

## **RELATIVISTIC ELECTROMAGNETIC BOUNDARY CONDITIONS**

H. Gür

Department of Engineering Physics  
Faculty of Sciences  
University of Ankara  
06100 Tandoğan, Ankara, Turkey

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# PART I: ANALYTICAL EXAMINATION OF ELECTROMAGNETIC BOUNDARY CONDITIONS

## 1. INTRODUCTION

As designated by Yeh [1], boundary conditions are the cornerstone for classical electrodynamics. However, a literature survey can show that electromagnetic boundary conditions are questioned from time to time [1–4], although their derivations are of standard textbook knowledge [5]. The present study proposes an analytical examination of the e.m. boundary conditions in an orthogonal curvilinear coordinate system which conforms locally and instantaneously to the boundary surface at a specified point on it. Subsequently, an original natural coordinate system introduced conforming again locally and instantaneously to the boundary surface at the same point on it; the latter removes the uncertainty in the choice of the basis unit vectors of an orthogonal curvilinear coordinate system, it reduces the number of the nonvanishing discontinuity relations, and determines natural aspects of the e.m. boundary conditions problem. The above mentioned natural coordinate system will be especially useful for the treatment of the relativistic e.m. boundary conditions which will be studied in Part II of this work.

## 2. MACROSCOPIC MAXWELL EQUATIONS

Maxwell equations governing the behavior of the time-varying electromagnetic fields are differential equations applying locally at each space-time point  $(\vec{x}, t)$ ; specifically,

$$\left. \begin{aligned} \vec{\nabla} \cdot \vec{D} &= 4\pi\rho \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{H} &= \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t} \\ \vec{\nabla} \times \vec{E} &= -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \end{aligned} \right\}, \quad (1)$$

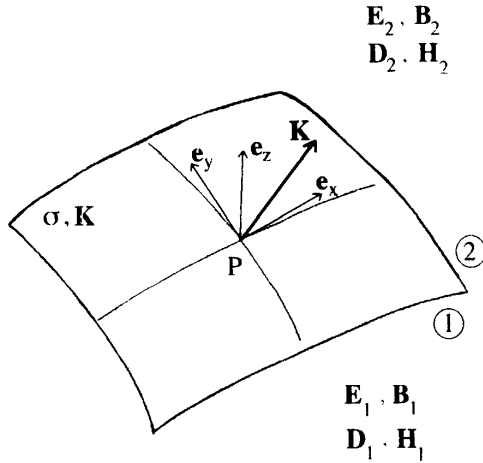
in Gaussian units. These equations take the following symmetric forms in source-free regions of a medium [1, 6], where  $\rho = 0$  and  $\vec{J} = 0$ :

$$\left. \begin{aligned} \vec{\nabla} \cdot \vec{D} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{H} &= \frac{1}{c}\frac{\partial\vec{D}}{\partial t} \\ \vec{\nabla} \times \vec{E} &= -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \end{aligned} \right\}. \quad (2)$$

According to these local and instantaneous differential equations, displacement current term  $(1/c)(\partial\vec{D}/\partial t)$  and its symmetric counterpart  $-(1/c)(\partial\vec{B}/\partial t)$  in the third and fourth relations of Eq. (2) concern only the source-free regions of the medium. This means that the displacement current term and its symmetric counterpart vanish where the sources are situated [7].

Let us consider also integral statements of the macroscopic Maxwell equations applying again locally at each space-time point  $(\vec{x}, t)$ :

$$\left. \begin{aligned} \oint_S \vec{D} \cdot \vec{n} \, da &= 4\pi \int_V \rho \, d^3x \\ \oint_S \vec{B} \cdot \vec{n} \, da &= 0 \\ \oint_C \vec{H} \cdot d\vec{\ell} &= \int_{S'} \left( \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t} \right) \cdot \vec{n}' \, da \\ \oint_C \vec{E} \cdot d\vec{\ell} &= - \int_{S'} \left( \frac{1}{c}\frac{\partial\vec{B}}{\partial t} \right) \cdot \vec{n}' \, da \end{aligned} \right\}. \quad (3)$$



**Figure 1(a).** A perspective view of an orthogonal curvilinear coordinate system, represented with the basis unit vectors  $\vec{e}_x$ ,  $\vec{e}_y$ ,  $\vec{e}_z$ , and conformed to a space-time point  $P(\vec{x}, t)$  on the boundary surface carrying surface charge and current densities  $\sigma(\vec{x}, t)$  and  $\vec{K}(\vec{x}, t)$ .

In the first two relations of Eq. (3),  $V$  is an infinitesimal volume at a space-time point  $(\vec{x}, t)$  and  $S$  is the closed surface bounding it;  $da$  is an element of area on the surface  $S$ , and  $\vec{n}$  is the unit normal to the surface at  $da$ , pointing outward from the enclosed volume  $V$ . In the last two relations of Eq. (3),  $C$  is an infinitesimal closed contour at the same space-time point  $(\vec{x}, t)$  and  $S'$  is an open surface extending across the contour;  $d\vec{\ell}$  is a line element on the contour  $C$ ,  $da$  is an area element on  $S'$ ; and  $\vec{n}'$  is the unit normal at  $da$ , pointing in the direction given by the right-hand rule concerning the sense of integration around the contour  $C$ .

As a continuation of the above discussion connected with Eq. (2), the integral terms  $\int_{S'} (1/c)(\partial\vec{D}/\partial t) \cdot \vec{n}' da$  and  $\int_{S'} (1/c)(\partial\vec{B}/\partial t) \cdot \vec{n}' da$  in the third and fourth relations of Eq. (3), which are the integral contributions of the displacement current and its symmetric counterpart, must vanish also where the sources are situated; because they concern again only the source-free regions of the medium [1].

### 3. ELECTROMAGNETIC BOUNDARY CONDITIONS IN AN ORTHOGONAL CURVILINEAR COORDINATE SYSTEM

The integral statements of the macroscopic Maxwell equations given in Eq. (3) are known to yield the discontinuity relations of the various electromagnetic field components at a boundary surface between two different media, assumed for example to carry surface charge and current densities  $\sigma(\vec{x}, t)$  esu/cm<sup>2</sup> and  $\vec{K}(\vec{x}, t)$  esu/(cm.s), respectively.

The purpose of the present study is an analytical examination of the e.m. boundary conditions problem in an orthogonal curvilinear coordinate system which conforms locally and instantaneously to the boundary surface at a specified point on it.

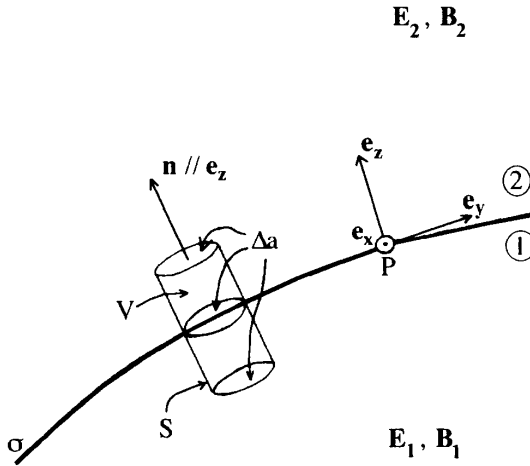
Let an orthogonal curvilinear coordinate system whose basis unit vectors are  $\vec{e}_x$ ,  $\vec{e}_y$ ,  $\vec{e}_z$  be locally and instantaneously joined to a specified space-time point  $P(\vec{x}, t)$  on the boundary surface (Fig. 1a) [1, 3], assumed also to be well-behaved, that is, to have a smooth shape. Let two of the basis unit vectors, for example  $\vec{e}_x$ ,  $\vec{e}_y$ , be chosen to lie on the tangential plane of the boundary surface at the specified space-time point  $P$ ; and let the third one, that is  $\vec{e}_z$ , be chosen to be perpendicular to the boundary surface at the same point  $P$ . The term "curvilinear," which qualifies the coordinate system, means that the directions of the basis unit vectors of the coordinate system vary according to the topological properties of the boundary surface, from a space-time point on it to another [3]. And, an orthogonal curvilinear coordinate system belonging to any other point on the boundary surface can be determined from the topological properties of the boundary, with reference to the specified point  $P$  on it.

Let us begin with the surface charge and current densities  $\sigma(\vec{x}, t)$  and  $\vec{K}(\vec{x}, t)$  at the space-time point  $P$  on the boundary; furthermore, we let the surface current density  $\vec{K}(\vec{x}, t)$  have two components:

$$\vec{K} = K_x \vec{e}_x + K_y \vec{e}_y . \quad (4)$$

Now, we can analyze e.m. boundary conditions in the local and instantaneous vicinities of the space-time point  $P(\vec{x}, t)$  on the boundary surface.

By applying the first two integral relations of Eq. (3) to the infinitesimal Gaussian cylinder placed at the specified space-time point  $P$  and illustrated in Fig. 1b, one can easily obtain following discontinuities in



**Figure 1(b).** In an infinitesimal cross-sectional view of  $x = \text{Constant}$  cut of the boundary surface, volume  $V$  is a small cylinder, half in one medium and half in the other, with the unit normal  $\vec{n} // \vec{e}_z$  to its top pointing from Medium 1 into Medium 2, and with the cross-sectional area  $\Delta a$ .

the normal components of the e.m. fields:

$$D_{2z} - D_{1z} = 4\pi\sigma, \quad (5)$$

$$B_{2z} - B_{1z} = 0. \quad (6)$$

Next, by taking also in consideration the vanishing of the integral contributions of the displacement current and its symmetric counterpart on the boundary carrying surface charge and current densities, applications of the last two integral relations of Eq. (3) to the infinitesimal Stokesian rectangles placed at the specified space-time point  $P$  and illustrated in Figs. 1c and 1d yield immediately the following discontinuities in the tangential components of the e.m. fields:<sup>1, 2</sup>

<sup>1</sup>, In comparison with our present problem, it is assumed that  $\sigma = 0$  and  $\vec{K} = 0$  in Yeh's work [1]. When  $\sigma \neq 0$  and  $\vec{K} \neq 0$ , Yeh's proof still remains valid for the fields  $\vec{E}$  and  $\vec{B}$ ; but the independent traditional proofs of the boundary conditions concerning the tangential components of the field  $\vec{H}$  and the normal component of the field  $\vec{D}$  are again necessary.

<sup>2</sup> Our present problem corresponds to take  $\vec{M}_f = 0$  in the paper by Stahl and Wolters [2].

$$-(H_{2y} - H_{1y}) = \frac{4\pi}{c} K_x , \quad (7)$$

$$-(E_{2y} - E_{1y}) = 0 ; \quad (8)$$

and

$$H_{2x} - H_{1x} = \frac{4\pi}{c} K_y , \quad (9)$$

$$E_{2x} - E_{1x} = 0 . \quad (10)$$

Here, the minus sign is retained in Eq. (8) in order to maintain its similarity with Eq. (7). The geometrical aspect implied in Eq. (7) and (9) is illustrated in Fig. 2.

Here, the choice of the directions of the tangential basis unit vectors  $\vec{e}_x$  and  $\vec{e}_y$  is uncertain, because there are infinite number orthogonal directions on the tangential plane of the boundary surface at the specified space-time point  $P$  on it. But this will be removed by the definition of the natural coordinate system considered in Sec. 5. Still, the decompositions obtained in Eqs. (5)–(10), related with Eq. (4), will be especially useful to understand relativistic e.m. boundary conditions which will be treated in Part II.

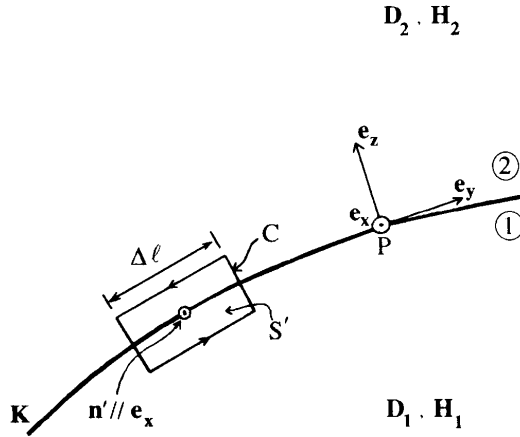
#### 4. COMPARISON WITH STANDARD FORMALISM

Eqs. (5)–(10) are the explicit expressions of the local and instantaneous e.m. field discontinuities at a specified space-time point  $P$  on an interface between two different media. And, as one can show by vector algebra from Eqs. (4), (7) and (8), and by referring to the vector diagram drawn in Fig. 2, the discontinuity in the tangential projection vector of the field  $\vec{H}$  occurs at right angle in a clockwise direction with respect to  $\vec{K}$  :

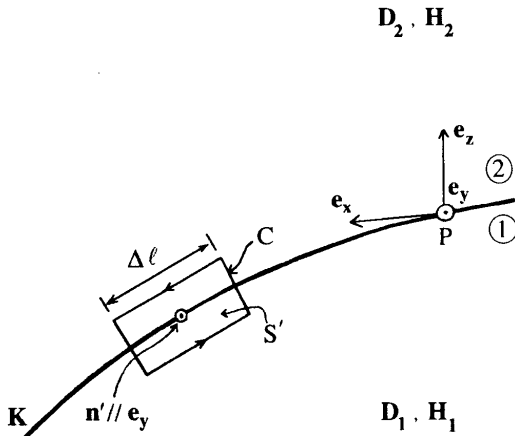
$$(\vec{H}_2 - \vec{H}_1)_{\text{tan}} \perp \vec{K} . \quad (11)$$

To compare with the standard formalism, it should be noted that Eq. (7) and (9) can be combined in a single vector equation, by accounting that the basis unit vector  $\vec{e}_z$  represents unit normal  $\vec{n}$  of the boundary surface, pointing from Medium 1 into Medium 2:

$$\vec{n} \times (\vec{H}_2 - \vec{H}_1) = \frac{4\pi}{c} \vec{K} . \quad (12)$$

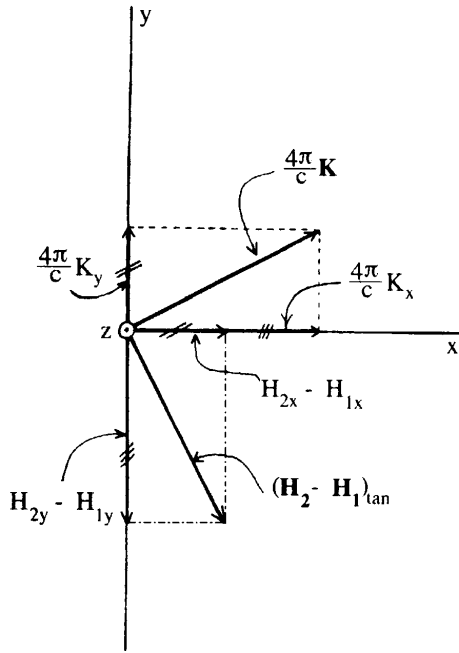


**Figure 1(c).** In the same infinitesimal cross-sectional view with the one illustrated in Fig. 1(b), contour  $C$  is a small rectangle, with its long sides  $\Delta\ell$  in either side of the boundary; plane of the contour  $C$  is oriented to be perpendicular to the boundary surface, so that its unit normal  $\vec{n}'$  coincides with the local and instantaneous direction of  $\vec{e}_x$  on the boundary surface.



**Figure 1(d).** And, in an infinitesimal cross-sectional view of  $y = \text{Constant}$  cut of the boundary surface, plane of a small rectangular contour  $C$  is oriented to be perpendicular to the boundary surface, so that its unit normal  $\vec{n}'$  coincides with the other local and instantaneous tangential direction of  $\vec{e}_y$  on the boundary surface.





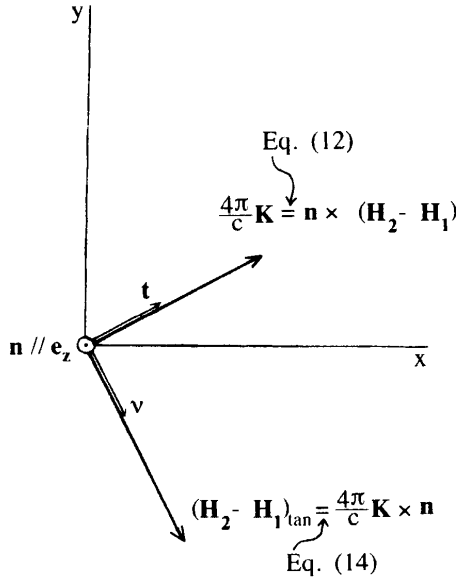
**Figure 2.** Discontinuity in the tangential projection vector of the field  $\vec{H}$  is drawn in an infinitesimal cross-sectional view of  $z = \text{Constant}$  cut of the boundary surface.

similarly, Eqs. (8) and (10) can be also combined in another single vector equation:

$$\vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0 ; \tag{13}$$

where, it is implied the minus sign which has been retained in Eq. (8). Here, one should note that Eq. (12) gives only the discontinuities in the components of the tangential projection vector of the field  $\vec{H}$ ; this means that the discontinuity in the tangential projection vector itself of the field  $\vec{H}$  can be only written by its components solved from Eq. (12). And, Eq. (13) has a similar meaning with Eq. (12).

Furthermore, by taking cross products of Eqs. (12) and (13) from right by the unit normal  $\vec{n}$ , these two equations can be expressed in another useful form, giving directly, by their left-hand sides, the discontinuities in the tangential projection vectors of the fields  $\vec{H}$  and



**Figure 3.** Basis unit vectors  $\vec{t}$ ,  $\vec{n}$ ,  $\vec{v}$  of the natural orthogonal curvilinear coordinate system and the discontinuity in the tangential projection vector of the field  $\vec{H}$  are drawn in the same cross-sectional view with the one illustrated in Fig. 2.

$\vec{E}$  :

$$(\mathbf{H}_2 - \mathbf{H}_1)_{\text{tan}} \equiv [\vec{n} \times (\vec{H}_2 - \vec{H}_1)] \times \vec{n} = \frac{4\pi}{c} \vec{K} \times \vec{n}, \quad (14)$$

$$(\vec{E}_2 - \vec{E}_1)_{\text{tan}} \equiv [\vec{n} \times (\vec{E}_2 - \vec{E}_1)] \times \vec{n} = 0. \quad (15)$$

In summary, by considering again that  $\vec{e}_z // \vec{n}$  at the considered space-time point  $P$  on the boundary surface, all the local and instantaneous discontinuity relations, including also the ones in the normal components of the fields  $\vec{D}$  and  $\vec{B}$ , can be collected in the standard vector notation:

$$\left. \begin{aligned} (\vec{D}_2 - \vec{D}_1) \cdot \vec{n} &= 4\pi\sigma \\ (\vec{B}_2 - \vec{B}_1) \cdot \vec{n} &= 0 \\ \vec{n} \times (\vec{H}_2 - \vec{H}_1) &= \frac{4\pi}{c} \vec{K} \\ \vec{n} \times (\vec{E}_2 - \vec{E}_1) &= 0 \end{aligned} \right\}. \quad (16)$$

## 5. NATURAL ORTHOGONAL CURVILINEAR COORDINATE SYSTEM FOR THE ELECTROMAGNETIC BOUNDARY CONDITIONS PROBLEM

From Eq. (16) and from Fig. 2, by taking also in consideration the forms seen in Eqs. (14) and (15) of the discontinuities in the tangential projection vectors of the fields  $\vec{H}$  and  $\vec{E}$ , it turns out that the e.m. boundary conditions problem possesses three local and instantaneous orthogonal directions, specifically,  $\vec{K}$ ,  $\vec{n}$  and  $\vec{K} \times \vec{n}$ . Thus, the surface current density  $\vec{K}$  and the unit normal  $\vec{n}$  of the boundary surface give the local and instantaneous defining directions of the problem.

By a borrowing from Michalski's notation [3], as illustrated in Fig. 3, three unit vectors  $\vec{t}$ ,  $\vec{n}$ ,  $\vec{\nu}$ , forming the basis of a local and instantaneous orthogonal curvilinear coordinate system at a specified space-time point  $P$  on the boundary surface, can be made respectively to coincide with the three local and instantaneous orthogonal directions  $\vec{K}$ ,  $\vec{n}$ ,  $\vec{K} \times \vec{n}$  of the problem itself. Then this new coordinate system, satisfying

$$\vec{t} \times \vec{n} = \vec{\nu} \quad (\text{with cyclic permutations}) \quad , \quad (17)$$

can be named as the *natural orthogonal curvilinear coordinate system* for the e.m. boundary conditions problem.

In this natural orthogonal curvilinear coordinate system, the surface current density  $\vec{K}$  at the space-time point  $P$  on the boundary becomes

$$\vec{K} = K\vec{t} \quad ; \quad (18)$$

and the discontinuity relations become consequently,

$$\left. \begin{aligned} D_{2n} - D_{1n} &= 4\pi\sigma \\ B_{2n} - B_{1n} &= 0 \\ H_{2\nu} - H_{1\nu} &= \frac{4\pi}{c}K \\ H_{2t} - H_{1t} &= 0 \\ E_{2\nu} - E_{1\nu} &= 0 \\ E_{2t} - E_{1t} &= 0 \end{aligned} \right\} , \quad (19)$$

which can be easily verified from Eq. (3) or from Eq. (16) by using Eqs. (17) and (18) and by decomposing the e.m. fields in terms of the

natural basis unit vectors  $\vec{t}$ ,  $\vec{n}$ ,  $\vec{v}$ . Comparison of Eq. (19) with the discontinuity relations already obtained in Eqs. (5) – (10) shows that the number of the nonvanishing discontinuities is reduced by the use of the natural orthogonal curvilinear coordinate system of the boundary conditions problem itself. Also, as illustrated in Fig. 3, Eq. (19) describes simply the natural aspect of the boundary conditions problem embodied in Eq. (11) and in Fig. 2.

Furthermore, the use of the natural orthogonal curvilinear coordinate system has removed the uncertainty question encountered in the choice of the tangential basis unit vectors of an orthogonal curvilinear coordinate system joined conformally to a specified space-time point  $P$  on the boundary surface, since one of the new tangential basis unit vectors, that is  $\vec{t}$ , has been chosen in the direction of the surface current density at the specified point  $P$ . And, the reduced form (19) of the discontinuities will be especially useful in the examination of the relativistic e.m. boundary conditions left to Part II.

## 6. SOME ADDITIONAL DISCUSSIONS

Once the natural orthogonal curvilinear coordinate system conforming locally and instantaneously to a specified space-time point  $P(\vec{x}, t)$  on a well-behaved boundary surface is determined, natural orthogonal curvilinear coordinate system belonging to any other space-time point on the boundary surface can be defined according to the topological and electrical properties of the boundary surface, with reference to the natural coordinates of the specified point  $P$  on it. At a different space-time point on the boundary surface, the surface current density can have two components in terms of the natural basis unit vectors of the specified space-time point  $P$ :

$$\vec{K} = K_t \vec{t} + K_\nu \vec{v} . \quad (20)$$

And, the discontinuity relations in the local and instantaneous vicinity of the new space-time point will have the forms equivalent to the ones already obtained in Eq. (5)–(10), with the following correspondence between the basis unit vectors used in Eqs. (5)–(10) and the natural basis unit vectors of the reference point  $P$  of the problem:

$$\vec{e}_x, \vec{e}_y, \vec{e}_z \quad \rightarrow \quad \vec{v}, \vec{t}, \vec{n} . \quad (21)$$

In this description, it is also necessary to take in consideration direction

variations of the unit normal  $\vec{n}$ , according to the topological properties of the boundary surface.

As another point, we can make use of some idealizations. For example, in most physical cases, encountered in nature or prepared in laboratory, boundary surfaces are ideal planes; and boundary surface charge and current densities are ideally uniform. Therefore, all points throughout the boundary surface and orthogonal coordinate systems belonging to them are identical. Hence, our original description (19) based on the natural orthogonal curvilinear coordinate system conformed to a specified space-time point  $P$  on the boundary surface becomes valid throughout the boundary.

## PART II: RELATIVISTIC ELECTROMAGNETIC BOUNDARY CONDITIONS

### 1. INTRODUCTION AND REVISION

In Part I, we had defined natural orthogonal curvilinear coordinate system locally and instantaneously joined to a specified space-time point  $P(\vec{x}, t)$  on a boundary surface being at rest between two media and carrying surface charge and current densities  $\sigma(\vec{x}, t)$  and  $\vec{K}(\vec{x}, t)$ . And, our present study aims to deduce relativistic electromagnetic boundary conditions concerning three distinct boundary surfaces, each having separately a local and instantaneous velocity in one of the three natural orthogonal directions at the same specified space-time point  $P(\vec{x}, t)$ .

During the last almost twenty five years, moving electromagnetic boundary conditions are studied from time to time [5, 8–12]. Differently from them, our present work will be based on the relativistic form invariance or covariance of the Maxwell equations, as a fundamental principle. Therefore, let us remember that, in spite of the independent historical developments of the Electomagnetism (1873) [13] and the Special Relativity (1905) [14], the latter is known to be implicit in the covariance of Maxwell equations [15–17].

By the fundamental covariance principle of electromagnetism, moving boundary conditions must not be different from the ones at rest, except for the different values of the surface charge and current densities because of the motion of the boundary surface. Indeed, there are in both cases, a surface charge density and a surface current density

on the interface which can be experimentally measured in both cases, for example, by using the vacuum Lorentz forces acting on the test charges being firstly at rest and moving then with a certain velocity in the laboratory. Our other two principal ideas (from Part I) are that we are again concerned with the evaluation of the integral forms of the Maxwell equations at the boundary surface being the place where the field sources are situated; and that the Maxwell equations are local and instantaneous differential equations. Consequently, the source densities  $\sigma'(\vec{x}, t)$  and  $\vec{K}'(\vec{x}, t)$ , which are measured at a specified point  $P(\vec{x}, t)$  having the local and instantaneous velocity  $\vec{v}$  of the moving interface, must be the Lorentz transformed values of the ones  $\sigma(\vec{x}, t)$  and  $\vec{K}(\vec{x}, t)$  which would be measured at the same specified point on the interface, when it would be at rest.<sup>1</sup>

## 2. RELATIVISTIC ELECTROMAGNETIC BOUNDARY CONDITIONS

Let us begin with a boundary surface being at rest between two media and carrying surface charge and current densities  $\sigma(\vec{x}, t)$  and  $\vec{K}(\vec{x}, t)$  at a specified space-time point  $P(\vec{x}, t)$  on it. As already obtained in Eq. (19) of Part I, discontinuity relations written in terms of the natural orthogonal curvilinear coordinate system represented by the basis unit vectors  $\vec{t}$ ,  $\vec{n}$ ,  $\vec{v}$  which are locally and instantaneously joined to the point  $P(\vec{x}, t)$  are the followings:

$$\left. \begin{aligned} D_{2n} - D_{1n} &= 4\pi\sigma \\ B_{2n} - B_{1n} &= 0 \\ H_{2\nu} - H_{1\nu} &= \frac{4\pi}{c}K \\ H_{2t} - H_{1t} &= 0 \\ E_{2\nu} - E_{1\nu} &= 0 \\ E_{2t} - E_{1t} &= 0 \end{aligned} \right\}; \quad (1)$$

where, following definition of the direction of the natural basis unit vector  $\vec{t}$  is used:

$$\vec{K} = K\vec{t}. \quad (2)$$

Now, let the boundary surface start to move with an arbitrary velocity, so that our specified space-time point  $P(\vec{x}, t)$  on it has a local and

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<sup>1</sup> Footnotes can be found in *Appendix II*.

instantaneous velocity  $\vec{v}$ , with the following decomposition in terms of the natural orthogonal basis unit vectors:

$$\vec{v} = v_t \vec{t} + v_n \vec{n} + v_\nu \vec{\nu} = c\beta_t \vec{t} + c\beta_n \vec{n} + c\beta_\nu \vec{\nu}. \quad (3)$$

Our present purpose is to examine what the discontinuity relations (1) become at our specified space-time point  $P(\vec{x}, t)$ , but now taken on the moving boundary surface. Let  $\sigma'(\vec{x}, t)$  and  $\vec{K}'(\vec{x}, t)$  represent the source densities on the moving interface. According to the covariance principle of electromagnetism, revised in Section 1, these  $\sigma'$  and  $\vec{K}'$  must be the Lorentz transformed values of the source densities  $\sigma$  and  $\vec{K}$  on the boundary surface as if it were at rest.

And, to calculate these Lorentz transformed values  $\sigma'$  and  $\vec{K}'$ , we must decide the sign of the Lorentz factor  $\vec{\beta}$ . To this aim, let us remember that the “fixed point”<sup>2</sup> of the Lorentz transformations is the point where the centers of the following spherical wave fronts coincide with each other at the instant  $t = t' = 0$  [20–24]:

$$\left. \begin{aligned} x^2 + y^2 + z^2 &= c^2 t^2 && \text{(in rest frame),} \\ x'^2 + y'^2 + z'^2 &= c^2 t'^2 && \text{(in moving frame)} \end{aligned} \right\}. \quad (4)$$

For the present, from an intuitive view point<sup>3</sup> based on the above definition of the fixed point of the Lorentz transformations, our boundary surface carrying the source densities  $\sigma$  and  $\vec{K}$  constitutes a moving interface, when it starts to move. Hence, this moving interface, with the source densities  $\sigma$  and  $\vec{K}$  on it, constitutes also the rest frame of the Lorentz transformations, where the fixed point of the transformations is placed. And, when observed from the rest frame of the moving interface, our laboratory where our test charges are found will constitute moving frame of the Lorentz transformations and will have the velocity opposite to (3):

$$\vec{v} = -v_t \vec{t} - v_n \vec{n} - v_\nu \vec{\nu} = -c\beta_t \vec{t} - c\beta_n \vec{n} - c\beta_\nu \vec{\nu}. \quad (5)$$

Thus, the rest and moving frames of the Lorentz transformations become uniquely defined. And, the source densities  $\sigma'$  and  $\vec{K}'$  measured by the test charges being found in the laboratory will be the source densities on the moving interface in the laboratory, because these test charges observe the velocity (3) of the moving interface, consistently with the physical reality.

After our intuitive decision<sup>3</sup> on the sign of the Lorentz factor  $\vec{\beta}$ , we can continue our examination. At the beginning, we know that the velocity direction of the moving frame gives a defining direction for the Lorentz transformations. But, e.m. boundary conditions problem has already its natural orthogonal defining directions represented by the basis unit vectors  $\vec{t}$ ,  $\vec{n}$ ,  $\vec{v}$  on the interface, when it is at rest (Sec. 5 of Part I). Therefore, it is necessary to treat three distinct Lorentz transformations for each separate component of the velocity (5), by leaving their combination, based on the method presented for example by Misner, Thorne and Wheeler [25], to a future study. Consequently, three distinct sets of discontinuity relations will be obtained, concerning each a boundary surface moving with a local and instantaneous velocity in one of the three natural orthogonal directions at our specified space-time point  $P(\vec{x}, t)$  on the interface.

Furthermore, without going to the combination of their Lorentz transformations, these three sets of moving electromagnetic discontinuity relations originally obtained in this work can be useful to treat some ideal plane boundary surfaces (Sec. 6 of Part I), again ideally, each moving with a uniform velocity in one of the three separate natural orthogonal directions, such as the one early treated by Noerdlinger [8].

### 3. BOUNDARY SURFACE MOVING WITH A VELOCITY

$$\vec{v} = v_t \vec{t}$$

Let the boundary surface be moving with the following local and instantaneous velocity at our specified space-time point  $P(\vec{x}, t)$  on it:

$$\vec{v} = v_t \vec{t} = c \beta_t \vec{t}. \quad (6)$$

For this velocity, we have the following Lorentz factors, convenient to our intuitive decision<sup>3</sup> on the sign of the Lorentz factor  $\vec{\beta}$ :

$$\vec{\beta} = -\beta_t \vec{t} \quad \text{and} \quad \gamma = \gamma_t = (1 - \beta_t^2)^{-1/2}. \quad (7)$$

#### 3.1 Surface Source Densities on the Moving Interface

Let us remember that the local and instantaneous surface current density  $\vec{K}(\vec{x}, t)$  at our specified space-time point  $P(\vec{x}, t)$  on the rest boundary surface had been used to define the natural basis unit vector



$\vec{t}$  (Sec. 5 of Part I), as also seen in Eq. (2). According to this definition, surface current density  $\vec{K}(\vec{x}, t)$  is parallel to the velocity (6). Hence, our interface carrying surface charge and current densities  $\sigma$  and  $\vec{K}$  is also locally and instantaneously parallel to the local and instantaneous velocity (6).

According to the elementary definition expressing that an electric current is charge in motion [26], a surface charge constitutes a surface current, when it moves parallelly to itself [27]. Thus,  $\sigma$  and  $\vec{K}$  are not independent of each other; and, they compose following four-vector:<sup>4</sup>

$$(c\sigma, \vec{K}) = (c\sigma, K\vec{t}) . \quad (8)$$

And, the components of the corresponding four-vector belonging to the moving interface can be found by using the Lorentz factors (7) in the four-vector Lorentz transformations [28]:

$$\left. \begin{aligned} c\sigma' &= \gamma_t [(c\sigma) + \beta_t K] \\ K' &= \gamma_t [K + \beta_t (c\sigma)] \\ \vec{K}'_{\perp} &= \vec{K}_{\perp} = 0, \end{aligned} \right\} \text{from Eq. (8)} . \quad (9)$$

Then, the transformed sources four-vector has the following form:

$$(c\sigma', \vec{K}') = (c\sigma', K'\vec{t}) ; \quad (10)$$

and, the source densities on the moving interface can be obtained from Eq. (9):

$$\left. \begin{aligned} \sigma' &= \gamma_t \left( \sigma + \frac{\beta_t}{c} K \right) \\ \vec{K}' &= K'\vec{t} = \gamma_t (K + c\beta_t \sigma) \vec{t} \end{aligned} \right\} . \quad (11)$$

### 3.2 Discontinuity Relations at the Moving Interface

Because of the fact that the transformed current density  $\vec{K}'$ , found in Eq. (11), has again only one component in the direction of the natural basis unit vector  $\vec{t}$  and because of the covariance principle of electromagnetism, local and instantaneous discontinuity relations at our specified space-time point  $P(\vec{x}, t)$  on the moving interface will

have the same form with the ones given in Eq. (1):

$$\left. \begin{aligned} D'_{2n} - D'_{1n} &= 4\pi\sigma' \\ B'_{2n} - B'_{1n} &= 0 \\ H'_{2\nu} - H'_{1\nu} &= \frac{4\pi}{c}K' \\ H'_{2t} - H'_{1t} &= 0 \\ E'_{2\nu} - E'_{1\nu} &= 0 \\ E'_{2t} - E'_{1t} &= 0 \end{aligned} \right\}; \quad (12)$$

where, the primed fields represent the fields measured in the laboratory, on either side of the moving boundary surface. So, only the magnitudes of the discontinuities are changed, because of the transformed values (11) of the surface source densities belonging to the moving interface.

#### 4. BOUNDARY SURFACE MOVING WITH A VELOCITY

$$\vec{v} = v_n \vec{n}$$

For an interface moving with the following local and instantaneous velocity

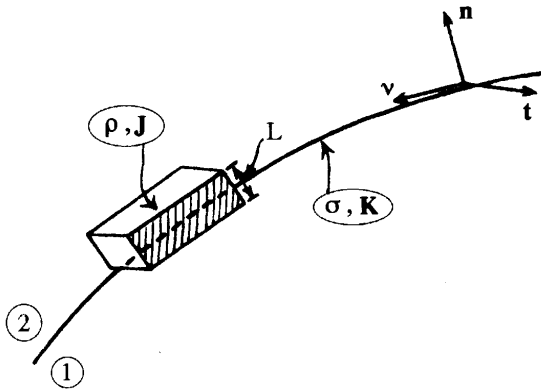
$$\vec{v} = v_n \vec{n} = c\beta_n \vec{n}, \quad (13)$$

at our specified space-time point  $P(\vec{x}, t)$  on it, we have the following Lorentz factors being convenient to our intuitive decision on the sign of the Lorentz factor  $\vec{\beta}$ :<sup>3</sup>

$$\vec{\beta} = -\beta_n \vec{n} \quad \text{and} \quad \gamma = \gamma_n = (1 - \beta_n^2)^{-1/2}. \quad (14)$$

##### 4.1 Surface Source Densities on the Moving Interface

Now, differently from the precedent Section 3, our interface carrying local and instantaneous surface charge and current densities  $\sigma$  and  $\vec{K} = K\vec{t}$  is perpendicular to the local and instantaneous velocity (13). Hence, the local and instantaneous motion of the boundary surface is locally and instantaneously perpendicular to its surface. And, contrarily to Section 3.1, surface charge and current densities  $\sigma$  and  $\vec{K}$  are independent of each other, in the physical meaning that they do not compose no longer a 4-vector. However, we can find again a sources 4-vector, but now, composed of the local and instantaneous volume charge and current densities  $\rho$  and  $\vec{J}$  corresponding to the local and instantaneous surface charge and current densities  $\sigma$  and  $\vec{K}$ ; and, we



**Figure 1.** Three dimensional illustration of the boundary surface, with an exaggerated section on it; where  $\rho$  and  $\vec{J}$  are the volume charge and current densities corresponding to the surface charge and current densities  $\sigma$  and  $\vec{K}$  on the boundary surface.

can treat its Lorentz transformations. Therefore, let us consider the elementary definitions connecting surface and volume charge [29] and current [8, 30] densities with each other:

$$\sigma = L\rho , \tag{15}$$

and

$$\vec{K} = L\vec{J}, \quad \text{with} \quad \vec{J} = J\vec{t} \quad [\text{because of Eq. (2)}] ; \tag{16}$$

where,  $L$  is the local thickness of the boundary surface (Fig. 1). We have then the following sources 4-vector:

$$(c\rho, \vec{J}) = (c\rho, J\vec{t}) . \tag{17}$$

And, the components of the corresponding 4-vector belonging to the moving interface can be obtained by using Lorentz factors (14) in the 4-vector Lorentz transformations [28]:

$$\left. \begin{aligned} c\rho' &= \gamma_n [(c\rho) - (-\beta_n \vec{n}) \cdot (J\vec{t})] \\ J'_n &= \gamma_n [J_n - (-\beta_n)(c\rho)] ; & \text{where, } J_n &= 0 \text{ from Eq. (17)} \\ J'_t &= J_t ; & \text{where, } J_t &= J \text{ from Eq. (17)} \\ J'_\nu &= J_\nu ; & \text{where, } J_\nu &= 0 \text{ from Eq. (17)} \end{aligned} \right\} ; \tag{18}$$

from where,

$$\left. \begin{aligned} c\rho' &= \gamma_n(c\rho) & \text{or} & & \rho' &= \gamma_n\rho \\ J'_n &= \gamma_n\beta_n(c\rho) \\ J'_t &= J \\ J'_\nu &= 0 \end{aligned} \right\} . \quad (19)$$

Then, the transformed 4-vector has the following form:

$$(c\rho', \vec{J}') = (c\rho', J'_t\vec{t} + J'_n\vec{n}) . \quad (20)$$

Volume current density component  $J'_n$  seen in Eqs. (19) and (20) does not cause to any discontinuity in the field  $\vec{H}'$ , when the boundary surface is crossed from the Medium 1 to the Medium 2. In fact, as easily understood by the use of the Ampère's law, the field  $\vec{H}'$  produced by the current density  $J'_n$  will be in the direction of the natural basis unit vector  $\vec{t}$ . Therefore, this field will remain within the local width of the boundary surface, where, we do not make any field discontinuity observation, because we cross the boundary surface in the direction of the natural basis unit vector  $\vec{n}$ . Hence, the 4-vector (20) can be considered as equivalent to the following reduced one:

$$(c\rho', \vec{J}') = (c\rho', J'_t\vec{t}) . \quad (21)$$

Subsequently, we can find surface source densities  $\sigma'$  and  $\vec{K}'$  on the moving interface; let us begin again with the elementary definitions similar to the ones seen in Eqs. (15) and (16):

$$\sigma' = L'\rho' , \quad (22)$$

and

$$\vec{K}' = L'\vec{J}' , \quad \text{with} \quad \vec{J}' = J'_t\vec{t} \quad \text{from Eq. (21)} ; \quad (23)$$

where,  $L'$  being the local thickness of the moving interface is the Lorentz contracted value of the local *proper* thickness  $L$  of the interface as if it were at rest:

$$L' = \frac{L}{\gamma_n} ; \quad (24)$$

and,  $\rho'$  and  $J'_t$  are recently calculated as seen in Eq. (19). Finally, we can substitute Eqs. (19) and (24) into Eqs. (22) and (23), and we can compare the results with Eqs. (15) and (16) to have the surface source densities measured on the moving interface:

$$\sigma' = \sigma , \quad (25)$$

and

$$\vec{K}' = \frac{\vec{K}}{\gamma_n}, \quad \text{or, by using Eq. (2),} \quad \vec{K}' = \frac{K}{\gamma_n} \vec{t}. \quad (26)$$

## 4.2 Discontinuity Relations at the Moving Interface

Again, because of the fact that the transformed surface current density  $\vec{K}'$ , found in Eq. (26) has again only one component in the direction of the natural basis unit vector  $\vec{t}$ , local and instantaneous discontinuity relations at our specified space-time point  $P(\vec{x}, t)$  on the moving boundary surface will not be different from the ones written in Eq. (12):

$$\left. \begin{aligned} D'_{2n} - D'_{1n} &= 4\pi\sigma' \\ B'_{2n} - B'_{1n} &= 0 \\ H'_{2\nu} - H'_{1\nu} &= \frac{4\pi}{c} K' \\ H'_{2t} - H'_{1t} &= 0 \\ E'_{2\nu} - E'_{1\nu} &= 0 \\ E'_{2t} - E'_{1t} &= 0 \end{aligned} \right\}; \quad (27)$$

and the discussions related with Eq. (12) will be again valid without change.<sup>5</sup>

## 5. BOUNDARY SURFACE MOVING WITH A VELOCITY

$$\vec{v} = v_\nu \vec{\nu}$$

Let the interface be moving with the following local and instantaneous velocity at our specified space-time point  $P(\vec{x}, t)$  on it:

$$\vec{v} = v_\nu \vec{\nu} = c\beta_\nu \vec{\nu}. \quad (28)$$

For this velocity, we have the following Lorentz factors being convenient to our intuitive decision on the sign of the Lorentz factor  $\vec{\beta}$ :<sup>3</sup>

$$\vec{\beta} = -\beta_\nu \vec{\nu} \quad \text{and} \quad \gamma = \gamma_\nu = (1 - \beta_\nu^2)^{-1/2}. \quad (29)$$

### 5.1 Surface Source Densities on the Moving Interface

Since our interface carrying local and instantaneous surface charge and current densities  $\sigma$  and  $\vec{K} = K\vec{t}$  is again moving locally and

instantaneously parallel to its surface, our present examination will be similar to Section 3.1.

Let us start by considering the sources 4-vector (8) belonging to the space-time point  $P(\vec{x}, t)$  on the boundary surface being at rest. Components of the corresponding sources 4-vector  $(c\sigma', \vec{K}')$  belonging to the same point  $P$ , but having now the velocity (28) because of the motion of the interface, can be found by using Eq. (29) in the 4-vector Lorentz transformations [28]:

$$\left. \begin{aligned} c\sigma' &= \gamma_\nu [(c\sigma) - (-\beta_\nu \vec{v}) \cdot (K\vec{t})] \\ K'_\nu &= \gamma_\nu [K_\nu - (-\beta_\nu)(c\sigma)] ; \\ K'_t &= K_t ; \\ K'_n &= K_n ; \end{aligned} \right\} \begin{array}{l} \text{where, } K_\nu = 0 \quad \text{from Eq. (8)} \\ \text{where, } K_t = K \quad \text{from Eq. (8)} \\ \text{where, } K_n = 0 \quad \text{from Eq. (8)} \end{array} ; \quad (30)$$

from where,

$$\left. \begin{aligned} c\sigma' &= \gamma_\nu(c\sigma) & \text{or} & & \sigma' &= \gamma_\nu\sigma \\ K'_\nu &= \gamma_\nu\beta_\nu(c\sigma) \\ K'_t &= K \\ K'_n &= 0 \end{aligned} \right\} . \quad (31)$$

Then, the transformed 4-vector has the following form:

$$(c\sigma', \vec{K}') = (c\sigma', K'_\nu \vec{v} + K'_t \vec{t}) . \quad (32)$$

## 5.2 Discontinuity Relations at the Moving Interface

Now, differently from our precedent Sections 3.1 and 4.1, an additional surface current density component  $K'_\nu \vec{v}$  is occurred at our specified space-time point  $P(\vec{x}, t)$  on the moving boundary surface. And, because of the covariance principle of electromagnetism, local and instantaneous discontinuity relations at the point  $P(\vec{x}, t)$  on the moving interface will have the same forms with the ones found in Eqs. (5)–(10) of Part I:

$$\left. \begin{aligned} D'_{2n} - D'_{1n} &= 4\pi\sigma' \\ B'_{2n} - B'_{1n} &= 0 \\ H'_{2\nu} - H'_{1\nu} &= \frac{4\pi}{c} K'_t \\ -(H'_{2t} - H'_{1t}) &= \frac{4\pi}{c} K'_\nu \\ E'_{2\nu} - E'_{1\nu} &= 0 \\ E'_{2t} - E'_{1t} &= 0 \end{aligned} \right\} ; \quad (33)$$

where, the following correspondence is used between the earlier Eqs. (5)–(10) and the present Eq. (33):<sup>6</sup>

$$\vec{e}_x, \vec{e}_y, \vec{e}_z \rightarrow \vec{\nu}, \vec{t}, \vec{n}; \quad (34)$$

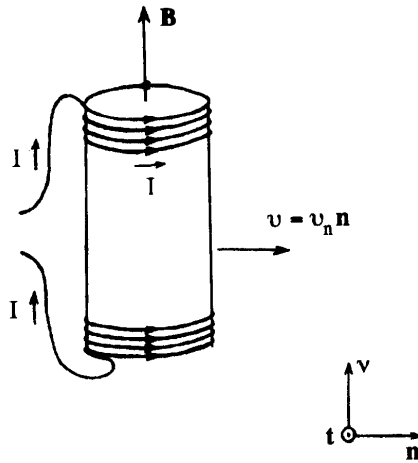
and the primed fields represent again the fields measured in the laboratory, in either side of the moving boundary surface. According to Eq. (33), an additional discontinuity is occurred as seen in its fourth line; and the discontinuity written in its third line is not changed, because of the unchanged value of  $K'_t$  found in the third line of Eq. (31).

## 6. CONCLUSIONS AND DISCUSSIONS

In this study, after the transformed values of the surface source densities belonging to the moving boundary surface are originally calculated in three separate special cases, each concerning one of the three boundary surfaces moving separately in one of the three local and instantaneous natural orthogonal directions, relativistic form invariance principle of electromagnetism is used to obtain three separate local and instantaneous sets of relativistic boundary conditions. For a boundary surface moving with a local and instantaneous velocity (3), combination of these three sets of moving e.m. boundary conditions is necessary. This essential problem is left to a future study; where, the combination of the three separate Lorentz transformed e.m. discontinuity relations will be essayed on the basis founded by the method presented for example by Misner, Thorne and Wheeler [25].

But, without going to the combination of their Lorentz transformations, these three sets of moving electromagnetic discontinuity relations originally calculated in this study can be useful to treat some ideal plane boundary surfaces (Sec. 6 of Part I), again ideally, each moving with a uniform velocity in one of the three separate natural orthogonal directions, such as the one early considered by Noerdlinger [8].

*As a further idealization*, not stated before, three separate sets of moving electromagnetic boundary conditions introduced by this work can be *perfectly valid at rigid plane boundary surfaces*, being between two rigid media, such as the ones being between two kinds of glass, and moving with a uniform velocity in one of the three separate natural orthogonal directions. In fact, an experiment of such kind can be ideally designed for laboratory purposes.



**Figure 2a.** A solenoid of nearly infinite length, placed in vacuum, with its axis lying in the  $\vec{v}$  direction, is moving with the velocity  $\vec{v} = v_n \vec{n}$ .

### APPENDIX I. LORENTZ TRANSFORMATION OF THE MAGNETIC FIELD $\vec{B}$ IN THE CASE OF A BOUNDARY SURFACE MOVING WITH A VELOCITY $\vec{v} = v_n \vec{n}$

Let us begin with a solenoid<sup>7</sup> placed in the vacuum, with its axis lying in the  $\vec{v}$  direction (Fig. 2a). If we assume that the solenoid is of nearly infinite length, nearly uniform magnetic field inside it will be the following:

$$\vec{B} = B_\nu \vec{v}, \quad \text{with} \quad B_\nu = \frac{4\pi}{c} \frac{dQ}{dt} N, \quad (\text{A.1})$$

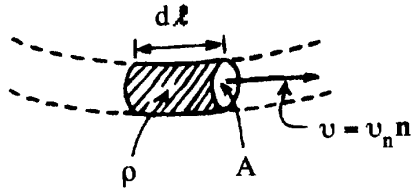
in Gaussian units; here,  $(dQ/dt) = I$  is the current flowing in its turns, and  $N$  is its number of turns per centimeter.

If the solenoid starts to move with the velocity (13), i.e., with a velocity being perpendicular to its axis, following nearly uniform magnetic field will be observed inside it,<sup>7</sup> because of the fundamental covariance principle:

$$\vec{B}' = B'_\nu \vec{v}, \quad \text{with} \quad B'_\nu = \frac{4\pi}{c} \frac{dQ'}{dt'} N'. \quad (\text{A.2})$$

Our present aim is to find the transformation rule between the fields  $\vec{B}$  and  $\vec{B}'$ .





**Figure 2b.** An exaggerated illustration of a volume element  $dV = A(dl)$  of the wire of the solenoid; this considered volume element is chosen so as to its cross sectional area  $A$  be perpendicular to the velocity  $\vec{v} = v_n \vec{n}$  of the solenoid.

Let us start by examining elementary definition of  $dQ$  seen in Eq. (A.1);  $dQ$  is the quantity of charge found in the volume element  $dV = A(dl)$  of the wire of the solenoid:

$$dQ = \rho A(dl) ; \quad (\text{A.3})$$

where,  $A$  is the cross sectional area of the wire,  $dl$  is its length element and  $\rho$  is the volume charge density inside the wire (Fig. 2b).

At first, let us consider Lorentz transformation of  $dQ$ ; in Eq. (A.3),  $\rho$  transform according to a similar rule found in Eq. (19):

$$\rho' = \gamma_n \rho ; \quad (\text{A.4})$$

cross section area  $A$ , being perpendicular to the velocity (13), transforms without changing:

$$A' = A ; \quad (\text{A.5})$$

and the proper length element  $dl$ , being parallel to the same velocity, will be Lorentz contracted:

$$dl' = \frac{dl}{\gamma_n} . \quad (\text{A.6})$$

Resultantly, by comparison also with Eq. (A.3),  $dQ$  transforms without changing:

$$dQ' = \rho' A'(dl') = (\gamma_n \rho) A \left( \frac{dl}{\gamma_n} \right) = \rho A(dl) = dQ . \quad (\text{A.7})$$

Secondly, let us take into account time dilation of the proper time  $dt$ :

$$dt' = \gamma_n dt . \quad (\text{A.8})$$

Next, let us find Lorentz transformation of the number of turns  $N$  per centimeter of the solenoid; since the solenoid is lying perpendicular to the velocity (13), its length remains the same under Lorentz transformation, and then, its number of turns per centimeter remains also the same:

$$N' = N . \quad (\text{A.9})$$

Finally, substitutions of Eqs. (A.7)–(A.9) into Eq. (A.2) and then comparison with Eq. (A.1) lead to the following transformation rule:

$$B'_\nu = \frac{B_\nu}{\gamma_n} . \quad (\text{A.10})$$

Consequently, by a good comparison, we can indicate that Noerdlinger [8] must have found that

$$B' = \frac{B_0}{\gamma} , \quad (\text{A.11})$$

instead of his Eq. (2).

Furthermore, in the problem studied by Noerdlinger [8], it is impossible to have any charge in his frame  $S'$ , obtained by the Lorentz transformations [28] of his sources 4-vector

$$(c\rho, \vec{J}) = (c\rho, J\vec{e}_y) , \quad \text{with free volume charge density satisfying} \quad \rho = 0 ; \quad (\text{A.12})$$

so that, there must not have been electric field  $\vec{E}'$ , either. Therefore, we must emphasize that Lorentz transformations of the electromagnetic fields must be used with great care, by taking also in consideration transformations of their sources. That is, Lorentz transformations of the sources and the fields are not independent of each other; and, if there are not the transformed sources, there will not be the corresponding transformed fields, either.<sup>1</sup>

## APPENDIX II. FOOTNOTES

<sup>1</sup> Verifying examples can be found in Section 6.7 of Ref. [18]. There, two parallel plane sheets of surface charge, moving parallelly to their surfaces, are considered to obtain Lorentz transformations of electromagnetic fields. From the view point of our present problem, each one of these plane sheets can be considered as defining a moving boundary surface.

In the first case treated there, these two parallel plane sheets are moving with the velocity  $v_o\vec{e}_x$ . In comparison with our problem, these moving surfaces are carrying surface charge and current densities  $\sigma'$  and  $\vec{K}'$  having respectively the values  $\pm\sigma$  and  $\pm\sigma v_o\vec{e}_x$  which are measured on the moving sheets in the laboratory. And, by using Eq. (11.22) of Ref. [5], we can show that these surface source densities  $\sigma'$  and  $\vec{K}'$  are the Lorentz transforms of the ones  $\sigma$  and  $\vec{K}$ , having respectively the values  $\pm(\sigma/\gamma_o)$  and 0 which would be measured on the same sheets when they would be at rest in the laboratory. For more details, see our Sections 3.1, 5.1 and *Footnote 4*.

- <sup>2</sup> Expression “fixed point” is borrowed from an essay by Ulam [19].  
<sup>3</sup> Our intuitive view point will be verified in *Footnote 4*.  
<sup>4</sup> Except that one spatial dimension is reduced, four-vector  $(c\sigma, \vec{K})$  is similar to the one written in Eq. (11.128) of Ref. [5]. Furthermore, present dependence between  $\sigma$  and  $\vec{K}$  based on the motion of the surface charge [27] provides a physical support to the mathematical postulate concerning Eq. (11.128) of Ref. [5]. This means that the surface charge and current densities, being both parallel to the velocity of the surface carrying them, mix with each other under the Lorentz transformations, just as the time and one space components of the four-vector  $(ct, \vec{x})$  mix with each other under similar Lorentz transformations.

Again, let us consider the plane sheets treated in the *Footnote 1*. 4-vectors constituted by the surface charge and current densities on these plane sheets are

$$(\pm c\sigma, \vec{K}) \equiv \left( \pm c \frac{\sigma}{\gamma_o}, 0 \right), \quad (\text{F.1})$$

and

$$(\pm c\sigma', \vec{K}') \equiv (\pm c\sigma, \pm\sigma v_o\vec{e}_x), \quad (\text{F.2})$$

when the sheets are respectively at rest and moving with the velocity  $v_o\vec{e}_x$ ; between these 4-vectors, convenient Lorentz factors are the followings:

$$\vec{\beta}_o = -\frac{v_o}{c}\vec{e}_x \quad \text{and} \quad \gamma_o = (1 - \beta_o^2)^{-1/2}. \quad (\text{F.3})$$

And, by using 4-vector Lorentz transformations [28], with the Lorentz factors (F.3), we can show that the four-vector (F.2) is the

Lorentz transform of the one (F.1); hence, the Lorentz factor  $\vec{\beta}_o$  is true. Therefore, our intuitive discussion on the sign of our Lorentz factors  $\vec{\beta}$  is also verified, by the obtainment of the right signs in the transformed sources 4-vector (F.2). In fact, if the surface charge densities  $\pm\sigma$  are the ones measured on the sheets moving with the velocity  $v_o\vec{e}_x$ , they must be also physically equivalent to the surface current densities  $\pm\sigma v_o\vec{e}_x$ , as also obtainable by the Lorentz transformations, as recently indicated. Thus, the sign of the Lorentz factor  $\vec{\beta}$  is uniquely defined; this means that the rest and moving frames of the Lorentz transformations are uniquely determined, as already expressed in Section 2.

This fact verifies also the mixing cited above between the components  $c\sigma$  and  $\vec{K}$  of the 4-vector  $(c\sigma, \vec{K})$ ; indeed, surface current densities  $\vec{K}' = \pm\sigma v_o\vec{e}_x$ , which are measured on the moving sheets, are occurred only because of the motion of the sheets; whereas, there are not surface current densities on the rest sheets.

- <sup>5</sup> Transformation rule given in Noerdlinger [8]'s Eq. (2) must be revised as seen in our *Appendix I*.
- <sup>6</sup> This correspondence is also used by Eq. (21), in Part I.
- <sup>7</sup> The use of a solenoid to obtain independent Lorentz transformation of the magnetic field is inspired from Section 6.7, p. 237 of Ref. [18].

## ACKNOWLEDGMENT

The author would like to thank the International Atomic Energy Agency, and UNESCO for hospitality at the International Center for Theoretical Physics, Trieste, Italy, where she studied partly this work during her visits. She wishes to acknowledge also the Turkish Physics Foundation at Ankara, Turkey, for supporting two of these visits.

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