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RELATIVISTIC COSMOLOGICAL MODELS WITH BOTH RADIATION AND MATTER

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Summary

The general theory is given for relativistic cosmological models which contain a mixture of radiation (pressure = $\frac{1}{3}$ radiation energy density) and dust (pressure zero) and which change from radiation like models for early time to dust like models for recent time. The basic parameter of the models is $\epsilon(t)$, the ratio of the pressure to the matter energy content of the universe. ϵ thus changes from $\frac{1}{3}$ at $t = 0$ to 0 for large t.

Special models are developed in which the initial rate of absorption of radiation by the dust is greater than the rate of emission and in which the present rate of emission is greater than the rate of absorption. These models have $\epsilon = \frac{1}{3}(1 + \mu t)^{-a}$ where μ is a positive constant and, for flat space, $0 < a < \frac{2}{3}$. These models have a radiation dominated early stage, a matter dominated present stage, a present value of radiation equivalent to that from a black-body at 3 °K and rates of emission and absorption of radiation near the observed values and are thus reasonable models of the Universe. The best value of a is about $\frac{7}{11}$.

1. *Introduction.* In relativistic cosmology, the assumptions are usually made that the universe is isotropic and homogeneous and is filled with matter or dust at the present epoch. The first assumption means that the Robertson–Walker metric

$$ds^2 = c^2dt^2 - R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\psi^2)\right\} \tag{1}$$

can be used to describe the space-time of the Universe; r, θ and ψ being dimensionless, co-moving co-ordinates and k the curvature, $+1$, 0 or -1. When this is substituted in Einstein's field equations with zero cosmological constant, the differential equations

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2 + kc^2}{R^2} = -\kappa p c^2 \tag{2}$$

and

$$3\frac{\dot{R}^2 + kc^2}{R^2} = \kappa\rho c^4 \tag{3}$$

result. p and ρ are the pressure and density respectively and $\kappa = 8\pi G c^{-4}$. The second assumption ($p = 0$) means that the solutions of equations (2) and (3) are

$$R = 2a \times \begin{cases}\sin^2\phi \\ \phi^2 \\ \sinh^2\phi\end{cases}, \quad ct = 2a \times \begin{cases}(\phi - \sin\phi\cos\phi) & k = +1 \\ \frac{2}{3}\phi^3 & k = 0 \\ (\sinh\phi\cosh\phi - \phi) & k = -1\end{cases} \tag{4}$$

and

$$\kappa\rho c^4 = 3H^2 \times \begin{cases}\sec^2\phi & k = +1 \\ 1 & k = 0 \\ \operatorname{sech}^2\phi & k = -1\end{cases} \tag{5}$$

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where a is a constant, ϕ is a parameter and H is Hubble's parameter;

$$H = \dot{R}/R. \tag{6}$$

Notice that the $k = +1$ model is the familiar cycloidal oscillating one and the $k = 0$ model is the Einstein–de Sitter one. The choice of a reasonably accurate model for the present epoch is usually restricted to one of these.

Models filled with radiation only, however, are generally chosen as reasonably accurate ones for early epochs, i.e. for times soon after the singularity at $t = 0$ which occurs in each of the models in (4). If this assumption is made, $p = \frac{1}{3}\rho c^2$ for this early time, and the solutions of equations (2) and (3) are now

$$R = b \times \begin{cases} \sin\phi \\ \phi \\ \sinh\phi \end{cases}, \quad ct = b \times \begin{cases} (1-\cos\phi) \\ \frac{1}{2}\phi^2 \\ (\cosh\phi - 1) \end{cases}, \quad \begin{matrix} k = +1 \\ k = 0 \\ k = -1 \end{matrix} \tag{7}$$

(where b is constant) where also equation (5) still holds. The radiation in these models is in thermal equilibrium with the matter and the photons, neutrons and electrons are not treated separately (as done, for example, by Wagoner *et al.* (1967)).

It is clear that there is no way so far of changing from one type of model to another, except by assuming such a change at a given time and by dealing with one type before this time and with a different type afterwards, as done, for example, by Gamow (1956). Analytical solutions which change from one to the other are, however, possible.

In recent years, an isotropic microwave background has been detected, first by Penzias & Wilson (1965), who gave the value $3{\cdot}5 \pm 1\,^\circ$K as the equivalent black-body temperature by determinations at one wavelength, then by Roll & Wilkinson (1966), who gave $3{\cdot}0 \pm 0{\cdot}5\,^\circ$K at a different wavelength and afterwards by many others, most of whom give $3\,^\circ$K. Welch *et al.* (1967) have recently reported a measurement of $2{\cdot}0 \pm 0{\cdot}8\,^\circ$K and say that a curve at $2{\cdot}5\,^\circ$K fits all measurements. The value assumed here is

$$T_0 = 3\,^\circ\text{K} \tag{8}$$

though one of $2{\cdot}5\,^\circ$K will be used as a comparison at one point of the paper. The value of $3\,^\circ$K may be a little high but there is also radiation of other kinds, which, though it has a density of about two orders of magnitude less than the black-body radiation, will help to raise the density of radiation slightly.

This other radiation results from the conversion of matter in galaxies and possible intergalactic sources. Bondi (1960) gave the value of the conversion rate as

$$10^{-30}\ \text{erg cm}^{-3}\,\text{s}^{-1} = 1{\cdot}1 \times 10^{-51}\ \text{g cm}^{-3}\,\text{s}^{-1}$$

but most writers give much smaller values. Davidson & Narlikar (1966) following Kiang suggest a conversion rate, S_0, in the optical range of

$$\begin{aligned} S_0 &= 3{\cdot}2 \times 10^{-32}\ \text{erg s}^{-1}\,\text{cm}^{-3} \\ &= 3{\cdot}6 \times 10^{-53}\ \text{g cm}^{-3}\,\text{s}^{-1} \end{aligned} \tag{9}$$

so that the total conversion rate must be higher than this. Felton (1966) argues for a total rate of

$$\begin{aligned} S_0 &= 1 \times 10^{-31}\ \text{erg cm}^{-3}\,\text{s}^{-1} \\ &= 1{\cdot}1 \times 10^{-52}\ \text{g cm}^{-3}\,\text{s}^{-1} \end{aligned} \tag{10}$$

and this will be the value used in this paper. But, as Felton says, ' Ginzburg has correctly emphasized that such an observation can only give a lower limit; additional emission in bands less well investigated than the optical (perhaps from objects other than galaxies) would increase ' S_0. He suggests an upper figure of seven times this amount.

Some of the radiation in the Universe is of course absorbed by the galaxies and dust and various writers suggest mean-free paths for radiation of different types. Whitrow & Yallop (1964, 1965), for example, discuss separately the contributions of galaxies and intergalactic dust in the absorption of radiation. In a model with present matter density of 10^{-29} g cm^{-3}, they give the mean-free time, χ_0(c^{-1} times the mean-free path), as

$$\chi_0^{-1} = 10^{-16}\,\text{s}^{-1} + 10^{-20}\,\text{s}^{-1} \tag{11}$$

where the first term results from the absorption by dust and the second from absorption by galaxies. The term for absorption by dust will be smaller if the density of the dust is smaller. Davidson & Narlikar (1966) suggest

$$\chi_0 \sim 10^{11}\,\text{yr} \sim 3\times 10^{18}\,\text{s} \tag{12}$$

for the absorption by galaxies and suggest this as a lower limit. As Chung-Chieh Cheng (1967) points out, if the mean-free path length for radiation at a particular wavelength is shorter than the radius of the universe, then there exists a critical distance such that sources emitting radiation beyond that distance from an observer will be invisible to that observer.

This paper deals with the possible types of analytical solutions which automatically change from the ' radiation ' model for early times to the ' dust ' model for recent times. Specific models in which radiation is absorbed by the matter and in which matter is converted to radiation are dealt with in some detail. It is shown that models with T_0, S_0 and χ_0 agreeing with the above values are possible.

2. *Basic parameters and equations.* Equation (6) has defined one of the basic parameters generally used in this paper; others are the deceleration parameter

$$q = -\ddot{R}(RH^2)^{-1} \tag{13}$$

and the pressure parameter

$$\epsilon = p(\rho c^2)^{-1}. \tag{14}$$

Since models are required which change from a radiation ($\epsilon = \frac{1}{3}$) dominated early stage to a dust ($\epsilon = 0$) dominated late stage, the limits on ϵ are

$$\tfrac{1}{3} \geqslant \epsilon \geqslant 0; \quad k = 0, -1$$

$$\tfrac{1}{3} \geqslant \epsilon > 0; \quad k = +1 \tag{15}$$

where ϵ would become zero only after an infinite time in the first two cases and would never become zero in the second case as R will reach a maximum value and then decrease to zero.

Now, when the energy content is split up into dust and radiation, the density and pressure can be written as

$$\rho = \rho_m + \rho_r \tag{16}$$

$$p = p_r = \tfrac{1}{3}aT^4 \tag{17}$$

where ρ_m and ρ_r are the densities of dust and radiation respectively, p_r is the pressure of the radiation, p_m, the pressure of the dust, is put equal to zero, a is

Stefan's constant and T is the temperature of the radiation. It follows from equation (14) that

$$\epsilon = \frac{\rho_r}{3(\rho_r+\rho_m)}; \quad \frac{\rho_r}{\rho_m} = \frac{3\epsilon}{1-3\epsilon} \tag{18}$$

where ρ_r is zero for $\epsilon = 0$ and ρ_m is zero for $\epsilon = \frac{1}{3}$.

With equations (16) and (17) holding, the conservation equation formed from Einstein's field equations or from equations (2) and (3) as

$$\frac{d}{dt}(\rho R^3) + \frac{p}{c^2}\frac{d}{dt}(R^3) = 0 \tag{19}$$

becomes

$$E_r + E_m = 0 \tag{20}$$

where E_r and E_m are first defined by Davidson (1962) as

$$E_m = \frac{1}{R^3}\frac{d}{dt}(\rho_m R^3)c^2 = (\dot{\rho}_m + 3H\rho_m)c^2 \tag{21}$$

$$E_r = \frac{1}{R^3}\left\{\frac{d}{dt}(\rho_r R^3)c^2 + p_r\frac{d}{dt}(R^3)\right\} = (\dot{\rho}_r + 4H\rho_r)c^2 \tag{22}$$

and are the total net rate of transfer of radiation energy per unit volume into matter energy and vice versa.

With Davidson & Narlikar (1966), equation (22) can be written as

$$\frac{1}{c^2}E_r = \dot{\rho}_r + 4H\rho_r = S - \rho_r/\chi \tag{23}$$

where $S(t)$ is the rate of emission from unit volume, from galaxies and possible intergalactic sources and $\chi(t)$ is a frequency-averaged mean-free time between the emission and absorption of a photon. This equation gives

$$\frac{d}{dt}\left(\rho_r R^4 \exp\int^t \frac{dt}{\chi}\right) = SR^4 \exp\int^t \frac{dt}{\chi} \tag{24}$$

or

$$\rho_{r,0}R_0{}^4 = \int_{t_*}^{t_0} SR^4 \exp\left(-\int_{t_*}^{t_0}\frac{dt}{\chi}\right)dt + \rho_r(t_*)R(t_*)\exp\left(-\int_{t_*}^{t_0}\frac{dt}{\chi}\right) \tag{25}$$

where the zero subscript denotes the present epoch so that t_0 is the time of the present epoch and $t = t_*$ is the local time on the observational horizon of the model. The first term gives the contribution of direct emission allowing for subsequent absorption and the second term gives the present background of radiation already existing at epoch t. For the exploding models considered in this paper, t_* is set at $t_* = 0$ so that the second term arises from the primeval black-body radiation associated with the singular state, hence $\rho_r(t_*)R^4(t_*)$ is required to be finite.

The parameters for models fitted with radiation and/or matter are found in McIntosh (1968) and the ones needed here are as follows. For

$$\Theta(\phi) = \begin{cases} \cot\phi & k = +1 \\ \phi^{-1} & k = 0 \\ \coth\phi & k = -1 \end{cases} \tag{26}$$

and

$$\Psi(\phi) = \begin{cases} \sec^2\phi & k = +1 \\ 1 & k = 0 \\ \mathrm{sech}^2\phi & k = -1, \end{cases} \tag{27}$$

$$H = \frac{c}{R}\Theta(\phi) = \frac{2\dot{\phi}}{1+3\epsilon}\Theta(\phi), \tag{28}$$

$$q = \frac{R\dot{\phi}}{c}\Psi(\phi) = \tfrac{1}{2}(1+3\epsilon)\Psi(\phi) \tag{29}$$

$$\kappa\rho_m c^4 = 3H^2(1-3\epsilon)\Psi(\phi) \tag{30}$$

and

$$\kappa\rho_r c^4 = 9H^2\epsilon\Psi(\phi). \tag{31}$$

It can also be shown that equations (28) and (30) give

$$\kappa\dot{\rho}_m c^4 = -9H^2\{\dot{\epsilon}+H(1-2\epsilon+3\epsilon^2)\}\Psi(\phi). \tag{32}$$

3. *The general theory of radiation–matter models.* By substituting from equations (30) and (33) into equation (21), E_m can now be written as

$$\kappa c^2 E_m = -9H^2\{\dot{\epsilon}+H\epsilon(1-3\epsilon)\}\Psi(\phi). \tag{33}$$

Thus E_m is positive, zero, or negative depending on whether $\{\dot{\epsilon}+H\epsilon(1-3\epsilon)\}$ or $H-\dot{\Sigma}/\Sigma$ is negative, zero or positive, where

$$\Sigma = A(1-3\epsilon)/\epsilon \tag{34}$$

(where A is constant) and

$$\dot{\Sigma}/\Sigma = -\dot{\epsilon}/(1-3\epsilon)\epsilon. \tag{35}$$

The assumption is made by most writers in this field that

$$E_m = E_r = 0, \tag{36}$$

that is, that

$$\rho_m R^3 = \text{const.}, \quad \rho_r R^4 = \text{const.} \tag{37}$$

In this case,

$$R = \Sigma \tag{38}$$

and details of models with R as this function of ϵ and $t = t(\epsilon)$ have been published elsewhere (McIntosh 1968). Heckmann (1931, 1932) discussed general properties of solutions of this kind but the first analytical solution was given by Alpher & Herman (1949) (see also Alpher, Gamow & Herman (1967)) for the case $k = -1$. The $k = +1$ solution is also given by equations (13)–(15) of their 1967 paper when $K_2{}^{1/2} = ic/R$, though the authors do not mention this as they are only interested in the $k = -1$ case. The corresponding $k = 0$ solution is also not mentioned in these papers. The three solutions are also given in a very different form by Chernin (1965), but Chernin does not mention that the solutions change from being radiation dominated to matter dominated. The $k = 0$ solution has been given recently by Jacobs (1967) and the $k = \pm 1$ solutions by Cohen (1967). Solutions, with equation (36) holding, mean that the radiation is cooled adiabatically as the universe expands, whilst the thermal character is preserved.

Because this case has been looked at in detail, it will not be discussed any further here.

The question now arises, if $\epsilon = \epsilon(t)$ is assumed known, can $R(t)$ be found? From the definitions of q in equations (13) and (29), it follows that

$$H = [t + \int q\, dt + \text{const.}]^{-1} \tag{39}$$

$$= 2[2t + \int \Psi(\phi)(1+3\epsilon)\, dt + \text{const.}]^{-1}. \tag{40}$$

Since the models are required to act like radiation models for small t, the constant in these last two equations must be chosen so that H is like $1/(2t)$ for small t. For a given function $\epsilon = \epsilon(t)$, $\phi = \phi(t)$ needs to be known in both the $k = \pm 1$ cases for a complete determination of the model. For $k = 0$,

$$H = \tfrac{2}{3}[t + \int \epsilon\, dt + \text{const.}]^{-1} \tag{41}$$

so that

$$\log R(t) = \tfrac{2}{3}\int [t + \int \epsilon\, dt + \text{const.}]^{-1}\, dt + \text{const.} \tag{42}$$

Notice that with the constant zero in equation (41), $\epsilon = 0$ and $\epsilon = \frac{1}{3}$ give the matter model $R \propto t^{2/3}$ and the radiation model $R \propto t^{1/2}$ respectively. Even with $\epsilon \sim 0$, H is still $\sim 2/(3t)$.

Given $\epsilon = \epsilon(t)$ and H_0, $\rho_{m,0}$ and $\rho_{r,0}$ can be found immediately from equations (30) and (31) and q_0 from equation (29).

From equation (18), it is clear that

$$\rho_r/\rho_m \to \infty \quad \text{as} \quad \epsilon \to \tfrac{1}{3} \tag{43}$$

and thus it would be expected that E_m is greater than zero for early t. For the rate of emission to be greater than the rate of absorption at the present, E_m must be negative, i.e. E_r must be positive and

$$S - \rho_r/\chi > 0 \tag{44}$$

from equation (23). Davidson (1962) showed that for $R \propto t^n$, $(k = 0)$

$$E_m \propto n(2n-1)(3n-2) \tag{45}$$

which is negative for $\frac{1}{2} < n < \frac{2}{3}$, from which it would appear that, as $R(t)$ changed from an $n \doteq \frac{1}{2}$ like model to an $n = \frac{2}{3}$ like model for t increasing, E_m would always be negative, but this is by no means true, as the $E_m = E_r = 0$ models show. With ϵ a constant, equation (33) shows that E_m is negative for $0 < \epsilon < \frac{1}{3}$, zero for $\epsilon = 0$ and positive for ϵ outside this range. Obviously equation (45) agrees with this.

Various models given by different forms of ϵ will be studied, with emphasis being laid on the $k = 0$ space.

4. *Models in the space* $k = 0$. One model in the $k = 0$ space with $E_r \neq E_m \neq 0$ is the one with

$$\epsilon = \tfrac{1}{3}\, e^{-\beta t}; \quad \text{where } \beta \text{ is constant} \tag{46}$$

which has been examined by McIntosh (1967a) but this is physically unsatisfactory as pointed out by Jacobs (1967) (see also McIntosh (1967b)). This model has $E_m > 0$ for all t. Many other simple forms of ϵ also give models with $E_m > 0$ for all t. A physically satisfactory form of ϵ, and the one which will be concentrated on here, is

$$\epsilon = \tfrac{1}{3}(1+\mu t)^{-a} \tag{47}$$

(where μ and a are positive constants).

For $a \neq 0, 1$,

$$H = 2\mu[3\mu t+(1-a)^{-1}(1+\mu t)^{(1-a)}-(1-a)^{-1}]^{-1} \tag{48}$$

from equation (41), where the constant has been chosen so that $H \sim 1/(2t)$ for μt small. From equations (33), (47) and (48),

$$\begin{aligned}\kappa c^2 E_m &= -\kappa c^2 E_r = -9H^2\{\dot{\epsilon}+\epsilon H(1-3\epsilon)\}\\ &= 3H^3\frac{[(1+\mu t)^{-a}\{(a-1)(3a-2)\mu t+2-a\}+a-2]}{2(a-1)(1+\mu t)^{2a}}.\end{aligned} \tag{49}$$

When μt is small

$$\kappa c^2 E_m \sim \tfrac{3}{8}a\mu t^{-2} \tag{50}$$

so that obviously E_m is greater than zero for early t for all (positive) a. For μt large'

$$\kappa c^2 E_m \sim \tfrac{3}{2}H^3(3a-2)(\mu t)^{-a} \quad a \neq \tfrac{2}{3} \tag{51}$$

and E_m is thus negative for a less than $\frac{2}{3}$ and positive for a greater than $\frac{2}{3}$. Also E_m is obviously positive for $a = \frac{2}{3}$ for all t..

When $a = 1$,

$$H = 2\mu[3\mu t+\log(1+\mu t)]^{-1} \tag{52}$$

and it can be shown that E_m is greater than zero for all t. When $a = 0$, $\epsilon = \frac{1}{3}$ for all t and E_m is obviously zero for all t.

From equation (31),

$$\epsilon = \kappa\rho_r c^4/9H^2 \tag{53}$$

so that, given $\rho_{r,0}$ and H_0, ϵ_0 can be calculated. For a given a, equation (47) will thus give μt_0 in the form

$$\mu t_0 = (3\epsilon_0)^{-1/a}-1. \tag{54}$$

For μt_0 and H_0 known, equation (48) yields μ so that t_0 can be calculated. With μ known, t can be calculated for any ϵ. For example, when $\epsilon = \frac{1}{6}$, i.e. $\rho_r = \rho_m$.

$$t = \mu^{-1}(2^{1/m}-1). \tag{55}$$

For a given ϵ_0 and for t_0 approximately constant, it is obvious that the smaller a is, the larger μ is and the smaller t at $\epsilon = \frac{1}{6}$ is.

The two cases $a = \frac{1}{2}$ and $a = \frac{7}{11}$ will now be examined in some detail.

5. *The models with $a = \frac{1}{2}$ and $a = \frac{7}{11}$*. With $a = \frac{1}{2}$ in equation (47)

$$\epsilon = \tfrac{1}{3}(1+\mu t)^{-1/2}. \tag{56}$$

In this case, most of the parameters of model can be calculated explicitly as the calculations become straightforward. It is thus a good model in which to determine the general properties of the models with $a < \frac{2}{3}$. Thus, from equation (48),

$$\begin{aligned}H &= 2\mu[3\mu t+2(1+\mu t)^{1/2}-2]^{-1}\\ &= \frac{6\mu\epsilon^2}{(1+5\epsilon)(1-3\epsilon)}.\end{aligned} \tag{57}$$

Since

$$\dot{\epsilon} = -\tfrac{9}{2}\mu\epsilon^3 = -\tfrac{1}{6}\mu(1+\mu t)^{-3/2}, \qquad (58)$$

this can be integrated to give

$$R = B\{5+3(1+\mu t)^{1/2}\}^{5/6}\{(1+\mu t)^{1/2}-1\}^{1/2}$$
$$= \frac{B(1+5\epsilon)^{5/6}(1-3\epsilon)^{1/2}}{3^{1/2}\epsilon^{4/3}}. \qquad (59)$$

Obviously,

$$R \sim 4B(\mu t)^{1/2} \quad \text{and} \quad \sim 3^{5/6}B(\mu t)^{2/3} \qquad (60)$$

for μt small and large respectively.

The redshift, z, of a light source is given by

$$1+z = R_0/R_1, \qquad (61)$$

where $R_1 = R(t_1)$, t_1 being the time of emission of light from that source and $R_0 = R(t_0)$, t_0 being the time of reception of that light by the observer. In this case,

$$1+z = \left(\frac{1+5\epsilon_0}{1+5\epsilon_1}\right)^{5/6}\left(\frac{1-3\epsilon_0}{1-3\epsilon_1}\right)^{1/2}\left(\frac{\epsilon_1}{\epsilon_0}\right)^{4/3}, \qquad (62)$$

where $\epsilon_0 = \epsilon(t_0)$ and $\epsilon_1 = \epsilon(t_1)$ with t_0 and t_1 as above.

Also

$$q = \tfrac{1}{2}(1+3\epsilon) = \tfrac{1}{2}\left\{\frac{1+(1+\mu t)^{1/2}}{(1+\mu t)^{1/2}}\right\}, \qquad (63)$$

$$\kappa\rho_m c^4 = 3H^2(1-3\epsilon) = 3H^2\left\{\frac{(1+\mu t)^{1/2}-1}{(1+\mu t)^{1/2}}\right\}, \qquad (64)$$

$$\kappa\rho_r c^4 = 9H^2\epsilon = \frac{3H^2}{(1+\mu t)^{1/2}} \qquad (65)$$

and

$$\kappa c^2 E_m = \frac{3H^3[6(1+\mu t)^{1/2}-6-\mu t]}{4(1+\mu t)^{3/2}}. \qquad (66)$$

E_m is zero when

$$\mu t = 24 \qquad (67)$$

and is positive for μt smaller than this and negative for μt larger than this.

Notice that

$$\kappa\rho_m c^4 R^3 = 12\sqrt{3}B^3\mu^2(1-3\epsilon)^{1/2}(1+5\epsilon)^{1/2} \qquad (68)$$

which is zero for $\epsilon = \frac{1}{3}$ and approaches a finite constant for $\epsilon \to 0$, and

$$\kappa\rho_r c^4 R^4 = 36B^4\mu^2(1+5\epsilon)^{4/3}\epsilon^{-1/3} \qquad (69)$$

so that $E_m = E_r = 0$ certainly does not hold. One other parameter that can be found is ϕ of equation (28) as

$$\phi = (c/2\mu B)\{5+3(1+\mu t)^{1/2}\}^{1/6}\{(1+\mu t)^{1/2}-1\}^{1/2} \qquad (70)$$

The relationships between ϕ, R and t for small and large t can be easily shown to be those of equations (7) and (4) respectively.

The reason for discussing the model with $a = \frac{7}{11}$ will become apparent when comparison is made between the values of $E_{r,0}$ in the two models and observational data. For $a = \frac{7}{11}$

$$H = 8\mu[12\mu t + 11(1+\mu t)^{4/11} - 11]^{-1}. \tag{71}$$

The values of various constants and variables for the two models are listed in Table I; the assumptions being made that

$$\rho_{r,0} = 6{\cdot}8 \times 10^{-34} \text{ g cm}^{-3} \tag{72}$$

which is the value of ρ_r corresponding to a background microwave flux as from a black-body at 3·0 °K and

$$H_0 = 3{\cdot}2 \times 10^{-18} \text{ s}^{-1} \sim 100 \text{ km s}^{-1} \text{ Mpc}^{-1}. \tag{73}$$

The steps followed in the calculation of the constants are those described in the previous section. The values of ϵ_0 $\rho_{m,0}$ and q_0 are independent of a.

TABLE I

Values of parameters in the model with $k = 0$, $\epsilon = \frac{1}{3}(1+\mu t)^{-a}$ for $a = \frac{1}{2}$ and $a = \frac{7}{11}$ when $T_0 = 3$ °K, $H_0 = 100$ km s^{-1} Mpc^{-1}

Parameter	$a = \frac{1}{2}$ model	$a = \frac{7}{11}$ model	Units
ϵ_0	$1{\cdot}2 \times 10^{-5}$	$1{\cdot}2 \times 10^{-5}$	—
$\rho_{m,0}$	$1{\cdot}8 \times 10^{-29}$	$1{\cdot}8 \times 10^{-29}$	g cm^{-3}
$\frac{1}{c^2} E_{r,0}$	$5{\cdot}5 \times 10^{-52}$	$9{\cdot}9 \times 10^{-53}$	g cm^{-3} s^{-1}
μt_0	$7{\cdot}2 \times 10^{8}$	$9{\cdot}1 \times 10^{6}$	—
μ	$3{\cdot}5 \times 10^{-9}$	$4{\cdot}4 \times 10^{-11}$	s^{-1}
t_0	$2{\cdot}1 \times 10^{17}$	$2{\cdot}1 \times 10^{17}$	s
q_0	0·500018	0·500018	—
$\mu t(\rho_m = \rho_r)$	3·0	1·9	—
$t(\rho_m = \rho_r)$	$8{\cdot}7 \times 10^{8}$	$4{\cdot}3 \times 10^{10}$	s
$\mu t(E_m = 0)$	24	279	—
$t(E_r = 0)$	$6{\cdot}9 \times 10^{9}$	$6{\cdot}4 \times 10^{12}$	s

Notice that the time when the radiation and matter densities were equal in both models is less than in the model with $E_r = E_m = 0$ and that the times for the two events $\rho_m = \rho_r$ and $E_m = 0$ are greater in the model with the larger value of a.

Since ρ_r is proportional to T^4, $\rho_{r,0}$ is fairly sensitive to changes in T_0. If $\rho_{r,0}(x)$ is written for the present value of the radiation density at $T_0 = x$ °K, then at $T = 2{\cdot}5$ °K,

$$\rho_{r,0}(2{\cdot}5) = \left(\frac{2{\cdot}5}{3}\right)^4 \rho_{r,0}(3) = 0{\cdot}48\, \rho_{r,0}(3). \tag{74}$$

Also ϵ_0 is proportional to $\rho_{r,0}$ from equation (31), so that

$$\epsilon_0(2{\cdot}5) = 0{\cdot}48\, \epsilon_0(3). \tag{75}$$

From equations (20), (56) and (66), for $a = \frac{1}{2}$,

$$\kappa c^2 E_r = \frac{486\mu^3\epsilon^7(1-15\epsilon)}{(1+5\epsilon)^3(1-3\epsilon)^2} \tag{76}$$

so that $E_{r,0}$ is also sensitive to changes in ϵ, though not as much as might be expected, since μ will also change. In the present epoch, $\mu\epsilon_0{}^2$ is approximately constant from equation (56), so that the change in $E_{r,0}$ is roughly proportional to the change in ϵ. Here,

$$\mu(2{\cdot}5) = 4{\cdot}3\mu(3), \tag{77}$$

so that

$$\frac{1}{c^2} E_{r,0}(2{\cdot}5) = 0{\cdot}48 \frac{1}{c^2} E_{r,0}(3) = 2{\cdot}6 \times 10^{-52} \text{ g cm}^{-3} \text{ s}^{-1}. \tag{78}$$

Thus, in equation (23), for $T_0 = 3\,°\text{K}$,

$$5{\cdot}45 \times 10^{-52} = S_0 - 6{\cdot}8 \times 10^{-34} \chi_0^{-1} \tag{79}$$

and, for $T_0 = 2{\cdot}5\,°\text{K}$ (Welch *et al.* (1967)),

$$2{\cdot}63 \times 10^{-52} = S_0 - 6{\cdot}8 \times 10^{-34} \chi_0^{-1}. \tag{80}$$

For either of these to be possible, S_0 must be much higher than Felton's value of $1{\cdot}1 \times 10^{-52}$ g cm^{-3} s^{-1} (equation (10)). A value of three times Felton's figure could mean that the second model could be possible. In this case,

$$\chi_0(2{\cdot}5) = 10^{19} \text{ s} = 3 \times 10^{11} \text{ yr}, \tag{81}$$

a value three times the lowest limit mentioned by Davidson & Narlikar (equation (12)).

With $a = \frac{7}{11}$, however, and for $T_0 = 3\,°\text{K}$,

$$9{\cdot}89 \times 10^{-53} = S_0 - 6{\cdot}8 \times 10^{-34} \chi_0^{-1} \tag{82}$$

so that with S_0 given by $1{\cdot}1 \times 10^{-52}$ g cm^{-3} s^{-1},

$$\chi_0 = 5{\cdot}6 \times 10^{19} \text{ s} = 1{\cdot}8 \times 10^{12} \text{ yr} \tag{83}$$

and the mean-free path, $c\chi_0$, is

$$c\chi_0 = 1{\cdot}7 \times 10^{30} \text{ cm}. \tag{84}$$

For t greater than about 10^{13} s, it can be shown that for both models $H = 2/(3t)$ to a good approximation so that the Einstein–de Sitter model is accurate enough compared with these models in this range. This is a much smaller time than $t = 4{\cdot}8 \times 10^{15}$ s which Partridge & Peebles (1967a, b) assume to be the earliest time at which distant, newly formed galaxies may be seen. (See also the remarks on the accuracy of the $E_m = E_r = 0$ model by McIntosh (1967b).)

Thus tests like the (m, z) relation need only be worked out for the Einstein–de Sitter and other zero-pressured models as the light from the sources being studied has only come during the time when the models have acted like the pressure free ones. The usual assumption that L (total luminosity of source) has remained constant does not make sense for very early times.

6. *Models in the $k = \pm 1$ spaces.* As mentioned in Section 3, when a particular function $\epsilon = \epsilon(t)$ is defined, a relationship $\phi = \phi(t)$ also needs to be known or assumed in the spaces with $k = \pm 1$. Possible forms of ϕ could be

$$\cosh \phi = 1 + \lambda^{-1}\mu t, \quad k = -1 \tag{85}$$

$$\cos \phi = 1 - \lambda^{-1}\mu t, \quad k = +1 \tag{86}$$

(λ constant). For example, when $k = -1$ and

$$\epsilon = \tfrac{1}{3}(1 + \mu t)^{-1/2} \tag{87}$$

and when ϕ is given by equation (85),

$$q = \tfrac{1}{2}\{1+(1+\mu t)^{-1/2}\}(1+\lambda^{-1}\mu t)^{-2} \tag{88}$$

and

$$H = 2\left\{2t-\lambda\mu^{-1}\left[(1+\lambda^{-1}\mu t)^{-1}-\frac{\lambda}{\lambda-1}\left\{\frac{(1+\mu t)^{1/2}}{\lambda+\mu t} + (\lambda-1)^{-1/2}\tan^{-1}\left(\frac{1+\mu t}{\lambda-1}\right)^{1/2}\right\}-\frac{\lambda^2-4\lambda+2}{(\lambda-1)^2}\right]\right\}^{-1}, \tag{89}$$

where λ is very large to make the coefficient of t in the denominator equal to $4t$ for μt small. For μt small, H is like $1/t$, which is ' Milne's ' model $R = ct$. For t zero, q is $\frac{1}{2}$ and as $\mu t \to \infty$, $q \to 0$. R could be evaluated from equation (28) as $R = c \tanh \phi H^{-1}$.

The model for $k = +1$ with ϕ given by equation (85) has a maximum value of R which then decreases to zero.

As in the usual pressureless models, the time scale is longest in the $k = -1$ models and shortest in the $k = +1$ models and the possible values of $\rho_{m,0}$ is smallest in the $k = -1$ models and largest in the $k = +1$ models.

7. *Conclusion.* Although the usual assumption in dealing with cosmological models that the pressure is zero is sufficient in constructing models for the present epoch, the assumption cannot be made when more detailed analysis is required. The requirements that a residual radiation of 3 °K is left over from the initial ' big-bang ' and that there is conversion of matter into radiation and absorption of radiation by the matter necessitate much more complicated models. The usual radiation–matter models of Alpher & Herman (1949) for the case of negative curvature or the models of Chernin (1965), McIntosh (1968), Jacobs (1967) and Cohen (1967) for the cases of zero and positive curvature as well as the case of negative curvature satisfy the first of these requirements only. Models with $\epsilon = \frac{1}{3}(1+\mu t)^{-a}$, $0 < a < \frac{2}{3}$ in a space of zero curvature satisfy both of these requirements as no doubt will many other models with different forms of ϵ. A value of $a = \frac{7}{11}$ means that the net rate of conversion of matter to radiation per unit volume over all frequencies is less than the observed rate of emission and allows for a mean-free path length of radiation of $1{\cdot}7 \times 10^{30}$ cm. The problem of radiation at different frequencies and the mean-free path length over various frequency ranges has not been dealt with in this paper.

It is apparent by comparison of a few models that the time at which $\rho_r = \rho_m$ is earliest for the models in which E_m is negative at the present epoch and latest in ones at which E_m is positive at the present epoch. For example, in the flat space model with $\epsilon = \frac{1}{3}\,e^{-\beta t}$ (McIntosh 1967a) ($E_{m,0} > 0$), the time was about 6·7 per cent of the present time and in the flat space model with $\epsilon = \frac{1}{3}(1+\mu t)^{-1/2}$ ($E_{m,0} < 0$), the time was about 4×10^{-7} per cent of the present time. In the few models considered here where $E_{m,0}$ is negative, the time when $E_m = 0$ is much later than when $\rho_r = \rho_m$ and it would seem that this is a characteristic of all such models. The time when the model is like a dust filled one rather than a radiation filled one is much larger in the models with $E_{m,0}$ negative than in those with $E_{m,0}$ zero or positive.

When the models act like dust filled ones, such formulae as

$$n(t)R^3(t) = \text{constant} = n(t_0)R^3(t_0) \tag{90}$$

and

$$S(t)R^3(t) = \text{constant} = S(t_0)R^3(t_0) \tag{91}$$

can be used; $n(t)$ being the number of galaxies per unit proper volume at time t and $S(t)$ being as defined before. It is obvious from equation (89) in the $k = 0$, $\epsilon = \frac{1}{3}(1+\mu t)^{-1/2}$ model that $\rho_m R^3$ has been almost constant in this time. Then the number-count redshift relations can be used and comparative tests can be made, but these tests do not hold too far back in time. They do hold, however, for most models in the time during which light has been coming to us from visible sources.

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References

Alpher, R. A., Gamow, G. & Herman, R. C., 1967. *Proc. U.S. natn. Acad. Sci.*, **58**, 2179.
Alpher, R. A. & Herman, R. C., 1949. *Phys. Rev.*, **75**, 1089.
Bondi, H., 1960. *Cosmology*, p. 46, Cambridge University Press.
Chernin, A. D., 1965. *Astr. Zh.*, **42**, 1124. Translation in 1966. *Soviet Astr.*, **9**, 871.
Chung-Chieh Cheng, 1967. *Nature, Lond.*, **215**, 1035.
Cohen, J. M., 1967. *Nature, Lond.*, **216**, 249.
Davidson, W., 1962. *Mon. Not. R. astr. Soc.*, **124**, 79.
Davidson, W. & Narlikar, J. V., 1966. *Rep. Prog. Phys.*, **29**, 539.
Felton, J. E., 1966. *Astrophys. J.*, **144**, 241.
Gamow, G., 1956. *Vistas in Astronomy*, Vol. II, p. 1726, ed. by A. Beer, Pergamon Press, London.
Heckmann, O., 1931. *Nachr. Ges. Wiss. Göttingen*, 126.
Heckmann, O., 1932. *Nachr. Ges. Wiss. Göttingen*, 97.
Jacobs, K. C., 1967. *Nature, Lond.*, **215**, 1156.
McIntosh, C. B. G., 1967a. *Nature, Lond.*, **215**, 36.
McIntosh, C. B. G., 1967b. *Nature, Lond.*, **216**, 1297.
McIntosh, C. B. G., 1968. *Mon. Not. R. astr. Soc.*, **138**, 423.
Partridge, R. B. & Peebles, P. J. E., 1967a. *Astrophys. J.*, **147**, 868.
Partridge, R. B. & Peebles, P. J. E., 1967b. *Astrophys. J.*, **148**, 377.
Penzias, A. A. & Wilson, R. W., 1965. *Astrophys. J.*, **142**, 419.
Roll, P. G. & Wilkinson, D. T., 1966. *Phys. Rev. Lett.*, **16**, 405.
Wagoner, R. V., Fowler, W. A. & Hoyle, F., 1967. *Astrophys. J.*, **148**, 3.
Welch, W. J., Keachie, S., Thornton, D. D. & Wrixon, G. 1967. *Phys. Rev. Lett.*, **18**, 1068.
Whitrow, G. J. & Yallop, B. D., 1964. *Mon. Not. R. astr. Soc.*, **127**, 301.
Whitrow, G. J. & Yallop, B. D., 1965. *Mon. Not. R. astr. Soc.*, **130**, 31.