Relativistic elasticity of rigid rods and strings

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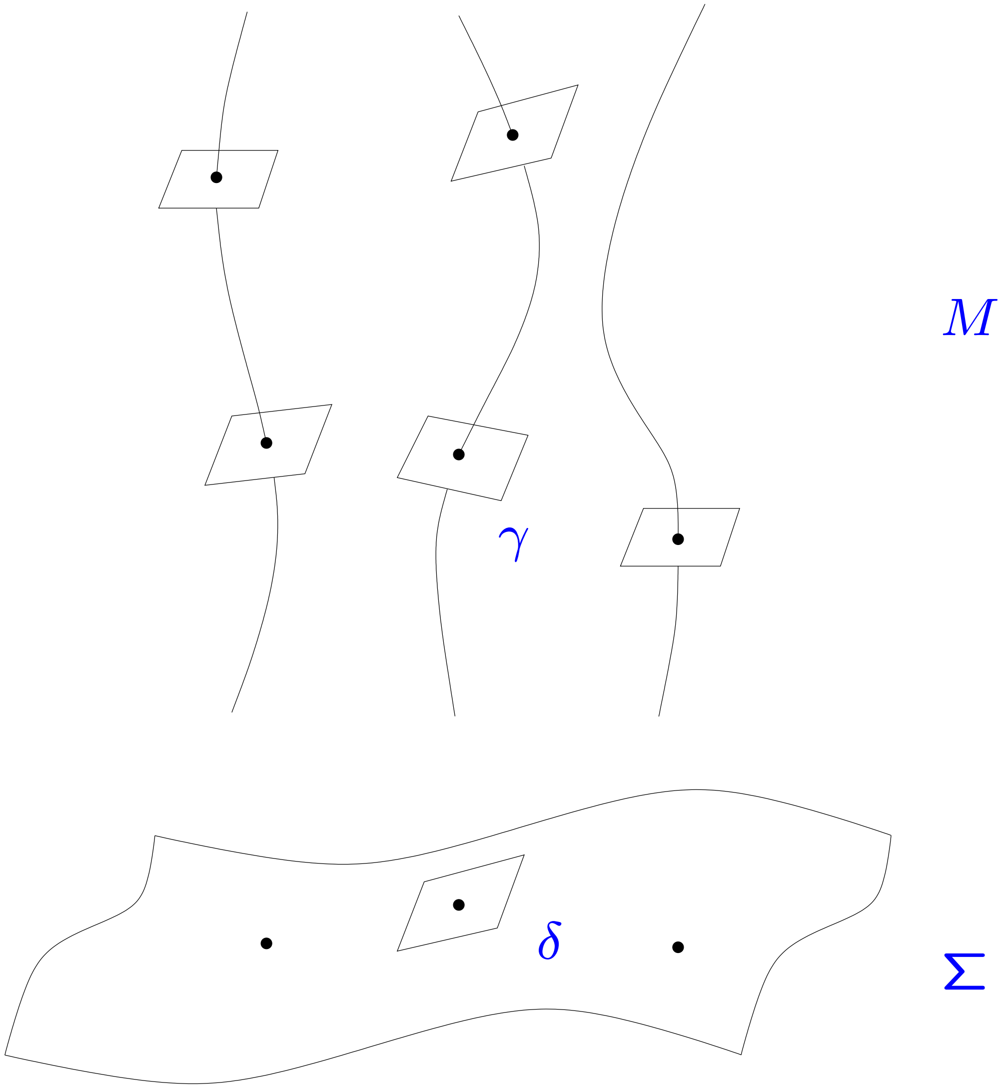
Outline

- Relativistic elasticity
- Rigid rods and strings
- Car and garage paradox
- Pushing a rigid rod
- Bell's spaceships paradox
- Fishing in black holes

Relativistic elasticity

- A continuous medium in General Relativity is described by:
  - A spacetime (M, g);
  - A Riemannian 3-manifold (Σ, δ) (relaxed configuration);
  - A projection map $\pi : M \to \Sigma$ whose level sets are timelike curves (the worldlines of the medium particles).


M
γ
δ
Σ

- If we choose local coordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ on Σ then we can think of π as a set of three scalar fields.

- We can complete $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ into coordinates $(\bar{t}, \bar{x}^1, \bar{x}^2, \bar{x}^3)$ for (M, g) yielding the rest frame of any given worldline:

$$g = -d\bar{t}^2 + \gamma_{ij} d\bar{x}^i d\bar{x}^j \qquad \text{(at that worldline)}.$$

- Note that

$$\gamma = \gamma_{ij} d\bar{x}^i d\bar{x}^j$$

is a (time-dependent) Riemannian metric on Σ, describing the local deformations of the medium along each worldline.

- We can compute the (inverse) metric γ from

$$\gamma^{ij} = g^{\mu\nu} \frac{\partial \bar{x}^i}{\partial x^\mu} \frac{\partial \bar{x}^j}{\partial x^\nu}.$$

- We must choose a Lagrangian density $\mathcal{L}$ for the action

$$S = \int_M \mathcal{L} \sqrt{-g}\, d^4 x.$$

- Assume $\mathcal{L} = \mathcal{L}(\bar{x}^i, \gamma^{ij})$. The energy-momentum tensor is then

$$T_{\mu\nu} = 2\,\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \mathcal{L}\, g_{\mu\nu} = 2\,\frac{\partial \mathcal{L}}{\partial \gamma^{ij}}\, \partial_\mu \bar{x}^i\, \partial_\nu \bar{x}^j - \mathcal{L}\, g_{\mu\nu}.$$

- Therefore

$$\mathcal{L} = T_{\bar{0}\bar{0}} = \rho$$

is the rest energy density.

- The choice of $\rho = \rho(\bar{x}^i, \gamma^{ij})$ is called the elastic law.

- Isotropic materials: ρ depends only on $({s_1}^2, {s_2}^2, {s_3}^2)$, the eigenvalues of γ_{ij} with respect to δ_{ij}. Note that (s_1, s_2, s_3) are the stretch factors along the principal directions.

- Assume that δ_{ij} is the Kronecker delta. In particular, we are assuming that the Riemannian 3-manifold (Σ, δ) is flat.

- More convenient variables:

$$\lambda_0 = \det(\gamma^{ij}) = \frac{1}{(s_1 s_2 s_3)^2};$$

$$\lambda_1 = \mathrm{tr}(\gamma^{ij}) = \frac{1}{{s_1}^2} + \frac{1}{{s_2}^2} + \frac{1}{{s_3}^2};$$

$$\lambda_2 = \mathrm{tr}\,\mathrm{cof}(\gamma^{ij}) = \frac{1}{(s_1 s_2)^2} + \frac{1}{(s_2 s_3)^2} + \frac{1}{(s_3 s_1)^2}.$$

- Examples:
  - Perfect fluid: $\rho = \rho(\lambda_0)$, yielding $p = 2\lambda_0 \dfrac{d\rho}{d\lambda_0} - \rho$.
  - Dust: $\rho = \rho_0 \sqrt{\lambda_0}$, yielding $p = 0$.

- Rigid fluid: $\rho = \frac{\rho_0}{2}(\lambda_0 + 1)$, yielding $p = \rho - \rho_0$.

- Stiff fluid: $\rho = A\lambda_0$, yielding $p = \rho$.

- John quasi-Hookean materials: $\rho = f(\lambda_0) + g(\lambda_0)\lambda_2$.

- Karlovini-Samuelsson quasi-Hookean materials: $\rho = f(\lambda_0) + g(\lambda_0)\lambda_1\lambda_2$.

- Stiff ultra-rigid equation of state: $\rho = A\lambda_2 + B$.

- Brotas rigid solid: $\rho = \frac{\rho_0}{8}(\lambda_0 + \lambda_1 + \lambda_2 + 1)$.

Rigid rods and strings

- For one-dimensional elastic bodies in a two-dimensional space-time (M, g) there is no difference between solids and fluids.

- Caution: these are not the strings of string theory – they have internal structure.

- The Lagrangian depends only on $\lambda_0 = \gamma^{11} = \partial_\alpha \bar{x} \partial^\alpha \bar{x}$.

- For a rigid elastic body (speed of sound = speed of light) we have $\rho = \frac{\rho_0}{2}(\lambda_0 + 1)$, yielding

$$T_{\mu\nu} = \rho_0 \left(\partial_\mu \bar{x}\, \partial_\nu \bar{x} - \frac{1}{2} \partial_\alpha \bar{x}\, \partial^\alpha \bar{x}\, g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \right).$$

- This is (essentially) just the energy-momentum tensor for a massless scalar field. So the equation of motion is just the wave equation:

$$\nabla^{\mu} T_{\mu\nu} = 0 \Leftrightarrow \Box \bar{x} = 0.$$

- We can always find a conjugated harmonic coordinate $\bar{t}$ such that

$$g = s^2 \left(-d\bar{t}^2 + d\bar{x}^2 \right).$$

This provides an an interesting interpretation for conformal coordinates in two-dimensional spacetimes.

- Static spacetimes:

$$g = -e^{2\phi(x)}dt^2 + dx^2 = e^{2\phi(\bar{x})}(-dt^2 + d\bar{x}^2)$$

(e.g. hanging strings in the Schwarzschild spacetime).

- Cosmological spacetimes:

$$g = -dt^2 + a^2(t)dx^2 = a^2(\bar{t})(-d\bar{t}^2 + dx^2)$$

(stretch factor equals the cosmological radius a).

Car and garage paradox

- Hitting a wall: before the collision

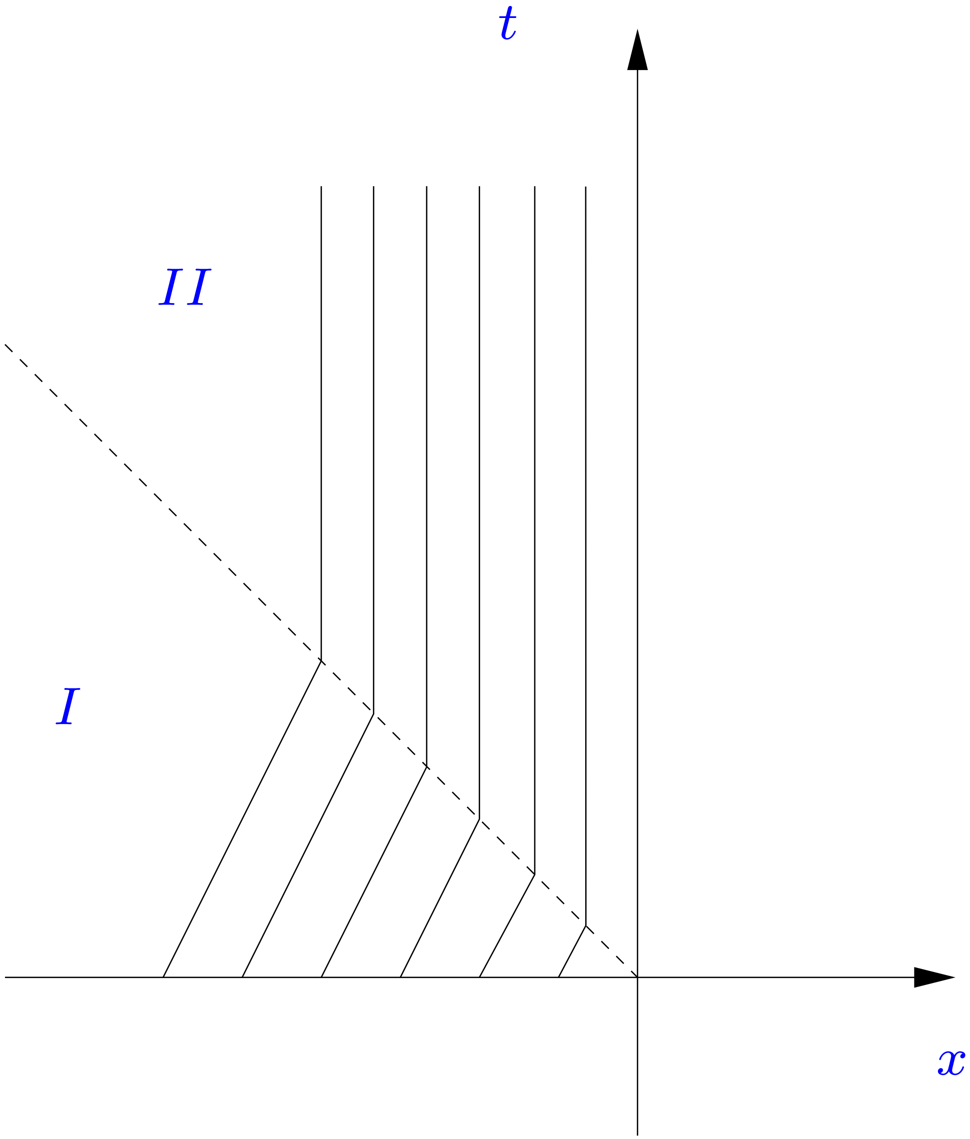
$$\bar{x} = \gamma(x - vt)$$

and so we have the initial-boundary value problem

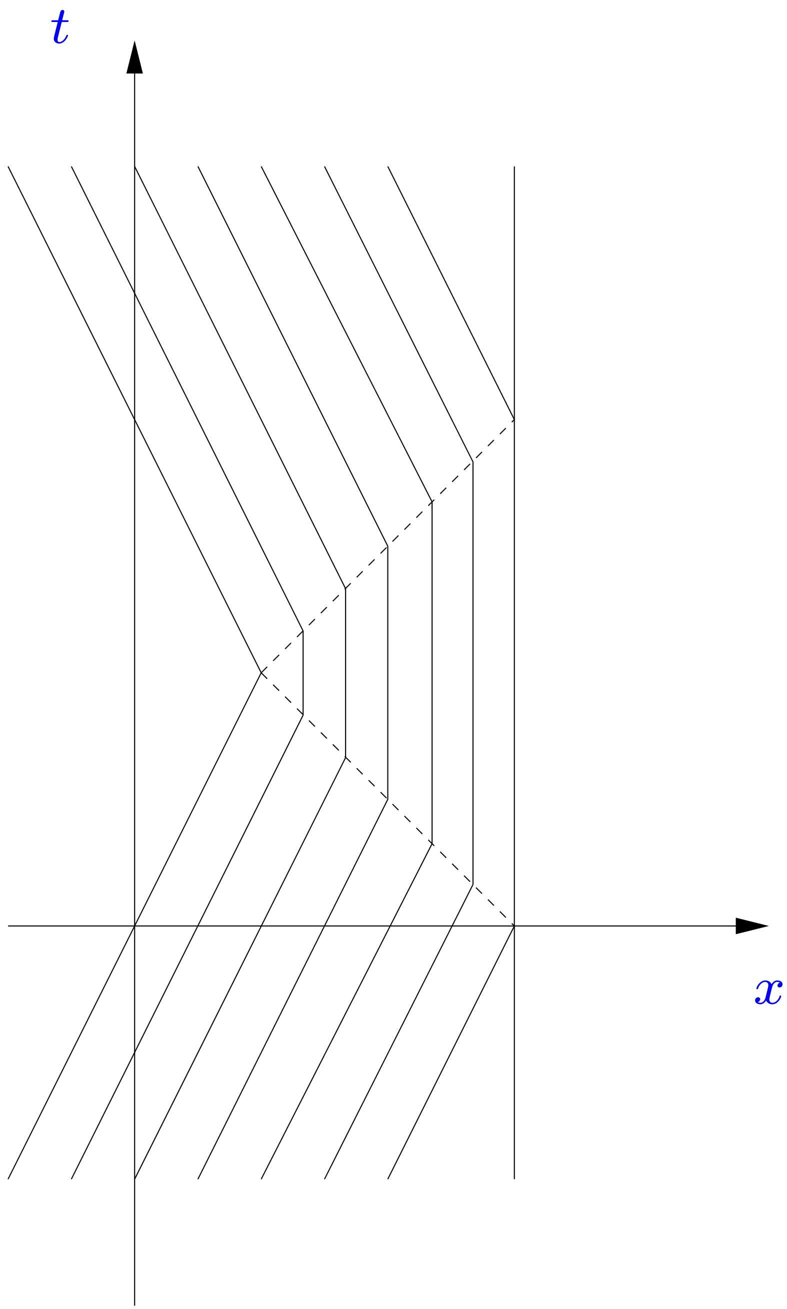
$$\begin{cases} \Box\bar{x} = 0 & (t > 0, x < 0) \\ \bar{x}(0, x) = \gamma x & (x < 0) \\ \frac{\partial\bar{x}}{\partial t}(0, x) = -v\gamma & (x < 0) \\ \bar{x}(t, 0) = 0 & (t > 0) \end{cases}$$

- Solution (on any conformal coordinate system):

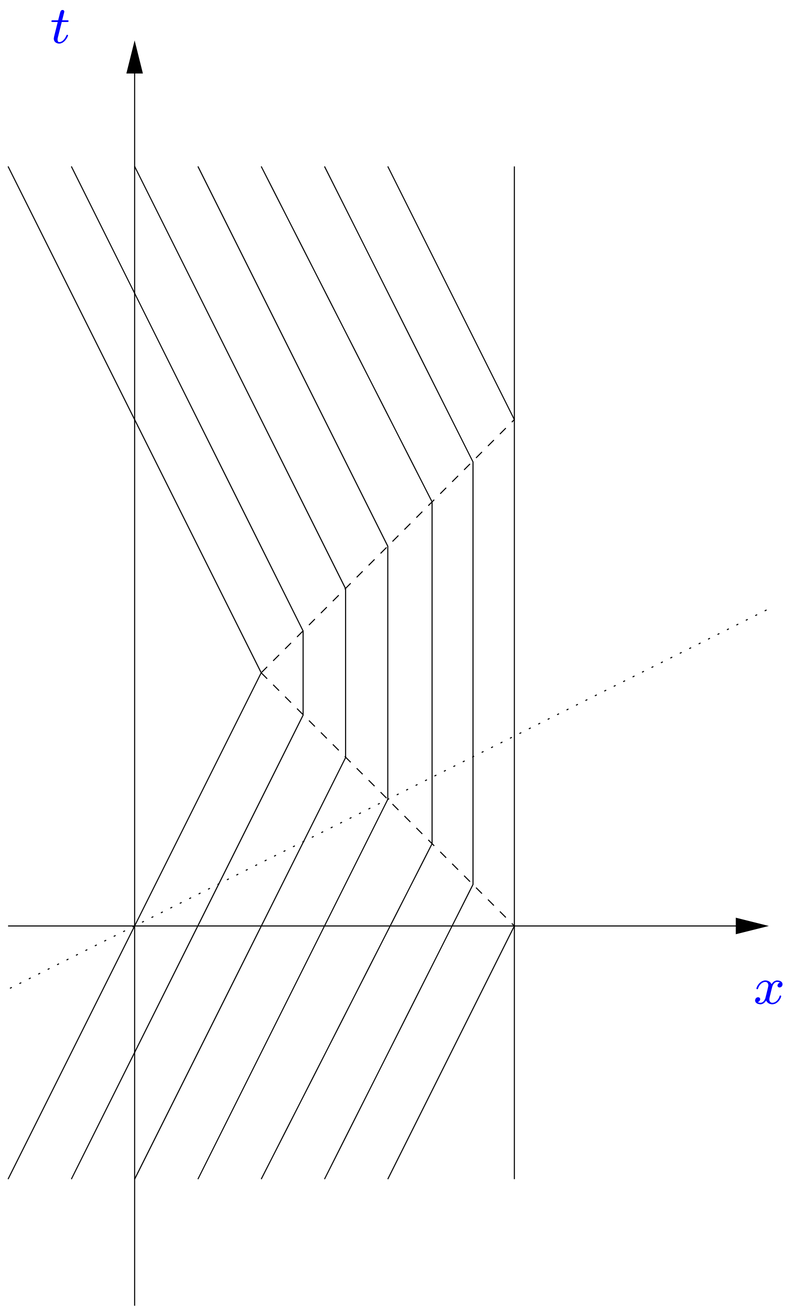
$$\bar{x}(t, x) = f(x - t) + g(x + t).$$



t
II
I
x



t
x

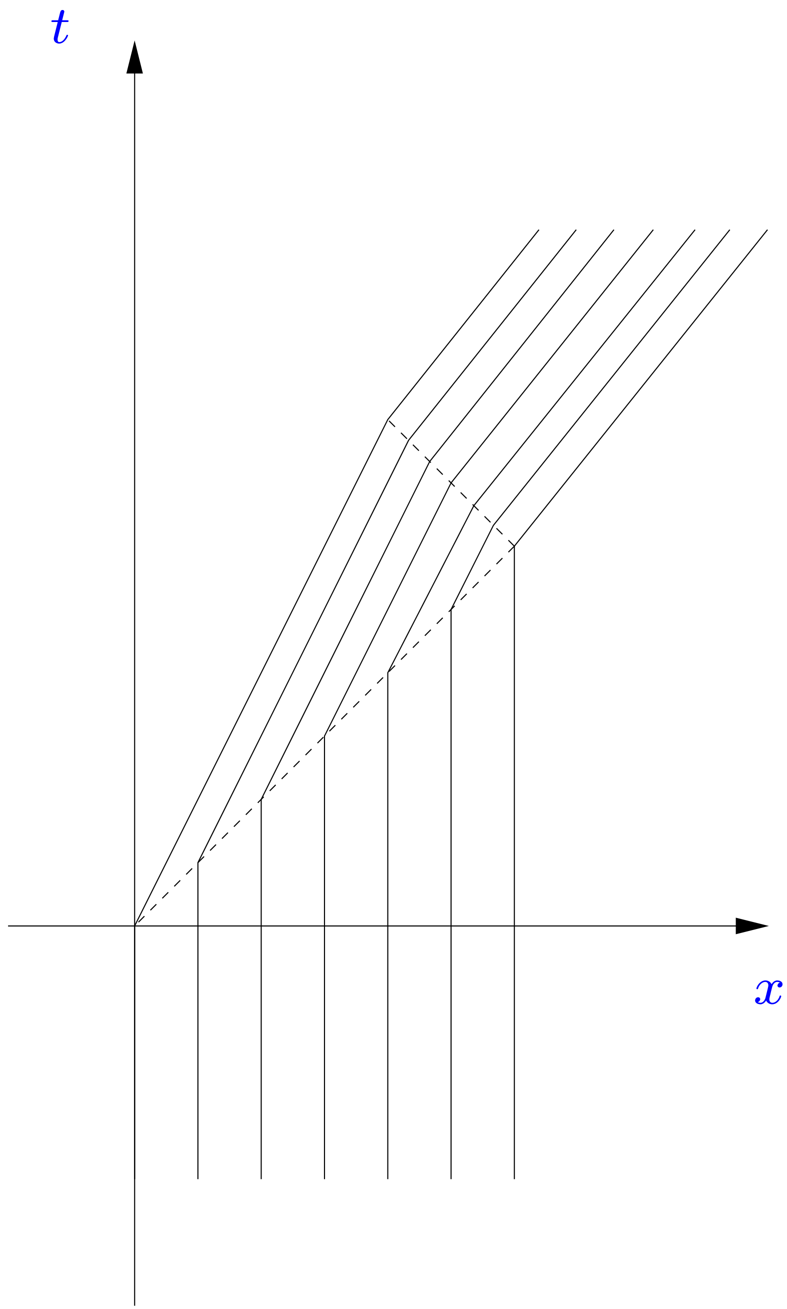


t
x

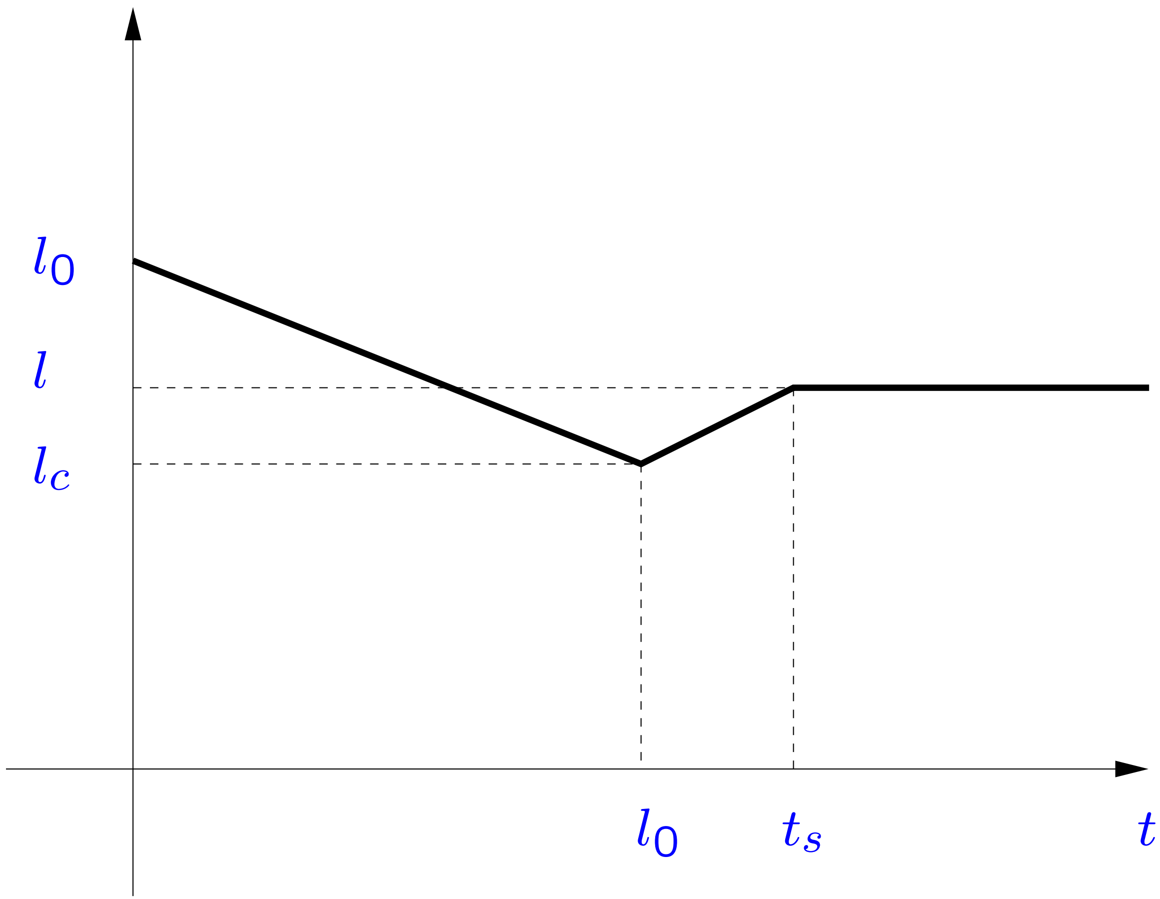
Pushing a rigid rod

- By changing frames we can reinterpret the solution above as describing a finite rod being pushed by a constant force.

- This offers a picture of how Lorentz contraction is attained in a physically realistic setting.



t
x



l_0
l
l_c
l_0
t_s
t

- If the force keeps acting then the rod will keep repeating these cycles of compression and rarefaction.

- If m_0 is the rest mass of the relaxed rod, a is the average proper acceleration and v is the velocity of the rear end as it starts being pushed then the tension is

$$p = m_0 a \, \frac{\gamma v}{\operatorname{arctanh} v}.$$

Bell's spaceships paradox

- Two identical spaceships connected by a string start moving simultaneously with the same acceleration profile. Does the string stretch?

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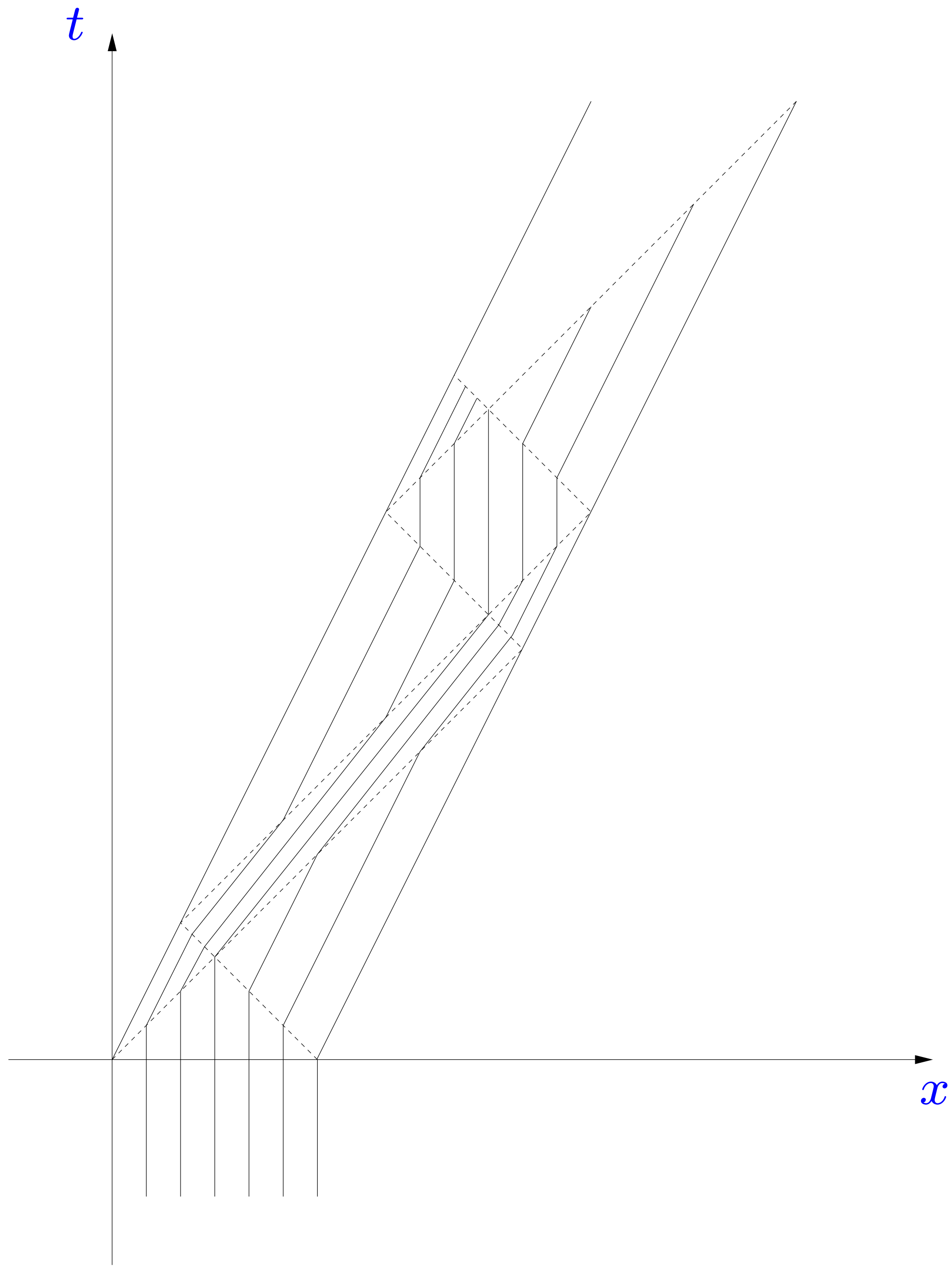
- We can construct a solution modeling a rigid string initially at rest whose endpoints start moving with velocity v at time $t = 0$.

Bell's spaceships paradox

- Two identical spaceships connected by a string start moving simultaneously with the same acceleration profile. Does the string stretch?

- Yes, because it must compensate length contraction.

- We can construct a solution modeling a rigid string initially at rest whose endpoints start moving with velocity v at time $t = 0$.



t
x

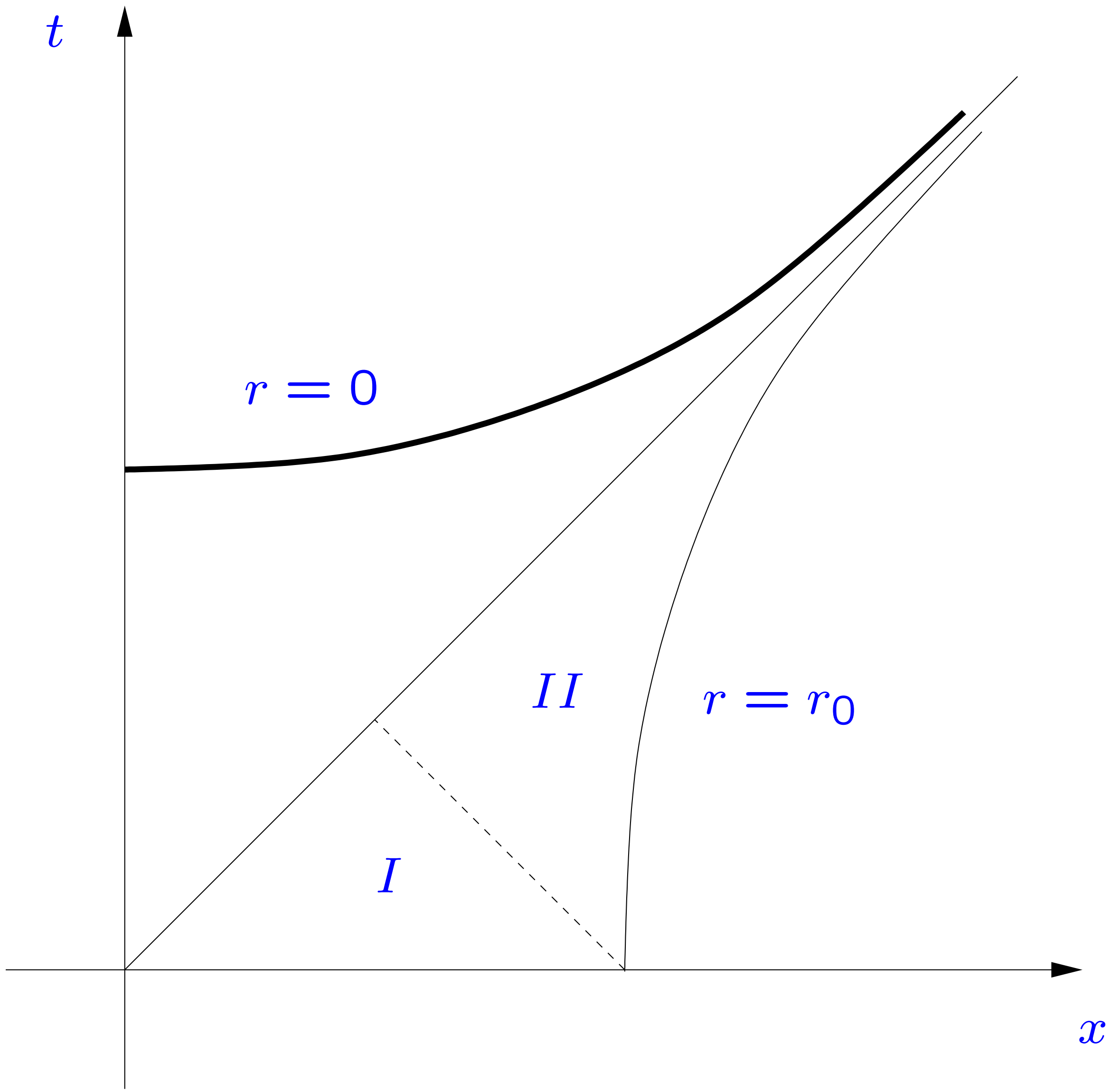
Fishing in black holes

- Kruskal-Szekeres coordinates ($2M = 1$):

$$g = 4r^{-1}e^{-r}\left(-dt^2 + dx^2\right), \qquad x^2 - t^2 = (r-1)e^r.$$

- If the string is being held at $r = r_0$ then we must solve the following initial-boundary value problem:

$$\begin{cases} \Box\bar{x} = 0 & (t > 0, 1 < r < r_0) \\ \bar{x}(0,x) = \int_0^x 2r^{-\frac{1}{2}}e^{-\frac{r}{2}}dx & (0 < x < x_0) \\ \frac{\partial\bar{x}}{\partial t}(0,x) = 0 & (0 < x < x_0) \\ \bar{x}(x_0 \sinh u, x_0 \cosh u) = \int_0^{x_0} 2r^{-\frac{1}{2}}e^{-\frac{r}{2}}dx & (u > 0) \end{cases}$$


t
r = 0
II
r = r0
I
x

- It is possible to compute an explicit exact solution. We find that:
  - Eventually the whole string will cross the horizon.
  - The force necessary to hold the string increases indefinitely.
  - More generally, the tension of the string increases along any future-pointing causal direction, and indeed approaches $+\infty$.
- Although our mathematical model does not contemplate the string breaking, any physical string will certainly do so.

Conclusion and outlook

- Elastic models are useful tools to model extended bodies in general relativity.

- Many questions to explore:

  - Motion of strings and other extended bodies in black hole backgrounds, and relation with cosmic censorship.

  - Oscillations, stability and collapse of elastic (neutron) stars.

  - Modeling supernovas through collapse of two-phase models: fluid atmosphere surrounding an elastic core.