the paper is twofold. The first one is to give description of the relativistic classical tops arising from the quantum R -matrices following (1.5). The second — is to derive new rational quantum R -matrix from the corresponding relativistic top via (1.9), which we obtain in a different way - by applying gauge transformation of Hecke type [6, 47–53, 75] to the rational Ruijsenaars-Schneider (RS) model [69, 70].

1. Relativistic classical tops from quantum R -matrices. Using local expansion of L -operator and R -matrix near $z = 0$ we get equations of motion related to the Hamiltonian $\mathcal{S}_0 = \text{tr}(S)$ for the relativistic top in the form:

$$\partial_{t_0} S = \{\mathcal{S}_0, S\} \stackrel{(1.5)-(1.6)}{=} [S, J^\eta(S)], \tag{1.10}$$

where the inverse inertia tensor J^η is the following linear functional:

$$J^\eta(S) = \text{tr}_2 \left(\left(R_{12}^{\eta,(0)} - r_{12}^{(0)} \right) S_2 \right) \tag{1.11}$$

with $R_{12}^{\eta,(0)}$ and $r_{12}^{(0)}$ be zero terms in the expansions (2.3), (2.13) near $z = 0$. See examples (3.36) and (1.23). These equations are presented in the Lax form

$$\partial_{t_0} L^\eta(z) = \{\mathcal{S}_0, L^\eta(z)\} \stackrel{(1.5)-(1.6)}{=} [L^\eta(z), M(z)] \tag{1.12}$$

with M -operator defined in terms of the classical r -matrix:

$$M(z) = -\text{tr}_2 (r_{12}(z) S_2). \tag{1.13}$$

The latter M -operator appears to be equal (up to sign) to the non-relativistic limit of the Lax matrix $L^\eta(z)$:

$$L^\eta(z) = \eta^{-1} \frac{\mathcal{S}_0}{N} 1_{N \times N} + l(z) + \eta \mathcal{M}(z) + O(\eta^2), \quad l(z) = \text{tr}_2 (r_{12}(z) S_2) = -M(z). \tag{1.14}$$

Moreover, the next term in the expansion is the M -operator

$$\partial_t l(z, S) = [l(z, S), \mathcal{M}(z, S)] \tag{1.15}$$

of the non-relativistic top given by equation

$$\partial_t S = [S, J(S)], \quad J(S) = \mathcal{M}(0, S), \tag{1.16}$$

where $\mathcal{M}(0)$ is the non-relativistic limit of $J^\eta(S)$ (1.11).

The model (1.16) is bihamiltonian. It means that it can be described by a pair of compatible Poisson structures. The first one (the Poisson-Lie) is generated by the linear r -matrix structure

$$\{l_1(z), l_2(w)\} = [l_1(z) + l_2(w), r_{12}(z - w)], \tag{1.17}$$

and the second — is by quadratic one

$$\{L_1(z), L_2(w)\} = [L_1(z) L_2(w), r_{12}(z - w)] \tag{1.18}$$

with

$$L(z, S) = \frac{s_0}{N} 1 + l(z, S) - \frac{\text{tr} l(z, S)}{N} 1, \tag{1.19}$$

where s_0 is additional generator (of the classical Sklyanin algebra). In the elliptic case (corresponding to the Belavin-Drinfeld classical r -matrix [17]) this type of bihamiltonian structure for (1.18)–(1.19) was described in [37].

Thus, we have two quadratic algebras — (1.6) with η -dependent Lax operator (1.5), and (1.18) with η -independent Lax operator (1.19). Both algebras are described by the same r -matrix. Then it is natural to expect a relation between $L^\eta(z, S)$ and $L(z, S)$. It can be written explicitly:

$$L^\eta(z + \eta_0, L(-\eta_0, S)) = \frac{\text{tr}L^\eta(z + \eta_0, S)}{\text{tr}S} L(z, S) \tag{1.20}$$

where $L(-\eta_0, S)$ is inserted into $L^\eta(z, S)$ as the second argument, $\eta_0 = \eta_0(\eta)$ is a zero of function $\text{tr}L^\eta(z, S)/\text{tr}S$. In notations of this paper $\eta_0 = -\eta$ in the elliptic case and $\eta_0 = -\eta/N$ in the rational one. Then the change of variables from the η -independent description (1.19) to the η -dependent description (1.5) can be written as

$$S \rightarrow -(\eta_0/\eta) L(-\eta_0, S). \tag{1.21}$$

Relation (1.20) allows also to find the M -operator for (1.19) as

$$M(z, S) = -(\eta_0/\eta) J^{-(\eta/\eta_0)z}(L(z, S)). \tag{1.22}$$

The simplest example of the relativistic top is obtained in the elliptic case, where the quantum R -matrix is the Belavin's one [16]. In \mathfrak{gl}_2 case it coincides with the Baxter's one.

Then for $S = \sum_{a=0}^3 S_a \sigma_a$, where σ_a are the Pauli matrices ($\sigma_0 = 1$)

$$J^\eta(S) = \sum_{a=0}^3 J_a^\eta S_a \sigma_a, \quad J_0^\eta = E_1(\eta), \quad J_\alpha^\eta = E_1(\eta + \omega_\alpha) - E_1(\omega_\alpha), \quad \alpha = 1, 2, 3, \tag{1.23}$$

$$E_1(z) = \partial_z \log \vartheta(z|\tau), \quad \omega_1 = \tau/2, \quad \omega_2 = (1 + \tau)/2, \quad \omega_3 = 1/2.$$

In the elliptic case we also consider the large N limit to the elliptic hydrodynamics [37, 64] by passage to the noncommutative torus description:

$$\partial_t S(x) = \text{ad}_{J^\eta(S)(x)}^* S(x) = [S(x), J^\eta(S)(x)]_\theta \tag{1.24}$$

where $[f(x), g(x)]_\theta = \theta^{-1}(f \star g - g \star f)$ is the Moyal bracket (\star is the Moyal product) and $J^\eta(S)$ is the pseudo-differential operator given in (5.60) (cf. (5.23)).

2. Quantum rational R -matrix. We propose the factorized form for the rational Ruijsenaars-Schneider (RS) Lax matrix

$$L^{\text{RS}}(z) = g^{-1}(z) g(z + \eta) e^{P/c}, \tag{1.25}$$

where c is the light speed, P is a diagonal matrix of the RS particles momenta, and $g(z)$ is the matrix depending on the RS particles coordinates q . The latter was introduced in [6], where the non-relativistic rational top was constructed similarly starting from the rational Calogero-Moser (CM) model [19–22]. The transformation $g(z)$ is known for the quantum elliptic and trigonometric RS models [7, 36] (where the quantum IRF-Vertex

correspondence was described). The rational one was mentioned in [6].¹ By performing the gauge transformation

$$L^\eta(z) = g(z)L^{\text{RS}}(z)g^{-1}(z) = g(z + \eta) e^{P/c} g^{-1}(z) \tag{1.26}$$

and re-expressing $L^\eta(z)$ in terms of its residue we come to the relativistic rational top. It corresponds to some special values of the Casimir functions, while arbitrary values are related in the same way to the spin RS model [39] (see also [11]). The answer is given in section 3.2.

Then using (1.5) we obtain rational unitary quantum R-matrix. In \mathfrak{gl}_2 case it is the 11-vertex R-matrix

$$R^{\hbar}(z) = \begin{pmatrix} \hbar^{-1} + z^{-1} & 0 & 0 & 0 \\ -\hbar - z & \hbar^{-1} & z^{-1} & 0 \\ -\hbar - z & z^{-1} & \hbar^{-1} & 0 \\ -\hbar^3 - 2z\hbar^2 - 2\hbar z^2 - z^3 & \hbar + z & \hbar + z & \hbar^{-1} + z^{-1} \end{pmatrix} \tag{1.27}$$

obtained previously in [76].² In section 4 we obtain \mathfrak{gl}_N generalization of (1.27). Introduction of ϵ parameter as

$$R^{\hbar,\epsilon}(z) = \epsilon R^{\epsilon\hbar}(\epsilon z) \tag{1.28}$$

allows to interpret it as deformation of the XXX R-matrix

$$\lim_{\epsilon \rightarrow 0} R^{\hbar,\epsilon}(z) = R^{\text{XXX}}(z) = \frac{1}{\hbar} 1 \otimes 1 + \frac{1}{z} P_{12}, \quad P_{12} = \sum_{i,j=1}^N E_{ij} \otimes E_{ji}. \tag{1.29}$$

Notice that in this limit the relativistic top (1.10), (1.11) becomes free mechanical system in the sense that $L^\eta(z) = \eta^{-1} \mathcal{S}_0 1 + z^{-1} S$, and equations of motion are trivial $\hat{S} = 0$. Therefore, the parameter ϵ can be also treated as an alternative definition of the coupling constant.

The Lax matrix (1.26) (which is gauge equivalent to the RS model) emerge from explicit change of variables:

$$L^\eta(z) = \text{tr}_2 (R_{12}^\eta(z) S_2), \tag{1.30}$$

$$S_{ij}(\mathbf{q}, \mathbf{p}) = \sum_{m=1}^N \frac{(q_m + \eta)^{\varrho(i)} e^{p_m/c}}{\prod_{k \neq m} (q_m - q_k)} (-1)^{\varrho(j)} \sigma_{\varrho(j)}(\mathbf{q}),$$

where $\varrho(i) = \delta_{i \leq N-1}(i-1) + \delta_{iN}N$ (see (3.2)), while $\sigma_j(q)$ are elementary symmetric functions (3.21)–(3.23). The case (1.30) corresponds to rank one matrix S and to special values of the Casimir function $\det L^\eta(z)$ of Poisson brackets (1.6). In the quantum elliptic and trigonometric cases the (1.30)-type formulae for $\hat{S} = \hat{S}(\frac{\partial}{\partial \mathbf{q}})$ can be found in [7, 36] (see also [39]).

¹These gauge transformations underly the Symplectic Hecke Correspondence [47] (see also [48–51, 75]) for the classical integrable systems in the Hitchin approach.

²It can be also obtained [74] by applying special limiting procedure to the Baxter elliptic R-matrix [15].

In general case $L^\eta(z) = \text{tr}_2(R_{12}^\eta(z)S_2)$, where all S_{ij} are independent variables. For non-relativistic models it was shown in [47] that the top models on the special coadjoint orbit are gauge equivalent to Calogero-Moser (CM) systems [19–22] while generic orbits correspond to their spin generalizations. In the same way, the generic relativistic top can be treated as alternative form of the spin RS model [39]. The gauge transformations used in (1.26) are of the same form as in non-relativistic case, where they play the role of modifications of the underlying Higgs bundles. Hence, we deal with the relativistic version of the Symplectic Hecke Correspondence. It allows us to obtain the non-dynamical quantum R -matrix instead of direct usage of the quantum IRF-Vertex Correspondence [65, 66]. In this respect, we realize the latter correspondence by means of the relativistic version of the classical (Symplectic Hecke) one. It is also interesting to mention that in view of (1.20), (1.21) we obtain the same form of equations (1.16) for the (spin) RS and (spin) CM models

Spin chains and Gaudin models related to the 11-vertex rational R -matrix (1.27) and its classical limit are obtained straightforwardly. As an example we get the \mathfrak{gl}_2 Gaudin model Hamiltonians emerging from non-relativistic limit of the inhomogeneous chain:

$$\begin{aligned}
 h_a &= \sum_{c \neq a}^n \text{tr} \left(\hat{S}^a l(z_a - z_c, \hat{S}^c) \right) = \sum_{c \neq a}^n \text{tr}_{12} \left(r_{12}(z_a - z_c) \hat{S}_1^a \hat{S}_2^c \right) = \\
 &= \sum_{c \neq a}^n \frac{\text{tr}(\hat{S}^a \hat{S}^c)}{z_a - z_c} - (z_a - z_c) \left(\hat{S}_{12}^a (\hat{S}_{11}^c - \hat{S}_{22}^c) + \hat{S}_{12}^c (\hat{S}_{11}^a - \hat{S}_{22}^a) \right) - (z_a - z_c)^3 \hat{S}_{12}^a \hat{S}_{12}^c.
 \end{aligned}
 \tag{1.31}$$

Notice that the first term corresponds to the standard rational (XXX) Gaudin Hamiltonians.

The 11-vertex model is defined by the quantum local Hamiltonian of the homogeneous periodic spin (1/2) chain on n sites is of the form:

$$H^{\text{local}} = \sum_{k=1}^n P_{k,k+1} - \eta^2 E_{21}^k \otimes (E_{11}^{k+1} - E_{22}^{k+1}) - \eta^2 (E_{11}^k - E_{22}^k) \otimes E_{21}^{k+1} - \eta^4 E_{21}^k \otimes E_{21}^{k+1},
 \tag{1.32}$$

where $E_{ij}^{n+1} = E_{ij}^1$ (here we use the dual generators E_{ij} : $(E_{ij})_{ab} = \delta_{ia} \delta_{jb}$, $S_{ji} = \text{tr}(E_{ij} \hat{S})$). It is a deformation of the XXX spin chain described by the only first term in (1.32):

$$H^{\text{XXX}} = \sum_{k=1}^n P_{k,k+1}, \quad P_{k,k+1} = E_{11}^k \otimes E_{11}^{k+1} + E_{12}^k \otimes E_{21}^{k+1} + E_{21}^k \otimes E_{12}^{k+1} + E_{22}^k \otimes E_{22}^{k+1}.
 \tag{1.33}$$

Acknowledgments

The work was supported by RFBR grants 12-02-00594 (A.L. and M.O.) and 12-01-00482 (A.Z.). The work of A.L. was partially supported by AG Laboratory GU-HSE, RF government grant, ag. 11 11.G34.31.0023. The work of A.Z. was partially supported by the D. Zimin’s fund “Dynasty” and by the Program of RAS “Basic Problems of the Nonlinear Dynamics in Mathematical and Physical Sciences” III9.

2 Sklyanin algebras and classical integrable systems

2.1 Quantum Sklyanin algebra

Let the quantum L -operator (1.1) has the following expansions in spectral parameter near $z = 0$:

$$\hat{L}^\eta(z) = L^\eta(z, \hat{S}) = \sum_{k=-1}^{\infty} z^k L^{\eta,(k)}(\hat{S}) = \frac{1}{z} \hat{S} + L^{\eta,(0)}(\hat{S}) + z L^{\eta,(1)}(\hat{S}) + O(z^2), \quad (2.1)$$

where the residue \hat{S} is \mathfrak{gl}_N -valued operator and $L^{\eta,(k)}$ are linear functionals of \hat{S} , i.e.

$$\hat{L}^\eta(z) = \sum_{a,b} \sum_{k=-1}^{\infty} z^k \mathcal{R}_{a,b}^{\eta,k} T_a \hat{S}_b, \quad L^{\eta,(k)}(\hat{S}) = \sum_{a,b} \mathcal{R}_{a,b}^{\eta,k} T_a \hat{S}_b \quad (2.2)$$

in some basis $\{T_a\}$ of \mathfrak{gl}_N . The coefficients $\mathcal{R}_{a,b}^{\eta,k}$ are functions of a free constant parameter η which role is explained below. Due to (1.1) the matrix elements \hat{S}_b satisfy quadratic relations of $\mathcal{A}_{\hbar,\eta}^{\text{SkI}}$ (see (2.7)) such as Sklyanin algebra [72, 73] or its different extensions [23, 63, 67, 68]. Notice that the representation space of operators \hat{S}_a is not fixed yet.

Similarly to (2.1) and (2.2) let the R -matrix be of the form:

$$R_{12}^{\hbar}(z) = \sum_{k=-1}^{\infty} z^k R_{12}^{\hbar,(k)} = \frac{1}{z} P_{12} + R_{12}^{\hbar,(0)} + z R_{12}^{\hbar,(1)} + O(z^2), \quad R_{12}^{\hbar,(k)} \in \mathfrak{gl}_N^{\otimes 2}, \quad (2.3)$$

$$R_{12}^{\hbar,(k)} = \sum_{a,b} \mathcal{R}_{a,b}^{\hbar,k} T_a \otimes T_{-b}, \quad (2.4)$$

where the generators T_{-b} are dual to T_b : $\text{tr}(T_a T_b) = \delta_{a+b}$, and $R_{12}^{\hbar,(-1)} = P_{12} = \sum_a T_a \otimes T_{-a}$ is the permutation operator. Formulae (2.3) and (2.4) imply the following simple link between L -operator and R -matrix:

$$\hat{L}^\eta(z) = \text{tr}_2 \left(R_{12}^\eta(z) \hat{S}_2 \right). \quad (2.5)$$

It is important to mention that we deal with two constants \hbar and η (1.1). While \hbar plays the role of the Planck constant, the parameter η will be shown to describe relativistic deformation in the sense of Ruijsenaars.

Using notations of (2.1) and (2.3) it easy to write down the quadratic relations of $\mathcal{A}_{\hbar,\eta}^{\text{SkI}}$. Indeed, consider residue of (1.2) at $w = 0$:

$$R_{12}^{\hbar}(z) \hat{L}_1^\eta(z) \hat{S}_2 = \hat{S}_2 \hat{L}_1^\eta(z) R_{12}^{\hbar}(z). \quad (2.6)$$

Expanding this equation near $z = 0$ we get identity $P_{12} \hat{S}_1 \hat{S}_2 = \hat{S}_2 \hat{S}_1 P_{12}$ for z^{-2} terms while the coefficients behind z^{-1} give rise to the Sklyanin algebra:³

$$\mathcal{A}_{\hbar,\eta}^{\text{SkI}} : \quad P_{12} L^{\eta,(0)}(\hat{S})_1 \hat{S}_2 + R_{12}^{\hbar,(0)} \hat{S}_1 \hat{S}_2 = \hat{S}_2 L^{\eta,(0)}(\hat{S})_1 P_{12} + \hat{S}_2 \hat{S}_1 R_{12}^{\hbar,(0)}. \quad (2.7)$$

³In his original paper [72, 73] Sklyanin used $\eta = \hbar$.

A typical representative for (2.3)-type of solutions of the Yang-Baxter equation (1.2) is the Belavin's elliptic R -matrix. It is considered in section 5.

It follows from (2.4) that the Sklyanin algebra $\mathcal{A}_{\hbar,\hbar}^{\text{SkI}}$ (with $\eta = \hbar$) has finite-dimensional representation⁴

$$\rho(\mathcal{A}_{\hbar,\hbar}^{\text{SkI}}) : \quad \rho(\hat{S}_a) = T_{-a} \in \mathfrak{gl}_N, \quad (2.8)$$

Then

$$R_{12}^{\hbar,(k)} = \rho(L^{\hbar,(k)}(\hat{S})) . \quad (2.9)$$

With this definition the quantum Yang-Baxter equation (1.2) coincides with exchange relations (1.1) in representation (2.8).

2.2 Classical limit

Quantum R -matrix. In the classical limit $\hbar \rightarrow 0$ the operators \hat{S}_a become \mathbb{C} -valued coordinates on the phase space of an integrable system described by the Lax matrix $L^\eta(z, S)$. Notice that relation (2.5) remains intact at classical level, i.e.

$$L^\eta(z, S) = \text{tr}_2(R_{12}^\eta(z)S_2) = \sum_{a,b} \sum_{k=-1}^{\infty} z^k \mathcal{R}_{a,b}^{\eta,k} T_a S_b . \quad (2.10)$$

Therefore, having the classical Lax matrix $L^\eta(z)$ of the described type we can compute the quantum R -matrix in the following way:

$$R_{12}^\hbar(z) = \sum_b \frac{\partial L^\hbar(z, S)}{\partial S_b} \otimes T_{-b} \quad (2.11)$$

We will use this formula in section 3 for derivation of the rational R -matrix.

Classical r -matrix. Let the quantum R -matrix has the following expansion in the Planck constant \hbar :

$$R_{12}^\hbar(z) = \frac{1}{\hbar} 1 \otimes 1 + r_{12}(z) + \hbar r'_{12}(z) + O(\hbar^2) \in \mathfrak{gl}_N^{\otimes 2}, \quad (2.12)$$

where

$$r_{12}(z) = \frac{P_{12}}{z} + r_{12}^{(0)} + O(z), \quad r'_{12}(z) = r'_{12}(0) + O(z) . \quad (2.13)$$

and $r_{12}^{(0)}$ comes from $R_{12}^{\hbar,(0)}$ in (2.3):

$$R_{12}^{\hbar,(0)} = \frac{1}{\hbar} 1 \otimes 1 + r_{12}^{(0)} + O(\hbar) . \quad (2.14)$$

The term $r_{12}(z)$ is the classical r -matrix. It is skew-symmetric

$$r_{12}(z) = -r_{21}(-z), \quad (2.15)$$

$$\mathcal{R}_{a,b}^0 = (-1)^{k+1} \mathcal{R}_{b,a}^k, \quad \mathcal{R}^0 = \lim_{\hbar \rightarrow 0} (\mathcal{R}^\hbar - \hbar^{-1} 1 \otimes 1) \quad (2.16)$$

⁴In \mathfrak{gl}_2 case the original Sklyanin algebra has simple representation in terms of the Pauli matrices $\hat{S}_a = \sigma_{-a} = \sigma_a$, $a = 0, 1, 2, 3$, with $\sigma_0 = 1_{2 \times 2}$, which are used as basis $\{T_a\}$.

and satisfies the classical Yang-Baxter equation:

$$[r_{12}(z-w), r_{13}(z)] + [r_{12}(z-w), r_{23}(w)] + [r_{13}(z), r_{23}(w)] = 0. \quad (2.17)$$

The latter arises from (1.2) and (2.12). Similarly, by substituting (2.12) into (1.1) we come to quadratic Poisson structure

$$\lim_{\hbar \rightarrow 0} \frac{1}{\hbar} [\hat{L}_1^\eta(z), \hat{L}_2^\eta(w)] := \{L_1^\eta(z), L_2^\eta(w)\} = [L_1^\eta(z) L_2^\eta(w), r_{12}(z-w)], \quad (2.18)$$

where $L^\eta(z)$ is the classical L -operator (the Lax matrix).

2.3 Relativistic top

Let us define the *relativistic top* as an integrable model described by the Lax matrix (2.10) and the r -matrix structure (2.18):

$$\begin{aligned} \{L_1^\eta(z), L_2^\eta(w)\} &= [L_1^\eta(z) L_2^\eta(w), r_{12}(z-w)], \\ L^\eta(z) &= \text{tr}_2 (R_{12}^\eta(z) S_2) = \frac{1}{z} S + L^{\eta,(0)}(S) + z L^{\eta,(1)}(S) + O(z^2). \end{aligned} \quad (2.19)$$

We will see that equations of motion have the form of the integrable multidimensional Euler (or Euler-Arnold) top. On the other hand, it will be shown below that the parameter η plays the same role as the relativistic deformation parameter in the Ruijsenaars-Schneider generalization of Calogero-Moser models. This is why we call these type of models relativistic tops.⁵

Classical Sklyanin algebra. The phase space is parameterized by N^2 coordinates $\{S_a\}$. It is equipped with the following quadratic Poisson structure:

$$\mathcal{A}_{\hbar=0,\eta}^{\text{skl}} : \quad \{S_1, S_2\} = [S_1 S_2, r_{12}^{(0)}] + [L^{\eta,(0)}(S)_1 S_2, P_{12}], \quad (2.20)$$

where $r_{12}^{(0)}$ is defined in (2.13) and $L^{\eta,(0)}(S)$ in (2.1) and (2.19). The brackets (2.20) can be obtained both — from the quantum algebra (2.7) (by taking the classical limit (2.12)) or from (2.19) by computing residue at $w = 0$

$$\{L_1^\eta(z), S_2\} = [L_1^\eta(z) S_2, r_{12}(z)] \quad (2.21)$$

and evaluating the coefficient in front of z^{-1} . The Poisson brackets are degenerated. In order to restrict it on a symplectic leaf we need to fix Casimir functions $C_k(S)$. They appear as coefficients in the expansion of $\det L^\eta(z)$ which is known to be central element for the Poisson brackets (2.18):

$$\det L^\eta(z) = \sum_{k=-N}^{\infty} z^k C_k(S). \quad (2.22)$$

⁵Our approach is in agreement with the one considered in [24–26] for relativistic particles and rotators (in an external fields and on curved spaces), where the authors also used the term relativistic top. Our case corresponds to their free top.

The number of independent Casimir functions (in general) equals N . They can be accumulated from coefficients in front of nonpositive powers of z in (2.22) (others are dependent). The Hamiltonians (including the Casimir functions) can be computed from the expansion near $z = 0$ of

$$\begin{aligned} \frac{1}{k} \text{tr} (L^\eta(z))^k &= \frac{1}{k} \text{tr}_{0,1,\dots,k} (R_{01}^\eta(z) \dots R_{0k}^\eta(z) S_1 \dots S_k) = \\ &= \frac{1}{z^k} H_{k,k} + \frac{1}{z^{k-1}} H_{k,k-1} + \dots + H_{k,0} + \dots, \quad k = 1 \dots N \end{aligned} \quad (2.23)$$

or from the spectral curve

$$\det (\lambda - L^\eta(z)) = 0. \quad (2.24)$$

The number of Casimir functions N should be subtracted from the number of independent Hamiltonians (2.23) $N(N + 1)/2$. This gives $N(N - 1)/2$ for the Hamiltonians only. It equals to the half of dimension of a general symplectic leaf. The Poisson commutativity of the Hamiltonians $H_{k,l}$ is guaranteed by (2.19). Therefore, the model is integrable in the Liouville-Arnold sense.

Equations of motion and Lax pair. The simplest Hamiltonian is given by

$$\mathcal{S}_0 := H_{1,1} = \text{tr}(S). \quad (2.25)$$

To get equations of motion let us compute the trace over the second component (in tensor product $\mathfrak{gl}_N^{\otimes 2}$) of (2.20). It leads to the top-like equations

$$\partial_{t_0} S = \{\mathcal{S}_0, S\} = [S, J^\eta(S)], \quad (2.26)$$

where the inverse inertia tensor J^η is the following linear functional of S :

$$J^\eta : S \rightarrow J^\eta(S)_1 = -\text{tr}_2(r_{12}^{(0)} S_2) + L^{\eta,(0)}(S)_1 \stackrel{(2.10)}{=} \text{tr}_2 \left(\left(R_{12}^{\eta,(0)} - r_{12}^{(0)} \right) S_2 \right). \quad (2.27)$$

In a similar way, by applying tr_2 to the both parts of (2.21) we get equations of motion (2.26) in the Lax form:

$$\partial_{t_0} L^\eta(z) = \{\mathcal{S}_0, L^\eta(z)\} = [L^\eta(z), M(z)], \quad (2.28)$$

where the M -operator equals

$$M_1(z) = -\text{tr}_2 (r_{12}(z) S_2). \quad (2.29)$$

Notice that it is independent of η . As we will see below the M -operator in this description coincides with non-relativistic Lax matrix.

2.4 Non-relativistic limit

The non-relativistic limit $\eta \rightarrow 0$ is similar to the classical one due (2.10). It follows from (2.12) and (2.10) that

$$L^\eta(z) = \frac{\mathcal{S}_0}{N\eta} 1_{N \times N} + \text{tr}_2 (r_{12}(z) S_2) + \eta \text{tr}_2 (r'_{12}(z) S_2) + O(\eta^2), \quad (2.30)$$

$$L^{\eta,(0)} = \frac{\mathcal{S}_0}{N\eta} 1_{N \times N} + \text{tr}_2 \left(r_{12}^{(0)} S_2 \right) + \eta \text{tr}_2 (r'_{12}(0) S_2) + O(\eta^2). \quad (2.31)$$

Plugging (2.30) and (2.31) into (2.19) and (2.20) respectively we get

$$\{L_1^\eta(z), L_2^\eta(w)\} = \frac{\mathcal{S}_0}{N\eta} [l_1(z) + l_2(w), r_{12}(z-w)] + [l_1(z)l_2(w), r_{12}(z-w)] + O(\eta), \quad (2.32)$$

where

$$l_1(z) := \text{tr}_2(r_{12}(z)S_2), \quad (2.33)$$

$$l(z) = \frac{1}{z}S + l^{(0)}(S) + z l^{(1)}(S) + O(z), \quad l^{(0)}(S)_1 = \text{tr}_2(r_{12}^{(0)}S_2). \quad (2.34)$$

and

$$\{S_1, S_2\} = \frac{\mathcal{S}_0}{N\eta} [S_2, P_{12}] + [S_1S_2, r_{12}^{(0)}] + [\text{tr}_3(r_{13}^{(0)}S_3)S_2, P_{12}] + O(\eta), \quad (2.35)$$

where $\text{tr}_3(r_{13}^{(0)}S_3)$ is just $l^{(0)}(S)_1$ as in (2.34).

When $\eta \rightarrow 0$ the leading term in (2.35) is the linear Poisson-Lie structure on \mathfrak{gl}_N^* Lie coalgebra. The generator $\mathcal{S}_0 = \text{tr}S$ is the Casimir function of the latter brackets. Let us fix it as $\mathcal{S}_0 = N$ and set

$$\{S_1, S_2\}_{\text{Lie}} := \lim_{\eta \rightarrow 0} \eta \{S_1, S_2\}. \quad (2.36)$$

Then

$$\{S_1, S_2\}_{\text{Lie}} = [S_2, P_{12}]. \quad (2.37)$$

In the same way the linear r -matrix structure is obtained at the level of Lax matrices:

$$\{l_1(z), l_2(w)\}_{\text{Lie}} = [l_1(z) + l_2(w), r_{12}(z-w)]. \quad (2.38)$$

Non-relativistic top. We will refer to an integrable model described by the Lax matrix (2.33) and the Poisson structure (2.37), (2.38) as the *non-relativistic top*. The phase space is the coadjoint orbit of GL_N Lie group. It is equipped with the linear Poisson-Lie structure on \mathfrak{gl}_N^* . For example, using the standard basis of matrices $(E_{ij})_{ab} = \delta_{ia}\delta_{jb}$ (2.37) acquires the form $\{S_{ij}, S_{kl}\} = \delta_{il}S_{kj} - \delta_{kj}S_{il}$. A general symplectic leaf is obtained by fixation of eigenvalues of S or the Casimir functions $c_k = \frac{1}{k}\text{tr}(S^k)$, $k = 1 \dots N$. The Hamiltonians appear as in (2.23):

$$\begin{aligned} \frac{1}{k}\text{tr}(l(z))^k &= \frac{1}{k}\text{tr}_{0,1,\dots,k}(r_{01}(z) \dots r_{0k}(z) S_1 \dots S_k) = \\ &= \frac{1}{z^k}h_{k,k} + \frac{1}{z^{k-1}}h_{k,k-1} + \dots + h_{k,0} + \dots, \quad k = 1 \dots N. \end{aligned} \quad (2.39)$$

It is easy to see that $h_{k,k} = c_k$. The Poisson commutativity of the Hamiltonians $h_{k,l}$ is guaranteed by (2.38). The M -operators corresponding to the Hamiltonians $h_{k,l}$ are evaluated by expansion of $-\text{tr}_2(r_{12}(z-w)l_2^{k-1}(w))$ (see [14]). An alternative way is given in (2.54)–(2.56).

Notice that the Poisson brackets (2.37) follows from (2.38) and local expansion from (2.34). To get (2.37) one should substitute expansion (2.34) into (2.38) and compute the residue at $w = 0$ and then at $z = 0$.

Similarly to (2.10), we have a simple link between the Lax matrix and classical r -matrix given by (2.33). Substitution of (2.33) into (2.38) gives rise to the classical Yang-Baxter equation (2.17) (it follows from the Jacobi identity for the Poisson brackets (2.38) as well). By analogy with (2.11) we have

$$r_{12}(z) = \sum_b \frac{\partial l(z)}{\partial S_b} \otimes T_{-b}. \tag{2.40}$$

This relation was used in [6] for computation of the rational classical r -matrix.

2.5 η -independent quadratic Poisson brackets

Let us consider another limit of brackets (2.32) and (2.35). Set

$$S_0 = \eta s_0. \tag{2.41}$$

With this rescaling $S_0 \rightarrow 0$ when $\eta \rightarrow 0$. Then the residue S becomes traceless, i.e.

$$S \xrightarrow{\eta \rightarrow 0} \bar{S} = S - \frac{1}{N} \text{tr}(S) 1_{N \times N}, \tag{2.42}$$

$$l(z) \xrightarrow{\eta \rightarrow 0} \bar{l}(z) = l(z) - \frac{1}{Nz} \text{tr}(S) 1_{N \times N} = \frac{1}{z} \bar{S} + l^{(0)}(S) + O(z). \tag{2.43}$$

Applying this limit to (2.32) we get

$$\{L_1(z), L_2(w)\} = [L_1(z) L_2(w), r_{12}(z-w)], \tag{2.44}$$

where the Lax matrix

$$L(z) := \frac{s_0}{N} 1 + \bar{l}(z). \tag{2.45}$$

The Poisson brackets (2.35) acquire the following form in the limit:

$$\mathcal{A}_{\hbar=0, \eta=0}^{\text{SkI}} : \{ \bar{S}_1, \bar{S}_2 \} = \frac{s_0}{N} [\bar{S}_2, P_{12}] + [\bar{S}_1 \bar{S}_2, r_{12}^{(0)}] + [\text{tr}_3(r_{13}^{(0)} \bar{S}_3) \bar{S}_2, P_{12}]. \tag{2.46}$$

The missing brackets $\{\bar{S}, s_0\}$ can be found by taking the limit in (2.26). Plugging (2.41) into (2.26), (2.27) we get:

$$\{s_0, \bar{S}\} = \lim_{\eta \rightarrow 0} \frac{[S, J^\eta(S)]}{\eta} = [\bar{S}, J(\bar{S})], \tag{2.47}$$

where the inverse inertia tensor J is defined as

$$J : \bar{S}_1 \rightarrow J(\bar{S})_1 = \text{tr}_2 (r'_{12}(0) \bar{S}_2) \tag{2.48}$$

with $r'_{12}(0)$ from (2.31). Equations (2.47), (2.48) also play the role of equations of motion generated by the Hamiltonian s_0 :

$$\partial_{t_0} \bar{S} = \{s_0, \bar{S}\} = [\bar{S}, J(\bar{S})], \tag{2.49}$$

The corresponding Lax equation can be obtained in two ways. The first one [14] — is by taking tr_2 in (2.44). This yields

$$\partial_{t_0} L(z) = \{s_0, L(z)\} = [L(z), M(z)], \tag{2.50}$$

where

$$M_1(z) = -\text{tr}_2(r_{12}(z-w)L_2(w)) . \tag{2.51}$$

The latter matrix should be w -independent up to some element from the kernel of $\text{ad}_{L(z)}$. Alternatively, one can consider the limit of (2.28): $\{\mathcal{S}_0, L^\eta(z)\} = [L^\eta(z), M(z)]$. Notice again that $M(z)$ given by (2.29) is η -independent. Moreover, it coincides with $L(z)$ up to sign and some scalar - element from $\text{Ker}(\text{ad}_{L(z)})$, i.e.

$$\{\mathcal{S}_0, L^\eta(z)\} = [L^\eta(z), M(z)] = -[L^\eta(z), L(z)] . \tag{2.52}$$

Then, substituting rescaling (2.41) and using expansion (2.30) we get

$$\partial_{t_0} L(z) = \{s_0, L(z)\} = \lim_{\eta \rightarrow 0} \frac{[L(z), L^\eta(z)]}{\eta} = [L(z), \mathcal{M}(z)] , \tag{2.53}$$

where

$$\mathcal{M}(z) = \text{tr}_2(r'_{12}(z)\bar{S}_2) . \tag{2.54}$$

with $r'_{12}(z)$ defined in (2.30). Thus, the roles of L and M -operators are interchanged while taking the limit. In addition, $r'_{12}(z)$ has no singularities at $z = 0$. Then

$$\mathcal{M}(0) = \text{tr}_2(r'_{12}(0)\bar{S}_2) \stackrel{(2.48)}{=} J(\bar{S}) . \tag{2.55}$$

Finally, we see that the expansion (2.30) of $L^\eta(z)$ in η provides M -operators for both — η -dependent and η -independent descriptions:

$$L^\eta(z) = \eta^{-1}\mathcal{S}_0/N - M(z) + \eta\mathcal{M}(z) + O(\eta^2) . \tag{2.56}$$

Notice also that the M -operator (2.54) is also valid for the linear r -matrix structure (2.38) since the Lax pairs for the linear and quadratic (η -independent) r -matrix structures are the same (up to scalar terms). The formulae obtained in this section can be considered as an extension of [14] for the class of integrable systems under consideration.

Relation between $\mathcal{A}_{\eta \neq 0}^{\text{SkI}}$ and $\mathcal{A}_{\eta=0}^{\text{SkI}}$. We have two different description of the classical Sklyanin algebras (and related integrable models). In the first one the quadratic Poisson structure (2.20) $\mathcal{A}_{h=0, \eta}^{\text{SkI}}$ is η -dependent. The second $\mathcal{A}_{h=0, \eta=0}^{\text{SkI}}$ (2.46), (2.47) — is η -independent. The same happens at quantum level. One can quantize the Lax matrix (2.45) as

$$\hat{L}(z) := \frac{1}{N} \hat{s}_0 + \text{tr}_2(r_{12}(z)\hat{S}_2) . \tag{2.57}$$

Then the exchange relations (1.1) gives η -independent quadratic algebra $\mathcal{A}_{h, \eta=0}^{\text{SkI}}$.

Algebras $\mathcal{A}_{\eta \neq 0}^{\text{SkI}}$ and $\mathcal{A}_{\eta=0}^{\text{SkI}}$ are related. The relation is easy to demonstrate explicitly in the elliptic case (see section 5). The idea is the following. There exists a linear functional φ^η on gl_N (depending on the boundary conditions) such that (2.5) and (2.57) are related as follows:

$$g^\eta(z) e^{\eta_0 \partial_z} \hat{L}(z, \varphi^\eta(\hat{S})) = \hat{L}(z, \hat{S}) , \tag{2.58}$$

where $g^\eta(z)$ is some function with a simple zero at $z = -\eta_0$ and simple pole at $z = 0$. Finally, the relation can be written as

$$L^\eta(z + \eta_0, L(-\eta_0, S)) = \frac{\text{tr} L^\eta(z + \eta_0, S)}{\text{tr} S} L(z, S). \tag{2.59}$$

It holds true in the rational case as well. We may use this relation to get explicit change of variables from the η -independent description (2.45) to the η -dependent (2.19):

$$S \rightarrow cL(c\eta, S), \quad c = -\frac{\eta_0}{\eta}. \tag{2.60}$$

The coefficient $-\eta_0/\eta$ is chosen in order to have $\text{Res}_{\eta=0} \left(-\frac{\eta_0}{\eta} L(-\eta_0, S) \right) = \text{Res}_{z=0} (z, S)$. It is interesting to note that plugging this change of variables to the equations of motion (2.26) gives

$$\partial_t L(c\eta, S) = [L(c\eta, S), cJ^\eta(L(c\eta, S))], \tag{2.61}$$

i.e. the Lax equations (2.50), where $c\eta$ plays the role of the spectral parameter. Hence, we get an alternative definition for the M -operator

$$M(z, S) = cJ^{z/c}(L(z, S)). \tag{2.62}$$

3 Relativistic rational top

In this section we obtain explicit answer for the Lax pair of the relativistic top. As it was already mentioned this model is a top-like form of the spin Ruijsenaars-Schneider (RS) model. To get the answer we represent the Lax matrix of RS model in the factorized form (3.6) which is convenient for the gauge transformation. The dynamical variables of the top are the components of the residue (3.18), (3.26) of the gauge transformed RS Lax matrix (3.16). We express the gauge transformed L -operator in terms of its residue. This gives the correct answer for generic top since it is independent of the Casimir functions values.

3.1 Factorized L -operators for classical Ruijsenaars-Schneider model

In this paragraph we propose factorized forms of L -operators for the rational RS model [69, 70].

Following [6] for the set of variables $\{q_j\}$, $j = 1 \dots N$ such that $\sum_{j=1}^N q_j = 0$ let us introduce the matrix

$$\Xi_{ij}(\mathbf{q}, z) := (z + q_j)^{\varrho(i)}, \quad i, j = 1 \dots N, \tag{3.1}$$

where

$$\varrho(i) = \begin{cases} i - 1 & \text{for } 1 \leq i \leq N - 1, \\ i & \text{for } i = N. \end{cases} \quad \varrho^{-1}(i) = \begin{cases} i + 1 & \text{for } 0 \leq i \leq N - 2, \\ i & \text{for } i = N. \end{cases} \tag{3.2}$$

It has the property

$$\det \Xi(\mathbf{q}, z) = Nz \prod_{1 \geq i > j \geq N} (q_i - q_j), \quad (3.3)$$

i.e. the matrix is degenerated at $z = 0$. It can be also treated as the rational analogue of the modification of bundles over elliptic curves used in [47] for the description of the elliptic top.

Rational \mathfrak{sl}_N RS model with spectral parameter is defined by the following Lax matrix:

$$L_{ij}^{\text{RS}}(z, \eta) = \eta \left(\frac{1}{q_i - q_j + \eta} - \frac{1}{Nz} \right) e^{p_j/c} \prod_{k \neq j}^N \frac{q_j - q_k - \eta}{q_j - q_k}. \quad (3.4)$$

where

$$\sum_{k=1}^N q_k = \sum_{k=1}^N p_k = 0. \quad (3.5)$$

The classical r -matrix structure was found in [12, 13].

Proposition 3.1 *The Lax matrix (3.4) can be written in the following form.*⁶

$$L^{\text{RS}}(z, \eta) = D_0(\mathbf{q}) \Xi^{-1}(\mathbf{q}, z) \Xi(\mathbf{q}, z - \eta) D_0^{-1}(\mathbf{q}) e^{P/c}, \quad (3.6)$$

where $D_0(\mathbf{q})$ is diagonal matrix $(D_0)_{ij} = \delta_{ij} \prod_{k \neq i}^N (q_i - q_k)$.

Let us also write the similar answer for⁷

Rational \mathfrak{sl}_N RS model without spectral parameter. The Lax matrix

$$L_{ij}^{\text{RS}} = \frac{\eta e^{p_j/c}}{q_i - q_j + \eta} \prod_{k \neq j}^N \frac{q_j - q_k - \eta}{q_j - q_k}. \quad (3.7)$$

is represented in the form:

$$\begin{aligned} L^{\text{RS}} &= D_0(\mathbf{q}) V^{-1}(\mathbf{q}, z) V(\mathbf{q}, z - \eta) D_0^{-1}(\mathbf{q}) e^{P/c} = \\ &= D_0(\mathbf{q}) V^{-1}(\mathbf{q}, z) C_{-\eta} V(\mathbf{q}, z) D_0^{-1}(\mathbf{q}) e^{P/c}, \end{aligned} \quad (3.8)$$

where

$$V_{ij}(\mathbf{q}, z) := (z + q_j)^{i-1}, \quad i, j = 1 \dots N, \quad (3.9)$$

and

$$(C_\lambda)_{ij} = \begin{cases} \frac{(i-1)! \lambda^{i-j}}{(j-1)! (i-j)!}, & j \leq i, \\ 0, & j > i. \end{cases} \quad (3.10)$$

⁶The prove is direct. See formulae in section 3.2 and appendix in [6].

⁷It is not used in the subsequent sections.

It easy to verify that

$$C_\lambda = \exp(\lambda C_0), \quad (C_0)_{ij} = \begin{cases} j, & i = j + 1, \quad i = 2, \dots, N, \\ 0, & \text{otherwise} \end{cases} \quad (3.11)$$

and $\partial_z V = C_0 V$. The limit to Calogero-Moser model is obtained as follows:

$$L_{ij}^{\text{CM}} = \lim_{c \rightarrow \infty} c L_{ij}^{\text{RS}} |_{\eta=\nu/c} - \delta_{ij} = P - \nu D_0 V^{-1} C_0 V D_0^{-1}. \quad (3.12)$$

Notice that we can also define the Lax matrix as

$$L'_{ij}{}^{\text{RS}} = \frac{\eta e^{p_j/c}}{q_i - q_j + \eta} \prod_{k \neq j}^N \frac{q_j - q_k + \eta}{q_j - q_k}. \quad (3.13)$$

It differs from (3.7) by the canonical map

$$e^{p_j/c} \longrightarrow e^{p_j/c} \prod_{k \neq j} \left(\frac{q_j - q_k + \xi}{q_j - q_k - \xi} \right)^a \quad (3.14)$$

with $a = 1$ and $\xi = \eta$. Then

$$\begin{aligned} L'^{\text{RS}} &= D_\eta^{-1}(\mathbf{q}) (V^T)(\mathbf{q}, z + \eta) (V^T)^{-1}(\mathbf{q}, z) D_\eta(\mathbf{q}) e^{P/c} = \\ &= D_\eta^{-1}(\mathbf{q}) (V^T)(\mathbf{q}, z) C_\eta^T (V^T)^{-1}(\mathbf{q}, z) D_\eta(\mathbf{q}) e^{P/c}, \quad (D_\lambda)_{ij} = \delta_{ij} \prod_{k \neq i}^N (q_i - q_k + \lambda). \end{aligned} \quad (3.15)$$

Let us mention that the transformation (3.9) was used in the classical [30] and the quantum [27–29] IRF-Vertex transformations. In this way the (Jordanian) R -matrices of the Cremmer-Gervais type were obtained.

3.2 Lax pair

Apply the gauge transformation $g(z) = \Xi(\mathbf{q}, z) D_0^{-1}(\mathbf{q})$ to the RS Lax matrix (3.4), (3.6) with $\eta := -\eta$. Then it follows from (3.6) that

$$\begin{aligned} \tilde{L}^{\eta,c}(z, \mathbf{q}, \mathbf{p}) &:= \Xi(\mathbf{q}, z) D_0^{-1}(\mathbf{q}) L_{ij}^{\text{RS}}(z, -\eta) D_0(\mathbf{q}) \Xi^{-1}(\mathbf{q}, z) = \\ &= \Xi(\mathbf{q}, z + \eta) e^{P/c} \Xi^{-1}(\mathbf{q}, z). \end{aligned} \quad (3.16)$$

Let $\sum_{j=1}^N p_j = 0$. Then from (3.3) it follows that

$$\det \tilde{L}^{\eta,c}(z, \mathbf{q}, \mathbf{p}) = \frac{z + \eta}{z}. \quad (3.17)$$

Our purpose now is to express this matrix in terms of its residue at $z = 0$. Set

$$S = N \operatorname{Res}_{z=0} \tilde{L}^{\eta,c}(z, \mathbf{q}, \mathbf{p}) = \frac{1}{\prod_{1 \geq i > j \geq N} (q_i - q_j)} \Xi(\mathbf{q}, \eta) e^{P/c} \operatorname{adj}(\Xi(\mathbf{q}, 0)), \quad (3.18)$$

where the adjugate is transpose of the cofactor matrix. To find matrix components we need the inverse of Ξ :

$$\Xi_{kj}^{-1}(\mathbf{x}) = (-1)^{\varrho(j)} \frac{\sigma_{\varrho(j)}(\mathbf{x})}{\left(\sum_{s=1}^N x_s\right) \prod_{s \neq k}^N (x_k - x_s)} - (-1)^{\varrho(j)} \frac{\overset{k}{\sigma}_{\varrho(j)}(\mathbf{x})}{\prod_{s \neq k}^N (x_k - x_s)}, \quad (3.19)$$

where $x_j = q_j + z$. Expansion in powers of z gives

$$\begin{aligned} \Xi_{mj}^{-1}(z, \mathbf{q}) &= \frac{1}{Nz} \frac{(-1)^{\varrho(j)}}{\prod_{r \neq m}^N (q_m - q_r)} \left(\sigma_{\varrho(j)}(\mathbf{q}) + \sum_{s=1}^{N-j} z^s \left[\sigma_{s+j-1}(\mathbf{q}) \binom{s+j-1}{j-1} \right. \right. \\ &\quad \left. \left. - N \overset{m}{\sigma}_{s+j-2}(\mathbf{q}) \binom{s+j-2}{j-1} \right] - (N-j) z^{N-j+1} \overset{m}{\sigma}_{N-1}(\mathbf{q}) \binom{N}{j-1} \right). \end{aligned} \quad (3.20)$$

In (3.19) and (3.20) the elementary symmetric functions are used:

$$\prod_{k=1}^N (\zeta - x_k) = \sum_{k=0}^N (-1)^k \zeta^k \sigma_k(\mathbf{x}) \quad (3.21)$$

or

$$\sigma_{N-d}(\mathbf{x}) = (-1)^N \sum_{1 \leq i_1 < i_2 < \dots < i_d \leq N} x_{i_1} x_{i_2} \dots x_{i_d}, \quad d = 0, \dots, N \quad (3.22)$$

and their derivatives

$$- \prod_{m \neq k}^N (\zeta - x_m) = \sum_{s=0}^{N-1} (-1)^s \zeta^s \overset{k}{\sigma}_s(\mathbf{x}). \quad (3.23)$$

These functions satisfy the following set of identities:

$$\overset{m}{\sigma}_j(\mathbf{x}) = \sum_{c=0}^{N-j-1} (-x_m)^c \sigma_{j+1+c}(\mathbf{x}), \quad (3.24)$$

$$\overset{m}{\sigma}_j(\mathbf{x}) = - \sum_{c=0}^j (-x_m)^{-1-c} \sigma_{j-c}(\mathbf{x}). \quad (3.25)$$

From (3.19) we can easily find S from (3.18):

$$S_{ij} = N \operatorname{Res}_{z=0} L_{ij}^{\text{top}}(z) = \sum_{m=1}^N \frac{(q_m + \eta)^{\varrho(i)} e^{p_m/c}}{\prod_{k \neq m} (q_m - q_k)} (-1)^{\varrho(j)} \sigma_{\varrho(j)}(\mathbf{q}) \quad (3.26)$$

To take into account the light speed c in the quadratic brackets (2.20) one should put the common factor $1/c$ in front of r.h.s. of (2.20). It is equivalent to redefinition of the classical r -matrix $r_{12}(z) \rightarrow r_{12}(z)/c$.

Using (3.20)–(3.25) we can rewrite the Lax matrix (3.16) in terms of the variables S (3.20). The computation gives:

$$L^n(z) = N \tilde{L}^{\eta,c}(z, \mathbf{q}, \mathbf{p}) = \frac{1}{z} \sum_{i,j=1}^N E_{ij} \left\{ \sum_{\gamma=0}^{\varrho(i)} z^\gamma \binom{\varrho(i)}{\gamma} S_{\varrho^{-1}(\varrho(i)-\gamma),j} \right.$$

$$- \sum_{\gamma=0}^{\varrho(i)} z^{\gamma+N-j+1} (-1)^{\varrho(j)+N} (N-j) \binom{\varrho(i)}{\gamma} \binom{N}{j-1} S_{\varrho^{-1}(\varrho(i)-\gamma),N} \quad (3.27)$$

$$+ \sum_{\gamma=0}^{\varrho(i)} \sum_{s=1}^{N-j} z^{s+\gamma} (-1)^{\varrho(j)+s+j-1} \binom{\varrho(i)}{\gamma} \binom{s+j-1}{j-1} S_{\varrho^{-1}(\varrho(i)-\gamma),\varrho^{-1}(s+j-1)} -$$

$$- N \sum_{s=1}^{N-j} \sum_{b=0}^{\varrho(i)} (-1)^{\varrho(j)+s+j-1} z^s (z+\eta)^b \binom{s+j-2}{j-1} \binom{\varrho(i)}{b} \times$$

$$\left[\delta_{\varrho(i)-j-s-b+1 \leq 0} \sum_{c=0}^{N-s-j+1} \sum_{p=0}^{\varrho(i)-b+c} (-\eta)^p \binom{\varrho(i)-b+c}{p} S_{\varrho^{-1}(\varrho(i)-b-p+c),\varrho^{-1}(s+j+c-1)} \right.$$

$$\left. - \delta_{\varrho(i)-j-s-b+1 > 0} \sum_{c=0}^{s+j-2} \sum_{p=0}^{\varrho(i)-b-c-1} (-\eta)^p \binom{\varrho(i)-b-c-1}{p} S_{\varrho^{-1}(\varrho(i)-b-p-c-1),\varrho^{-1}(s+j-c-2)} \right] \quad (3.28)$$

$$+ \frac{1}{z} \left[zN E_{NN} - \sum_{j=1}^N z^{N-j+2} (-1)^{\varrho(j)+N} (N-j)N \binom{N}{j-1} E_{Nj} \right.$$

$$- N \sum_{i,j=1}^N \sum_{s=1}^{N-j} \sum_{b=0}^{\varrho(i)} (-1)^{\varrho(j)+s+j-1} z^s (z+\eta)^b \binom{s+j-2}{j-1} \binom{\varrho(i)}{b} \delta_{p \leq 1} \delta_{\varrho(i)-b-p-j-s+2,0} \times$$

$$\quad (3.29)$$

$$\left. \sum_{p=0}^{\varrho(i)-b+N-s-j+1} (-\eta)^p \binom{\varrho(i)-b+N-s-j+1}{p} E_{ij} \right] \times$$

$$\times \left(-\frac{1}{N(-\eta)} \sum_{j=1}^N S_{jj} - \frac{1}{N^2} \sum_{j=1}^N \left[\delta_{\varrho(j) \geq 1} \varrho(j) S_{\varrho^{-1}(\varrho(j)-1),\varrho^{-1}(j)} + (-1)^{\varrho(j)+j} j S_{j,\varrho^{-1}(j)} \right. \right.$$

$$\left. - N \sum_{b=0}^{\varrho(j)} (-1)^{\varrho(j)+j+b} \binom{\varrho(j)}{b} \sum_{c=0}^{N-j} \sum_{p=0}^{\varrho(j)-b+c} (-\eta)^{p+b} \binom{\varrho(j)-b+c}{p} S_{\varrho^{-1}(\varrho(j)-b-p+c),\varrho^{-1}(j+c)} \right] \quad (3.30)$$

Here and below we imply that the values of indices corresponding to the undefined argument value $(N-1)$ of function ϱ^{-1} (3.2) are skipped in the summations.

It is important to mention that the answer does not depend on the values of the Casimir functions (2.22). As we know it is defined by only quantum R -matrix which is the subject of the next section. Therefore, we can consider the obtained expression (3.27)–(3.30) as independent definition of the Lax matrix for the generic *rational relativistic top*. The case (3.16) which is gauge equivalent to the rational RS model appears for the particular

values of the Casimir functions C_k (2.22). From (3.17) we conclude that the RS case corresponds to:

$$\text{RS : } C_{-1} = \eta, \quad C_0 = 1, \quad C_k = 0, k \neq -1, 0. \quad (3.31)$$

In the non-relativistic limit this case corresponds to the rational top on the coadjoint orbit of minimal dimension $(2N - 2)$. In a general case the obtained model yields alternative description of the spin RS model [39].

The quantum Lax matrix is obtained from (3.27)–(3.30) by substitution $S \rightarrow \hat{S}$. In (3.26) it corresponds to $p_j := \hbar \partial_{q_j}$ with the choice of normal ordering.

M-operator from non-relativistic limit. As it was shown in (2.33) the M -operator (2.29) coincides (up to minus) with the non-relativistic limit of the Lax matrix. Hence, we can use the answer obtained in [6]:⁸

$$\begin{aligned} M_{ij}(z) = & -\frac{1}{z} \times \quad (3.32) \\ & \left[\sum_{\gamma=0}^{\varrho(i)} z^\gamma \binom{\varrho(i)}{\gamma} S_{e^{-1}(\varrho(i)-\gamma), j} - \sum_{\gamma=0}^{\varrho(i)} z^{\gamma+N-j+1} (-1)^{\varrho(j)+N(N-j)} \binom{\varrho(i)}{\gamma} \binom{N}{j-1} S_{e^{-1}(\varrho(i)-\gamma), N} \right. \\ & - N \sum_{\gamma=0}^{\varrho(i)} \sum_{s=1}^{N-j} \delta_{\varrho(i)-j+1 \leq s+\gamma} z^{s+\gamma} (-1)^{\varrho(j)+s+j-1} \binom{\varrho(i)}{\gamma} \binom{s+j-2}{j-1} \sum_{c=0}^{N-s-j+1} S_{e^{-1}(\varrho(i)-\gamma+c), e^{-1}(s+j+c-1)} \\ & - N \sum_{\gamma=0}^{\varrho(i)} \sum_{s=1}^{N-j} \delta_{\varrho(i)-j+1 > s+\gamma} z^{s+\gamma} (-1)^{\varrho(j)+s+j-2} \binom{\varrho(i)}{\gamma} \binom{s+j-2}{j-1} \sum_{c=0}^{s+j-2} S_{e^{-1}(\varrho(i)-\gamma-c-1), e^{-1}(s+j-c-2)} \\ & \left. + \sum_{\gamma=0}^{\varrho(i)} \sum_{s=1}^{N-j} z^{s+\gamma} (-1)^{\varrho(j)+s+j-1} \binom{\varrho(i)}{\gamma} \binom{s+j-1}{j-1} S_{e^{-1}(\varrho(i)-\gamma), e^{-1}(s+j-1)} - \frac{\delta_{i,j}}{N} \sum_{k=1}^N \sum_{c=0}^{N-k-2} S_{k+c, k+c+1} \right] \end{aligned}$$

Example: for $N = 2$ (3.27)–(3.30) yields the Lax matrix

$$L^\eta(z) = \frac{1}{z} S_{2 \times 2} + \frac{\text{tr}(S)}{\eta} 1_{2 \times 2} - (z + \eta) \begin{pmatrix} S_{12} & 0 \\ (S_{11} - S_{22}) + (\eta^2 + z^2 + \eta z) S_{12} & -S_{12} \end{pmatrix} \quad (3.33)$$

with $S_{2 \times 2} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ and $\text{tr} S = S_{11} + S_{22}$. The determinant defines the Casimir functions

$$\det L^\eta(z) = \frac{1}{z^2} C_0 + \left(\frac{1}{z\eta} + \frac{1}{\eta^2} \right) C_1, \quad (3.34)$$

$$C_0 = \det S = S_{11} S_{22} - S_{12} S_{21}, \quad C_1 = (S_{11} + S_{22} + \eta^2 S_{12})^2 - 4\eta^2 S_{12} S_{22}$$

of the Poisson structure (2.20) for \mathfrak{gl}_2 . The Hamiltonian $S_{11} + S_{22}$ generates equations of motion (2.26) with the M -operator (2.29):

$$M(z) = -\frac{1}{z} \begin{pmatrix} S_{11} - z^2 S_{12} & S_{12} \\ S_{21} - z^2 (S_{11} - S_{22}) - z^4 S_{12} & S_{22} + z^2 S_{12} \end{pmatrix} \quad (3.35)$$

⁸Notice that in [6] the answer is \mathfrak{sl}_N -valued. It differs from (2.33) by scalar matrix and factor $-N$.

In $N = 2$ case $r_{12}^{(0)}$ from (2.13) vanishes. Hence, $J^\eta(S)$ (2.27) is defined by only $L^{\eta,(0)}$ from (2.1):

$$J^\eta(S) = L^{\eta,(0)} = - \begin{pmatrix} \eta S_{12} & 0 \\ \eta^3 S_{12} + \eta(S_{11} - S_{22}) & -\eta S_{12} \end{pmatrix} + \frac{S_{11} + S_{22}}{\eta} \mathbf{1}_{2 \times 2}. \quad (3.36)$$

The last scalar term vanishes from the commutator in the equations (2.26): $\dot{S} = [S, J^\eta(S)]$.

3.3 Spin chains and Gaudin models

The Lax matrix (1.4), (3.27)–(3.30) allows to define a class of integrable \mathfrak{gl}_N spin chains with the transfer-matrix

$$\hat{T}_n(z) = L^\eta(\hat{S}^1, z - z_1) \dots L^\eta(\hat{S}^n, z - z_n). \quad (3.37)$$

on n sites with inhomogeneities z_k . The underlying quantum algebra consists of n copies of (2.7) with the quantum R -matrix (4.1)–(4.4). The non-relativistic limit gives rise to the Gaudin model defined by the Lax operator

$$\hat{L}^G(z) = \sum_{a=1}^n l(z - z_a, \hat{S}^a). \quad (3.38)$$

Its Hamiltonians are computed as residues of $\text{tr} \left(\hat{L}^G(z) \right)^2$:

$$\hat{h}_a = \sum_{c \neq a}^n h_{a,c}, \quad h_{a,c} = \text{tr}_{12} \left(r_{12}(z_a - z_c) \hat{S}_1^a \hat{S}_2^c \right) = \text{tr} \left(\hat{S}^a l(z_a - z_c, \hat{S}^c) \right). \quad (3.39)$$

For example, in \mathfrak{gl}_2 case the classical r -matrix (2.13) (classical limit of (1.27))

$$r_{12}(z) = \begin{pmatrix} 1/z & 0 & 0 & 0 \\ -z & 0 & 1/z & 0 \\ -z & 1/z & 0 & 0 \\ -z^3 & z & z & 1/z \end{pmatrix} \quad (3.40)$$

gives

$$h_{a,c} = \frac{\text{tr}(\hat{S}^a \hat{S}^c)}{z_a - z_c} - (z_a - z_c) \left(\hat{S}_{12}^a (\hat{S}_{11}^c - \hat{S}_{22}^c) + \hat{S}_{12}^c (\hat{S}_{11}^a - \hat{S}_{22}^a) \right) - (z_a - z_c)^3 \hat{S}_{12}^a \hat{S}_{12}^c. \quad (3.41)$$

In the limit (1.29) this formula reproduces the well-known rational Gaudin Hamiltonians (the first term in (3.41)).

Let us also compute the quantum local Hamiltonian of the homogeneous ($z_a = 0$) periodic spin chain on n sites. The quantum R -matrix

$$\tilde{R}^\eta(z) = z\eta R^\eta(z) \quad (3.42)$$

with R (1.27), satisfies

$$\tilde{R}^\eta(0)_{12} = \eta P_{12}. \quad (3.43)$$

Therefore, we can calculate the local Hamiltonian using standard approach of [15, 40–45].

The answer is given by $\sum_{k=1}^n H_{k,k+1}$, where $H_{k,k+1} = P_{k,k+1} \frac{d}{dz} \tilde{R}_{k,k+1}^\eta(z) |_{z=0}$:

$$H^{\text{local}} = \sum_{k=1}^n P_{k,k+1} - \eta^2 E_{21}^k \otimes (E_{11}^{k+1} - E_{22}^{k+1}) - \eta^2 (E_{11}^k - E_{22}^k) \otimes E_{21}^{k+1} - \eta^4 E_{21}^k \otimes E_{21}^{k+1}, \quad (3.44)$$

where $E_{ij}^{n+1} = E_{ij}^1$. It is a deformation of the XXX spin chain⁹. The latter is described by the only first term in (1.32):

$$H^{\text{XXX}} = \sum_{k=1}^n P_{k,k+1}, \quad P_{k,k+1} = E_{11}^k \otimes E_{11}^{k+1} + E_{12}^k \otimes E_{21}^{k+1} + E_{21}^k \otimes E_{12}^{k+1} + E_{22}^k \otimes E_{22}^{k+1}. \quad (3.45)$$

The generators E_{ij} , $(E_{ij})_{ab} = \delta_{ia} \delta_{jb}$ are dual to $\hat{S}_{ji} = \text{tr}(E_{ij} \hat{S})$.

We describe this type of models and related soliton equations in our next publication [54].

4 Quantum rational R-matrix

The quantum non-dynamical R -matrix can be found by the standard procedure of the IRF-Vertex Correspondence starting from the rational RS model. Here we use another approach based on (1.9). Applying it to (3.27)–(3.30) we get

$$R_{12}^{\hbar}(z) = \sum_{k,l=1}^N \frac{\partial L^{\hbar}(z)}{\partial S_{kl}} \otimes E_{lk} = \frac{1}{z} \sum_{i,j=1}^N E_{ij} \otimes \left\{ \sum_{\gamma=0}^{\varrho(i)} z^\gamma \binom{\varrho(i)}{\gamma} E_{j, \varrho^{-1}(\varrho(i)-\gamma)} \right. \quad (4.1)$$

$$- \sum_{\gamma=0}^{\varrho(i)} z^{\gamma+N-j+1} (-1)^{\varrho(j)+N} (N-j) \binom{\varrho(i)}{\gamma} \binom{N}{j-1} E_{N, \varrho^{-1}(\varrho(i)-\gamma)}$$

$$+ \sum_{\gamma=0}^{\varrho(i)} \sum_{s=1}^{N-j} z^{s+\gamma} (-1)^{\varrho(j)+s+j-1} \binom{\varrho(i)}{\gamma} \binom{s+j-1}{j-1} E_{\varrho^{-1}(s+j-1), \varrho^{-1}(\varrho(i)-\gamma)} -$$

$$- N \sum_{s=1}^{N-j} \sum_{b=0}^{\varrho(i)} (-1)^{\varrho(j)+s+j-1} z^s (z + \hbar)^b \binom{s+j-2}{j-1} \binom{\varrho(i)}{b} \times$$

$$\left[\delta_{\varrho(i)-j-s-b+1 \leq 0} \sum_{c=0}^{N-s-j+1} \sum_{p=0}^{\varrho(i)-b+c} (-\hbar)^p \binom{\varrho(i)-b+c}{p} E_{\varrho^{-1}(s+j+c-1), \varrho^{-1}(\varrho(i)-b-p+c)} \right.$$

$$\left. - \delta_{\varrho(i)-j-s-b+1 > 0} \sum_{c=0}^{s+j-2} \sum_{p=0}^{\varrho(i)-b-c-1} (-\hbar)^p \binom{\varrho(i)-b-c-1}{p} E_{\varrho^{-1}(s+j-c-2), \varrho^{-1}(\varrho(i)-b-p-c-1)} \right] \quad (4.2)$$

⁹Another deformation of the Heisenberg chain was found in [46]. It is interesting to find some relation (if any) between two deformations.

$$\begin{aligned}
 & + \frac{1}{z} \left[zN E_{NN} - \sum_{j=1}^N z^{N-j+2} (-1)^{\varrho(j)+N} (N-j)N \binom{N}{j-1} E_{Nj} \right. \\
 & - N \sum_{i,j=1}^N \sum_{s=1}^{N-j} \sum_{b=0}^{\varrho(i)} (-1)^{\varrho(j)+s+j-1} z^s (z+\hbar)^b \binom{s+j-2}{j-1} \binom{\varrho(i)}{b} \delta_{p \leq 1} \delta_{\varrho(i)-b-p-j-s+2,0} \times \quad (4.3) \\
 & \left. \sum_{p=0}^{\varrho(i)-b+N-s-j+1} (-\hbar)^p \binom{\varrho(i)-b+N-s-j+1}{p} E_{ij} \right] \otimes \\
 & \otimes \left(-\frac{1}{N(-\hbar)} \sum_{j=1}^N E_{jj} - \frac{1}{N^2} \sum_{j=1}^N \left[\delta_{\varrho(j) \geq 1} \varrho(j) E_{\varrho^{-1}(j), \varrho^{-1}(\varrho(j)-1)} + (-1)^{\varrho(j)+j} j E_{\varrho^{-1}(j), j} \right. \right. \\
 & \left. \left. - N \sum_{b=0}^{\varrho(j)} (-1)^{\varrho(j)+j+b} \binom{\varrho(j)}{b} \sum_{c=0}^{N-j} \sum_{p=0}^{\varrho(j)-b+c} (-\hbar)^{p+b} \binom{\varrho(j)-b+c}{p} E_{\varrho^{-1}(j+c), \varrho^{-1}(\varrho(j)-b-p+c)} \right] \right) \quad (4.4)
 \end{aligned}$$

As in (3.27)–(3.30) we imply that the values of indices corresponding to undefined argument value $N-1$ of ϱ^{-1} function are skipped in summations. Notice that the obtained R -matrix is unitary (1.3) with $f^h(z) = \frac{1}{\hbar^2} - \frac{1}{z^2}$:

$$R_{12}^h(z) R_{21}^h(-z) = \left(\frac{1}{\hbar^2} - \frac{1}{z^2} \right) 1 \otimes 1. \quad (4.5)$$

By redefinition

$$R_{12}^h(z, \epsilon) = \epsilon R_{12}^{\epsilon \hbar}(z\epsilon) \quad (4.6)$$

we can treat the answer as deformation of the standard XXX R -matrix. Indeed, one can verify that

$$\lim_{\epsilon \rightarrow 0} R_{12}^h(z, \epsilon) = \hbar^{-1} 1 \otimes 1 + z^{-1} P_{12}. \quad (4.7)$$

In the classical limit we get the rational skew-symmetric non-dynamical r -matrix from [6]:¹⁰

$$\begin{aligned}
 r^{\text{top}}(z) &= \frac{1}{Nz} 1 \otimes 1 + \frac{1}{z} \sum_{i,j=1}^N E_{ij} \otimes \quad (4.8) \\
 & \left[\sum_{\gamma=0}^{\varrho(i)} z^\gamma \binom{\varrho(i)}{\gamma} E_{\varrho^{-1}(\varrho(i)-\gamma), j} - \sum_{\gamma=0}^{\varrho(i)} z^{\gamma+N-j+1} (-1)^{\varrho(j)+N} (N-j) \binom{\varrho(i)}{\gamma} \binom{N}{j-1} E_{\varrho^{-1}(\varrho(i)-\gamma), N} \right. \\
 & - N \sum_{\gamma=0}^{\varrho(i)} \sum_{s=1}^{N-j} \delta_{\varrho(i)-j+1 \leq s+\gamma} z^{s+\gamma} (-1)^{\varrho(j)+s+j-1} \binom{\varrho(i)}{\gamma} \binom{s+j-2}{j-1} \sum_{c=0}^{N-s-j+1} E_{\varrho^{-1}(\varrho(i)-\gamma+c), \varrho^{-1}(s+j+c-1)} \\
 & - N \sum_{\gamma=0}^{\varrho(i)} \sum_{s=1}^{N-j} \delta_{\varrho(i)-j+1 > s+\gamma} z^{s+\gamma} (-1)^{\varrho(j)+s+j-2} \binom{\varrho(i)}{\gamma} \binom{s+j-2}{j-1} \sum_{c=0}^{s+j-2} E_{\varrho^{-1}(\varrho(i)-\gamma-c-1), \varrho^{-1}(s+j-c-2)} \\
 & \left. + \sum_{\gamma=0}^{\varrho(i)} \sum_{s=1}^{N-j} z^{s+\gamma} (-1)^{\varrho(j)+s+j-1} \binom{\varrho(i)}{\gamma} \binom{s+j-1}{j-1} E_{\varrho^{-1}(\varrho(i)-\gamma), \varrho^{-1}(s+j-1)} - \frac{\delta_{i,j}}{N} \sum_{k=1}^N \sum_{c=0}^{N-k-2} E_{k+c, k+c+1} \right]
 \end{aligned}$$

¹⁰Expression (4.8) differs from the one given in [6] by common factor N and scalar term $1 \otimes 1/Nz$.

Example: 11-vertex R-matrix. In $N = 2$ case we obtain the 11-vertex R-matrix:

$$R^h(z) = \begin{pmatrix} \hbar^{-1} + z^{-1} & 0 & 0 & 0 \\ -\hbar - z & \hbar^{-1} & z^{-1} & 0 \\ -\hbar - z & z^{-1} & \hbar^{-1} & 0 \\ -\hbar^3 - 2z\hbar^2 - 2\hbar z^2 - z^3 & \hbar + z & \hbar + z & \hbar^{-1} + z^{-1} \end{pmatrix} \quad (4.9)$$

This R -matrix was obtained previously in [76] and later rediscovered in [74] by applying special limiting procedure to the Baxter-Belavin elliptic R -matrix. Author of [74] suggested the algorithm which allowed to get the answer using computer calculations. In this paper we obtain explicit answer for \mathfrak{gl}_N case using different approach. It is based on the relations to the classical RS model.

Example: rational \mathfrak{gl}_3 R-matrix. In \mathfrak{gl}_3 case (4.1)–(4.4) gives the following 9×9 quantum R -matrix:

$$R^h(z) = \begin{pmatrix} \hbar^{-1} + z^{-1} & 0 & 0 \\ 1 & \hbar^{-1} & 0 \\ 2\hbar^2 + 3z\hbar + 2z^2 & -3\hbar - 3z & \hbar^{-1} \\ -1 & z^{-1} & 0 \\ 2\hbar + 2z & 0 & 0 \\ 2z^3 + 3z\hbar^2 + 2\hbar^3 + 3z^2\hbar & -3\hbar^2 - 3z\hbar - z^2 & 1 \\ -2\hbar^2 - 3z\hbar - 2z^2 & -3\hbar - 3z & z^{-1} \\ 2z^3 + 3z\hbar^2 + 2\hbar^3 + 3z^2\hbar & 3z^2 + 3z\hbar + \hbar^2 & -1 \\ 2\hbar^5 + 3z^4\hbar + 3z^2\hbar^3 + 2z^5 + 3z\hbar^4 + 3z^3\hbar^2 & 3z^4 - 3\hbar^4 - 3z\hbar^3 + 3z^3\hbar & -z^2 + \hbar^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ z^{-1} & 0 & 0 & 0 & 0 & 0 \\ -3\hbar - 3z & 3 & 0 & z^{-1} & 0 & 0 \\ \hbar^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \hbar^{-1} + z^{-1} & 0 & 0 & 0 & 0 \\ -3z\hbar - 3z^2 - \hbar^2 & 0 & \hbar^{-1} & 1 & z^{-1} & 0 \\ -3\hbar - 3z & -3 & 0 & \hbar^{-1} & 0 & 0 \\ z^2 + 3\hbar^2 + 3z\hbar & 0 & z^{-1} & -1 & \hbar^{-1} & 0 \\ 3z\hbar^3 + 3\hbar^4 - 3z^3\hbar - 3z^4 & -6\hbar^3 - 6z^3 - 9z\hbar^2 - 9z^2\hbar & 3z + 3\hbar & -\hbar^2 + z^2 & 3z + 3\hbar & \hbar^{-1} + z^{-1} \end{pmatrix} \quad (4.10)$$

Plugging it into (1.30) one gets gauge equivalent top-like form of 3-body rational RS model while for generic variables S it can be considered as alternative description of the spin RS model.

5 Belavin's R -matrix and elliptic models

5.1 Sklyanin algebra and relativistic elliptic tops

Belavin's elliptic \mathfrak{gl}_N R -matrix [16]

$$R_{12}^h(z) = \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} \varphi_a^h(z) T_a \otimes T_{-a} \quad (5.1)$$

is the central object of the section as well as related quantum L -operator:

$$\hat{L}^\eta(z) \stackrel{(2.5)}{=} \text{tr}_2 \left(R_{12}^\eta(z) \hat{S}_2 \right) = \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} \varphi_a^\eta(z) T_a \hat{S}_a. \quad (5.2)$$

In (5.1) and (5.2) the basis of \mathfrak{gl}_N and corresponding functions are chosen as¹¹

$$T_a = T_{a_1 a_2} = \exp\left(\frac{\pi i}{N} a_1 a_2\right) Q^{a_1} \Lambda^{a_2}, \quad Q_{kl} = \delta_{kl} \exp\left(\frac{2\pi i}{N} k\right), \quad \Lambda_{kl} = \delta_{k-l+1=0 \pmod N}, \quad (5.3)$$

$$\varphi_a^\eta(z) = \exp(2\pi i z \partial_\tau \omega_a) \phi(z, \omega_a + \eta), \quad \phi(z, u) = \frac{\vartheta'(0) \vartheta(u+z)}{\vartheta(z) \vartheta(u)}, \quad \omega_a = \frac{a_1 + a_2 \tau}{N}, \quad (5.4)$$

where $\vartheta(z)$ is the odd theta function and $a_1, a_2 \in \mathbb{Z}_N$.

The Sklyanin algebra (2.7) is defined by the local behavior of $\varphi_a^\eta(z)$ near $z = 0$:

$$\phi(z, u) = \left(\frac{1}{z} + E_1(u) + \frac{z}{2} (E_1^2(u) - \wp(u)) + \dots \right), \quad E_1(z) = \partial_z \log \vartheta(z). \quad (5.5)$$

Then

$$R_{12}^{\hbar, (0)}(z) = \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} E_1(\omega_a + \hbar) T_a \otimes T_{-a}, \quad (5.6)$$

and

$$\hat{L}^{\eta, (0)}(z) = \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} (E_1(\omega_a + \eta) + 2\pi i \partial_\tau \omega_a) T_a \hat{S}_a. \quad (5.7)$$

The last terms ($2\pi i \partial_\tau \omega_a$) are canceled out in the final answers.

Sklyanin algebra $\mathcal{A}_{\hbar, \eta}^{\text{SkI}}$ for $\hbar, \eta \neq 0$ in components $\hat{S}_a = \text{tr}(\hat{S} T_{-a})$ can be derived from either (2.7) together with (5.6), (5.7) or directly — by plugging (5.1) and (5.2) into exchange relations (1.1). The latter way requires identity

$$\varphi_{a-c}^\eta(z) \varphi_{b+c}^\eta(w) \varphi_c^\hbar(z-w) - \varphi_{b+c}^\eta(z) \varphi_{a-c}^\eta(w) \varphi_{a-b-c}^\hbar(z-w) = \mathbf{f}(a, b, c | \tau, \hbar, \eta) \varphi_a^{\eta+\hbar}(z) \varphi_b^{\eta-\hbar}(w),$$

where

$$\mathbf{f}(a, b, c | \tau, \hbar, \eta) = E_1(\omega_c + \hbar) - E_1(\omega_{a-b-c} + \hbar) + E_1(\omega_{a-c} + \eta) - E_1(\omega_{b+c} + \eta). \quad (5.8)$$

The quadratic relations $\mathcal{A}_{\tau, \hbar, \eta}^{\text{SkI}}$ in the $T_a \otimes T_b$ component of (1.1) read as follows:

$$\sum_{c \in \mathbb{Z}_N \times \mathbb{Z}_N} \mathbf{f}(a, b, c | \tau, \hbar, \eta) \left(\hat{S}_{a-c} \hat{S}_{b+c} \kappa_{c, a-b} - \hat{S}_{b+c} \hat{S}_{a-c} \kappa_{a-b, c} \right) = 0, \quad (5.9)$$

¹¹See for example review [75].

where $\kappa_{a,b} = \exp \frac{\pi i}{N}(a_2 b_1 - a_1 b_2)$ comes from $T_a T_b = \kappa_{a,b} T_{a+b}$. Notice that in the case $\eta = \hbar$ and $b = 0$ one should consider the limit $\eta \rightarrow \hbar$ in (5.8) which gives

$$\varphi_{a-c}^{\hbar}(z)\varphi_{b+c}^{\hbar}(w)\varphi_c^{\hbar}(z-w) - \varphi_{b+c}^{\hbar}(z)\varphi_{a-c}^{\hbar}(w)\varphi_{a-b-c}^{\hbar}(z-w) = \mathbf{f}(a, 0, c|\tau, \hbar, \hbar) \varphi_a^{2\hbar}(z),$$

$$\mathbf{f}(a, 0, c|\tau, \hbar, \hbar) = E_2(\omega_c + \hbar) - E_2(\omega_a - \omega_c + \hbar).$$
(5.10)

The case $\eta = \hbar$ and $N = 2$ in (5.9) gives rise to the Sklyanin algebra in its original form [72, 73].

It follows from (5.8) that the “structure constant” $\mathbf{f}(a, b, c|\tau\hbar, \eta)$ is double periodic with respect to the shifts

$$\hbar \rightarrow \hbar + \mathbb{Z} + \mathbb{Z}\tau, \quad \eta \rightarrow \eta + \mathbb{Z} + \mathbb{Z}\tau.$$

Therefore, we can consider the pair (\hbar, η) as points on two elliptic curves $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. Consider the upper half-plane $\mathcal{H}^+ \subset \mathbb{C}$. The moduli space \mathfrak{M} of elliptic curves is the result of the action of $SL_2(\mathbb{Z})$ on \mathcal{H}^+ by the Möbius transform

$$\mathfrak{M} = \mathcal{H}^+ / SL_2(\mathbb{Z}), \quad \left(\tau \rightarrow \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right).$$

The modular transformations acts on the theta-function as

$$\vartheta \left(\frac{v}{\alpha + \beta\tau} \middle| \frac{\gamma + \delta\tau}{\alpha + \beta\tau} \right) = \zeta(\alpha + \beta\tau)^{\frac{1}{2}} \exp \left(\frac{i\pi\alpha v^2}{\alpha + \beta\tau} \right) \vartheta(v|\tau),$$

where $\zeta^8 = 1$. Then we can find that $\mathbf{f}(a, b, c|\tau, \hbar, \eta)$ is modular invariant, and therefore, it is a well defined function on \mathfrak{M} . In this way the universal bundle

$$\mathcal{E}_\tau = \begin{array}{c} \Sigma_\tau \times \Sigma_\tau \\ \downarrow \\ \mathfrak{M} \end{array} \tag{5.11}$$

plays the role of the moduli space of the algebra $\mathcal{A}_{\tau, \hbar, \eta}^{\text{SkI}}$. Then the moduli space is

$$\text{Mod}(\mathcal{A}_{\tau, \hbar, \eta}^{\text{SkI}}) = \mathcal{E}_\tau. \tag{5.12}$$

Relation between $\mathcal{A}_{\eta \neq 0}^{\text{SkI}}$ and $\mathcal{A}_{\eta=0}^{\text{SkI}}$. Following [23]¹² let us clarify the relation between η -dependent and η -independent L -operators discussed above (2.58). Two descriptions are distinct from each other by quasiperiodic boundary conditions on the lattice $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$:

$$L^\eta(z+1) = Q^{-1}L^\eta(z)Q, \quad L^\eta(z+\tau) = \exp(-2\pi i \eta)\Lambda^{-1}L^\eta(z)\Lambda, \tag{5.13}$$

$$L(z+1) = Q^{-1}L(z)Q, \quad L(z+\tau) = \Lambda^{-1}L(z)\Lambda, \tag{5.14}$$

where Q and Λ are from (5.3). Notice that $L(z)$ (5.14) is a section of $\text{End}V$ -bundle, where V is the holomorphic vector bundle with the transition functions Q and $\tilde{\Lambda}(z) =$

¹²In that paper $\eta = \hbar$ was considered.

$\exp(2\pi i(\frac{z}{N} + \frac{\tau}{2N}))\Lambda$. In the same time $L^\eta(z)$ should be considered as a map between V and V' , where V' is defined by the transition functions Q and $\Lambda'(z) = \exp(2\pi i\eta)\Lambda(z)$.

Set $\eta_0 = -\eta$ and $g^\eta(z) = 1/\phi(z - \eta, \eta)$ in (2.58). Then

$$\frac{1}{\phi(z - \eta, \eta)} \hat{L}^\eta(z - \eta, \varphi(\hat{S})) = T_0 \hat{S}_0 + \sum_{a \neq 0} \varphi_a^0(z) T_a \hat{S}_a, \quad (5.15)$$

where

$$\varphi(\hat{S}) = T_0 \hat{S}_0 + \sum_{a \neq 0} \varphi_a^0(\eta) T_a \hat{S}_a, \quad (5.16)$$

i.e. the η -independent description is given by

$$L(z, \hat{S}) = T_0 \hat{S}_0 + \sum_{a \neq 0} \varphi_a^0(z) T_a \hat{S}_a = T_0 \hat{S}_0 + \text{tr}_2(r_{12}(z) \hat{S}), \quad (5.17)$$

The underlying identity is very simple:

$$\varphi_a^\eta(z - \eta) / \varphi_0^\eta(z - \eta) = \varphi_a^0(z) / \varphi_a^0(\eta). \quad (5.18)$$

The sum over $a \neq 0$ corresponds to \mathfrak{sl}_N part, i.e. \hat{S} can be replaced with \hat{S} in the r.h.s. of (5.15) and (5.16) as in (2.57). The classical r -matrix (2.12) emerging in the η -independent form (2.57) is the Belavin-Drinfeld \mathfrak{sl}_N r -matrix [17]:

$$r_{12}(z) = \sum_{a \neq 0} \varphi_a^0(z) T_a \otimes T_{-a}. \quad (5.19)$$

The scalar term $T_0 \otimes T_0 E_1(z)$ is not important here.

Relativistic top appears in the quasi-classical limit. The Lax matrix (2.10)¹³

$$L^\eta(z) = \text{tr}_2(R_{12}^\eta(z) S_2) = \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} \varphi_a^\eta(z) T_a S_a. \quad (5.20)$$

together with M -matrix (2.29)

$$M(z) = -\text{tr}_2(r_{12}(z) S_2) \stackrel{(5.19)}{=} - \sum_{a \neq 0} \varphi_a^0(z) T_a S_a \quad (5.21)$$

provides equations of motion (2.26) generated by the Hamiltonian $H = S_0$:

$$\partial_{t_0} S = \{S_0, S\} = [S, J^\eta(S)], \quad (5.22)$$

where the inverse inertia tensor J^η (2.27):

$$\begin{aligned} J^\eta(S) &= \text{tr}_2 \left(\left(R_{12}^{\eta, (0)} - r_{12}^{(0)} \right) S_2 \right) = T_0 S_0 E_1(\eta) + \sum_{a \neq 0} T_a S_a (E_1(\omega_a + \eta) - E_1(\omega_a)) = \\ &= S E_1(\eta) + \sum_{a \neq 0} T_a S_a (E_1(\omega_a + \eta) - E_1(\omega_a) - E_1(\eta)). \end{aligned} \quad (5.23)$$

The first (scalar) term $T_0 S_0 E_1(\eta)$ in the upper line of (5.23) vanishes from the commutator in (5.22) as well as the first term $S E_1(\eta)$ in the lower line. To verify (5.22), (5.23) one needs the following identity:

$$\varphi_a^\eta(z) \varphi_b^0(z) = \varphi_{a+b}^\eta(z) (E_1(z) + E_1(\omega_a + \eta) + E_1(\omega_b) - E_1(z + \omega_{a+b} + \eta)). \quad (5.24)$$

¹³In (2.10) $\mathcal{J}^\eta_{a,b}(z) = \delta_{a+b} \varphi_a^\eta(z)$.

Non-relativistic limit $\eta \rightarrow 0$ coincides with η -independent description at the level of equations of motion because the Lax matrices (2.33) and (2.45) are the same up to the scalar term $S_0 1$. This is due to existence of bihamiltonian structure, i.e. any linear combination of the linear and quadratic Poisson brackets are again some Poisson bracket (see [37] for details).

The equations of motion (5.22) keep the same form in the limit with

$$J(S) = \lim_{\eta \rightarrow 0} \eta^{-1} J^\eta(\bar{S}) = - \sum_{a \neq 0} T_a S_a E_2(\omega_a), \quad E_2(z) = -\partial_z^2 \log \vartheta(z), \quad (5.25)$$

where \bar{S} is the \mathfrak{sl}_N part of S . The equations of motion $\dot{S} = [S, J(S)]$ are generated by the Lax pair $l(z) = \sum_{a \neq 0} \varphi_a^0(z) T_a S_a$ and the M -matrix (2.54):

$$\mathcal{M}(z) = \sum_{a \neq 0} f_a(z) T_a S_a, \quad f_a(z) = \partial_\eta \varphi_a^\eta(z) |_{\eta=0}. \quad (5.26)$$

The underlying elliptic function identity is very well known [38]:

$$\varphi_a^0(z) f_b(z) - \varphi_b^0(z) f_a(z) = \varphi_{a+b}^0(z) (E_2(\omega_a) - E_2(\omega_b)) = \varphi_{a+b}^0(z) (\wp(\omega_a) - \wp(\omega_b)). \quad (5.27)$$

This model was introduced in [47]. Its phase space is the coadjoint orbit of Lie group GL_N . When dimension of the orbit is minimal ($2N - 2$) the Lax matrix is gauge equivalent to the one of elliptic Calogero-Moser model. More detailed description can be found in [75]. Higher rank Sklyanin algebras in the context of integrable systems were also discussed in [18, 23, 37].

5.2 Poincaré invariance

The Poincaré Lie algebra for the relativistic integrable systems [69, 70] is defined as

$$\{\mathcal{H}, \mathcal{P}\} = 0, \quad \{\mathcal{B}, \mathcal{H}\} = \mathcal{P}, \quad \{\mathcal{B}, \mathcal{P}\} = \mathcal{H}. \quad (5.28)$$

The RS models can be obtained by symplectic (or the Poisson) reduction procedures from the cotangent bundles to a certain loop groups [34, 35].¹⁴ For the top-like (elliptic) Lax operators (1.19) a similar procedure was suggested in [18] and [23]. In all the description the Lax matrix appears through reduction from the group element. It satisfies some moment map constraint generated by the symmetries of the (co)adjoint action.

Let us show that the mentioned above reductions provide naturally the Poincaré Lie algebra (5.28). As a preliminary, consider the finite-dimensional case, i.e. the cotangent bundle T^*G to $G = GL_N$ Lie group. Let $g \in G$ and $A \in \mathfrak{gl}_N^*$. The symplectic structure on T^*G is equal to

$$\omega = d \operatorname{tr} (A g^{-1} dg). \quad (5.29)$$

The corresponding Poisson brackets are of the form:

$$\{A_1, A_2\} = [P_{12}, A_2], \quad \{g_1, g_2\} = 0, \quad (5.30)$$

$$\{A_1, g_2\} = g_1 P_{12}, \quad (5.31)$$

¹⁴See also example of the relativistic Toda systems [71] in [31, 32].

Taking tr_1 of (5.31) we get

$$\{\text{tr}A, g\} = g. \tag{5.32}$$

Therefore, $\{\text{tr}A, g^{-1}\} = -g^{-1}$, and we have the following identification with (5.28):

$$\mathcal{H} := \text{tr}(g + g^{-1}), \quad \mathcal{P} := \text{tr}(g - g^{-1}), \quad \mathcal{B} := \text{tr}A. \tag{5.33}$$

Notice that the variable dual to the boost \mathcal{B} is $\log(\det g)^{1/N}$. Indeed, it follows from (5.32) that

$$\{\text{tr}A, \det g\} = N \det g. \tag{5.34}$$

After reduction by the action of the gauge group

$$g \rightarrow f^{-1}gf, \quad A \rightarrow f^{-1}Af, \quad f = f(A) \tag{5.35}$$

the trivial Poisson brackets $\{g_1, g_2\} = 0$ become those quadratic of the r -matrix form (1.18) [18]. The quantities \mathcal{H} , \mathcal{P} and \mathcal{B} from (5.33) are gauge invariants. The Poisson brackets (5.28) of the Poincaré Lie algebra keep the same form after reduction.

Similar reasoning is valid in more complicated cases. In [23] the classical Lax operator $L(z, S)$ (5.17) (where $\hat{S} \rightarrow S$) was derived starting from the affine space over the cotangent bundle $T^*LL(G)$ to the two-loop group $LL(G)$, $G = \text{GL}_N$. Here we deal with the holomorphic bundle on the elliptic curve Σ_τ (with moduli τ) given by transition functions Q and $\bar{\Lambda} = \exp(2\pi i (-\frac{z}{N} - \frac{\tau}{2N}))\Lambda$ from (5.3). \bar{A} — is the component of the $d_{\bar{A}}$ connection on Σ_τ . In the Dolbeault description the holomorphic structure of the vector bundle is defined by the operator

$$d_{\bar{A}} = \bar{\partial} + \bar{A} : \Omega^{(0,0)}(\Sigma_\tau, \mathfrak{gl}(N, \mathbb{C})) \rightarrow \Omega^{(0,1)}(\Sigma_\tau, \mathfrak{gl}(N, \mathbb{C})) \tag{5.36}$$

$$\bar{A}(z+1, \bar{z}+1) = Q\bar{A}(z, \bar{z})Q^{-1}, \quad \bar{A}(z+\tau, \bar{z}+\bar{\tau}) = \Lambda\bar{A}(z, \bar{z})\Lambda^{-1}.$$

A section $\xi(z, \bar{z})$ is holomorphic if $d_{\bar{A}}\xi(z, \bar{z}) = 0$. Two holomorphic structures $d_{\bar{A}}$ and $d_{\bar{A}'}$ are equivalent if they are related by the gauge transformation of the gauge group \mathcal{G}

$$\bar{A} \rightarrow f^{-1}\bar{A}f + f^{-1}\bar{\partial}f. \tag{5.37}$$

The quotient of the space of generic connections $\mathcal{A} = \{d_{\bar{A}}\}$ ¹⁵ by the gauge group action is the moduli space $Bun(\Sigma_\tau, \text{GL}(N, \mathbb{C})) = \mathcal{A}/\mathcal{G}$ of holomorphic bundles. The initial phase space P_{Σ_τ} is the Poisson algebra of holomorphic functionals on \mathcal{R} . The Poisson brackets are similar to (5.30), (5.31) (see [23]). The gauge transformations (5.37) along with

$$g \rightarrow f^{-1}gf, \quad f \in \mathcal{G}^s \subset \mathcal{G} : f(z, \bar{z})|_{z=0} = 1 \tag{5.38}$$

leads to the finite-dimensional reduced phase space P^{red} . The reduced Poisson algebra coincides with (2.46) for the elliptic r -matrix (5.19), and

$$g(z, \bar{z})|_{\text{Pred}} = L(z) = T_0S_0 + \text{tr}_2(r_{12}(z)S_2) = T_0S_0 + \sum_{\alpha \neq 0} \varphi_\alpha^0(z)T_\alpha S_\alpha. \tag{5.39}$$

¹⁵We don't discuss here the stability of the bundles, because we need only an open subset of \mathcal{A} .

From (5.37) it follows that

$$\mathrm{tr}\bar{A} \rightarrow \mathrm{tr}\bar{A} + \mathrm{tr}(f^{-1}\bar{\partial}f). \quad (5.40)$$

Since $\mathrm{tr}\bar{A}$ is double-periodic it can be gauge transformed to a constant a , i.e.

$$\mathrm{tr}\bar{A}(z, \bar{z})|_{\mathrm{pred}} = a. \quad (5.41)$$

Using the arguments similar to (5.30)–(5.32) one can see that

$$\{a, L(z)\} = L(z). \quad (5.42)$$

and reproduce the Poincaré algebra (5.28) in the form of (5.33) with $g = L(z)$. The variable a extends the phase space of the top. The variable dual to a is the one which acts on the top variables by dilatation $S \rightarrow \lambda S$. This action does not preserve the values of the Casimir functions generated by $\det L(z, S)$. Therefore, the Poincaré symmetry emerges on the top's phase space extended by the two-dimensional space (a, λ) — cotangent bundle to the one-dimensional center of the group.

5.3 Large N limit: 2d elliptic hydrodynamics

In this paragraph we consider the large N limit of the elliptic tops following and [37, 64] (see also [5]), where η -independent case was studied. This type of limit leads to 2d hydrodynamics [8–10]. The idea is to replace the generators (5.3) of \mathfrak{gl}_N with

$$T_a := \frac{i}{2\pi\theta} \exp\left(2\pi i \frac{a_1 a_2}{2} \theta\right) U_1^{a_1} U_2^{a_2} \quad a \in \mathbb{Z}^{(2)} = \mathbb{Z} \oplus \mathbb{Z}. \quad (5.43)$$

While Q and Λ from (5.3) gives the finite-dimensional representation of the Heisenberg group, the generators U_1 and U_2 satisfy commutation relation of the noncommutative torus \mathcal{T}_θ^2 :

$$U_1 U_2 = e^{-2\pi i \theta} U_2 U_1, \quad \theta \in [0, 1). \quad (5.44)$$

A generic element from \mathcal{T}_θ^2 is $X = \sum_{a_1, a_2 \in \mathbb{Z}} c_{a_1, a_2} U_1^{a_1} U_2^{a_2}$, $c_{a_1, a_2} \in \mathbb{C}$. This space is naturally identified with smooth functions on the two-dimensional torus $T^2 = \{\mathbb{R}^2 / \mathbb{Z} \oplus \mathbb{Z}\}$:

$$U_1 \rightarrow \exp(2\pi i x_1), \quad U_2 \rightarrow \exp(2\pi i x_2), \quad 0 < x_1, x_2 \leq 1 \quad (5.45)$$

with the Moyal multiplication

$$\exp(2\pi i x_1) \star \exp(2\pi i x_2) = e^{-2\pi i \theta} \exp(2\pi i x_2) \star \exp(2\pi i x_1), \quad (5.46)$$

or

$$(f \star g)(x) := fg + \sum_{n=1}^{\infty} \frac{(\imath\pi\theta)^n}{(2\pi\imath)^{2n} n!} \varepsilon_{r_1 p_1} \dots \varepsilon_{r_n p_n} (\partial_{x_{r_1} \dots x_{r_n}}^n f) (\partial_{x_{p_1} \dots x_{p_n}}^n g) \quad (5.47)$$

for functions

$$f(x) = \sum_{a \in \mathbb{Z}^{(2)}} f_a T_a(x), \quad T_a(x) = \frac{i}{2\pi\theta} \exp\left(2\pi i \frac{a_1 a_2}{2} \theta\right) \exp(2\pi i a_1 x_1) \exp(2\pi i a_2 x_2). \quad (5.48)$$

Then

$$U_1 f(x) = f(x - \theta), \quad U_2 f(x) = \exp(2\pi i x) f(x). \quad (5.49)$$

In other words, U_1 and U_2 are $\text{GL}(\infty)$ analogues of Q and Λ from (5.3). The finite-dimensional relations $T_a T_b = \kappa_{a,b} T_{a+b}$ are saved in the infinite-dimensional case with

$$\kappa_{a,b}^\theta = -2\pi i \theta \exp(\pi i \theta a \times b), \quad (a \times b = a_2 b_1 - a_1 b_2). \quad (5.50)$$

Then we can introduce the following generalization of the Belavin's R -matrix (5.1) [37]:

$$R_{12}^h(z|\theta, \epsilon) = \sum_{a \in \mathbb{Z} \times \mathbb{Z}} \varphi_a^h(z|\theta, \epsilon) T_a \otimes T_{-a} \quad (5.51)$$

Here

$$\begin{aligned} \varphi_a^h(z|\theta, \epsilon) &= \exp(2\pi i \epsilon_2 a_2 z \theta) \phi((\epsilon_1 a_1 + \tau \epsilon_2 a_2) \theta + \hbar, z), \\ \epsilon &= (\epsilon_1, \epsilon_2), \quad \epsilon_a \theta < 1. \end{aligned} \quad (5.52)$$

and T_a is the basis (5.43). It satisfies the Yang-Baxter equation (1.2). The related quantum L -operator is similar to (5.2):

$$\hat{L}^\eta(z|\theta, \epsilon) \stackrel{(2.5)}{=} \text{tr}_2 \left(R_{12}^\eta(z) \hat{S}_2 \right) = \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} \varphi_a^\eta(z|\theta, \epsilon) T_a \hat{S}_a, \quad (5.53)$$

where tr is defined as trace functional on \mathcal{T}_θ^2 :

$$\langle X \rangle = \text{tr}(X) = c_{00}, \quad \langle 1 \rangle = 1, \quad \langle XY \rangle = \langle YX \rangle. \quad (5.54)$$

In the Moyal representation: $\text{tr} f = -\frac{1}{4\pi^2} \int_{\mathcal{T}_\theta^2} f dx_1 dx_2 = f_{00}$.

The Lax operator (5.53) satisfies the quasi-periodicity conditions

$$\begin{cases} L^\eta(z+1) = U_1^{-\epsilon_2} L^\eta(z) U_1^{\epsilon_2}, \\ L^\eta(z+\tau) = \exp(2\pi i \eta) U_2^{-\epsilon_1} L^\eta(z) U_2^{\epsilon_1}, \end{cases} \quad (5.55)$$

Here ϵ_1 and ϵ_2 are arbitrary real numbers in the sense of (5.49). Conditions (5.55) mean that $L^\eta(z)$ is a section of the (twisted) Higgs bundle over elliptic curve Σ_τ with the structure group SIN_θ . The latter consists of invertible elements of \mathcal{T}_θ^2 (see e.g. [5]). The exchange relations (1.1) provide the direct generalization of the Sklyanin algebra (2.7):

$$\begin{aligned} &\mathcal{A}_{\tau, \hbar, \eta, \theta, \epsilon}^{\text{Skl}} : \\ &P_{12} L^{\eta, (0)}(\theta, \epsilon) (\hat{S})_1 \hat{S}_2 + R_{12}^{\hbar, (0)}(\theta, \epsilon) \hat{S}_1 \hat{S}_2 = \hat{S}_2 L^{\eta, (0)}(\theta, \epsilon) (\hat{S})_1 P_{12} + \hat{S}_2 \hat{S}_1 R_{12}^{\hbar, (0)}(\theta, \epsilon). \end{aligned} \quad (5.56)$$

with

$$R_{12}^{\hbar, (0)}(z|\theta, \epsilon) = \sum_{a \in \mathbb{Z} \times \mathbb{Z}} E_1((\epsilon_1 a_1 + \epsilon_2 a_2 \tau) \theta + \hbar) T_a \otimes T_{-a}, \quad (5.57)$$

and

$$\hat{L}^{\eta, (0)}(z|\theta, \epsilon) = \sum_{a \in \mathbb{Z} \times \mathbb{Z}} E_1((\epsilon_1 a_1 + \epsilon_2 a_2 \tau) \theta + \eta) T_a \hat{S}_a. \quad (5.58)$$

Notice that this algebra depends on five parameters: $\hbar, \eta, \tau, \theta \epsilon_1, \theta \epsilon_2$.

Classical limit $\hbar \rightarrow 0$ leads to 2d elliptic hydrodynamics written in the Euler-Arnold form [8–10]. First, mention that the parameters $\epsilon_{1,2}$ (5.52) allow to define the complex structure on the noncommutative torus \mathcal{T}_θ^2 . For element $X = \sum_a c_a T_a$ define

$$\bar{\partial}_{\epsilon,\tau} X = \sum_a (\epsilon_1 a_1 + \epsilon_2 a_2 \tau) c_a T_a. \tag{5.59}$$

The operator J^η (5.23) acts as the following pseudo-differential operator:

$$J^\eta(S)(x) = E_1(\eta + \theta \bar{\partial}_{\epsilon,\tau}) S(x). \tag{5.60}$$

Consider analogue of the classical finite-dimensional Lax pair (5.20), (5.21):

$$L^\eta(z, S(x)) = \sum_{a \in \mathbb{Z}^{(2)}} S_a \varphi_a^\eta(z|\theta, \epsilon) T_a, \quad M(z, S(x)) = \sum_{a \in \mathbb{Z}^{(2)}} S_a \varphi_a^0(z|\theta, \epsilon) T_a. \tag{5.61}$$

Then the Lax equations provide equations of motion:

$$\partial_t S(x) = \text{ad}_{J^\eta(S)(x)}^* S(x) = [S(x), J^\eta(S)(x)]_\theta, \tag{5.62}$$

where

$$[f(x), g(x)]_\theta = \theta^{-1} (f \star g - g \star f). \tag{5.63}$$

In components we have:

$$\begin{aligned} \partial_t S_\alpha &= \sum_{\gamma \in \mathbb{Z}^{(2)}} C_\theta(\alpha, \gamma) S_\gamma S_{\alpha-\gamma} E_1((\epsilon_1 \gamma_1 + \epsilon_2 \gamma_2 \tau) \theta + \eta), \\ C_\theta(\alpha, \beta) &= \frac{1}{\pi \theta} \sin(\pi \theta (\alpha \times \beta)). \end{aligned} \tag{5.64}$$

The obtained equations (5.62) can be treated as hydrodynamical limit of the elliptic spin Ruijsenaars-Schneider model. It is an interesting problem to find its relation to another type of hydrodynamical limit [1, 2]. The latter approach leads to the quantum \mathfrak{gl}_N Benjamin-Ono and KdV systems while our approach gives rise to the \mathfrak{gl}_N Sklyanin type algebra (5.56) and classical equations (5.62), (5.63).

In the *non-relativistic limit* $\eta \rightarrow 0$ we reproduce the answer from [37, 64]. Likewise the RS model goes into CM one, the relativistic top goes to non-relativistic (5.25). In the same way instead of J^η (5.60) we get

$$J(S)(x) = -E_2(\theta \bar{\partial}_{\epsilon,\tau}) S(x). \tag{5.65}$$

In a similar way one can describe the dispersionless limit $\theta \rightarrow 0$, $\epsilon_{1,2} \rightarrow \infty$, $\lim_{\theta \rightarrow 0} (\theta \epsilon_{1,2}) = \epsilon'_{1,2} < 1$. In the limit the Lie algebra \mathfrak{sin}_θ of the group SIN_θ becomes the Lie algebra $Ham(T^2)$ of Hamiltonian vector fields on the two-dimensional torus.

6 Conclusion

Let us briefly summarize the obtained results.

Theorem 1 *Let $\hat{L}^\eta(z) = \text{tr}_2 \left(R_{12}^\eta(z) \hat{S}_2 \right)$, $\hat{S} = \text{Res}_{z=0} \hat{L}(z)$ be \mathfrak{gl}_N solution of the quantum exchange relations (1.1) with the quantum non-dynamical R-matrix satisfying (1.2), (1.3) and (2.3), (2.12). Then*

- 1) *The quantum exchange relations (1.1) define the following \mathfrak{gl}_N Sklyanin algebra:*

$$\mathcal{A}_{\hbar,\eta}^{skl} : \quad P_{12} L^{\eta,(0)}(\hat{S})_1 \hat{S}_2 + R_{12}^{\hbar,(0)} \hat{S}_1 \hat{S}_2 = \hat{S}_2 L^{\eta,(0)}(\hat{S})_1 P_{12} + \hat{S}_2 \hat{S}_1 R_{12}^{\hbar,(0)}. \quad (6.1)$$

- 2) *The \mathfrak{gl}_N -valued Lax matrix*

$$L^\eta(z) = \text{tr}_2 \left(R_{12}^\eta(z) S_2 \right), \quad S = \text{Res}_{z=0} L(z) \quad (6.2)$$

defines the classical integrable system described by the Poisson structure

$$\begin{aligned} \{L_1^\eta(z), L_2^\eta(w)\} &= [L_1^\eta(z) L_2^\eta(w), r_{12}(z-w)], \\ \mathcal{A}_{\hbar=0,\eta}^{skl} : \quad \{S_1, S_2\} &= [S_1 S_2, r_{12}^{(0)}] + [L^{\eta,(0)}(S)_1 S_2, P_{12}]. \end{aligned} \quad (6.3)$$

- 3) *The simplest Hamiltonian $\text{tr}(S)$ generates top-like equations of motion*

$$\dot{S} = [S, J^\eta(S)] \quad (6.4)$$

with the inverse inertia tensor

$$J^\eta(S) = \text{tr}_2 \left(\left(R_{12}^{\eta,(0)} - r_{12}^{(0)} \right) S_2 \right), \quad (6.5)$$

where $R_{12}^{\eta,(0)}$ and $r_{12}^{(0)}$ are the coefficients of the local expansion of the quantum R-matrix (2.3) and the classical r-matrix (2.13) respectively.

- 4) *Equations (6.4) are presented in the Lax form $\dot{L}^\eta(z) = [L^\eta(z), M(z)]$ with the M-operator given in terms of the classical r-matrix:*

$$M(z) = -\text{tr}_2 \left(r_{12}(z) S_2 \right). \quad (6.6)$$

- 5) *Alternatively, the relativistic top can be described in η -independent form as bihamiltonian system with the quadratic Poisson structure described in section 2.5 and the linear Poisson structure given in section 2.4. The relation between η -dependent and η -independent descriptions are given by (2.59).*

Theorem 2 *The classical Lax matrix (3.27)–(3.30) provides an example of relativistic integrable top described in Theorem 1. If S is of rank one and $\det L^\eta(z) = \frac{z+\eta}{z}$ then the model is gauge equivalent to the rational \mathfrak{sl}_N RS model defined by the Lax matrix (3.4) with the change of variables*

$$S_{ij}(\mathbf{p}, \mathbf{q}) = \sum_{m=1}^N \frac{(q_m + \eta)^{\varrho(i)} e^{p_m/c}}{\prod_{k \neq m} (q_m - q_k)} (-1)^{\varrho(j)} \sigma_{\varrho(j)}(\mathbf{q}). \quad (6.7)$$

Theorem 3 *The quantum R -matrix (4.1)–(4.4) is a unitary solution of the Yang-Baxter equation (1.2) with $f^{\hbar}(z) = \hbar^{-2} - z^{-2}$ from (1.3). It can be obtained from the Lax matrix (3.27)–(3.30) as*

$$R_{12}^{\hbar}(z) = \sum_{k,l=1}^N \frac{\partial L^{\hbar}(z)}{\partial S_{kl}} \otimes E_{lk}. \tag{6.8}$$

In the limit (1.28)–(1.29) it gives the XXX R -matrix.

The latter statement can be obtained by the direct IRF-Vertex transformation starting from the quantum R -matrix for the rational RS model. We will give this proof elsewhere.

The obtained rational R -matrix allows to define new type of spin chains and Gaudin models. In the elliptic case, in addition to the relativistic top we describe the large N limit as the elliptic hydrodynamics.

Remarks

- Relations between the Ruijsenaars-Schneider (RS) systems and quantum integrable chains appeared recently in the context of the Quantum-Classical duality using the Bethe ansatz approach [33] or the τ -function approach [3, 4]. This duality, in particular, implies the substitution $\eta = \hbar$ into the Lax matrix of the RS model and provides an alternative (to the algebraic Bethe ansatz) method for computation of spectrum of the quantum spin chains transfer-matrices.

The phenomenon of the Quantum-Classical duality type was also observed at the level of gauge theories in the series of papers [58–62]. In this approach the Planck constant (in quantum integrable system) was identified with the twisted mass parameter in the $\mathcal{N} = 2^*$ SUSY Yang-Mills theory. The latter mass parameter is related to the action of the global U(1) group on the adjoint chiral multiplet field. It resembles the appearance of the η parameter in the twisted boundary conditions for the Lax operator (5.13), i.e. the twisted mass and the η play similar roles and can be closely related.

A relation between the classical and quantum systems arises also in studies of the spectral duality [55–57]. A general statement is that the spectral duality works in the same way both at classical and quantum levels, or the properly defined classical and quantum spectral curves for spin chains coincide. It also resembles the similarity of the quantum R -matrices and the classical Lax operators.

- Expression (4.1)–(4.4) for the rational R -matrix is complicated. In the same time, the computations of particular examples emerge a lot of cancellations. We hope that the answer can be simplified. It is an interesting problem to find some elegant form for the rational R -matrix of group-theoretical type.
- The classification of integrable systems of Hitchin type on elliptic curves can be naturally made in terms of characteristic classes of underlying Higgs bundles [48–51, 75]. In particular, it allows one to obtain intermediate solutions of the quantum Yang-Baxter equation (between pure dynamical for RS model and pure non-dynamical

one) [52, 53]. In the rational case we deal with degenerated (and punctured) elliptic curve $y^2 = z^3$. It is interesting to know whether elliptic classification survives in the rational limit.

- In this paper we do not consider the trigonometric case. However, it would appear reasonable that the obtained results are valid in this case as well. The corresponding non-dynamical quantum R -matrix was obtained in [7] from the trigonometric RS model via the IRF-Vertex transformation. It can be used for construction of trigonometric top-like models.

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