Progress of Theoretical Physics, Vol. 112, No. 2, August 2004

Relativistic Entanglements of Spin 1/2 Particles with General Momentum

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(Received May 17, 2004)

In this paper, the Lorentz transformations of entangled Bell states with general momentum not necessarily orthogonal to the boost direction and spin are studied. We extend quantum correlations and Bell's inequality to the relativistic regime by considering normalized relativistic observables. It is shown that quantum information along the direction perpendicular to the boost is eventually lost, and Bell's inequality is not always violated for entangled states in special relativity. This could impose restrictions on certain quantum information processing, such as quantum cryptography using massive particles.

§1. Introduction

Relativistic quantum information processing is of growing interest not only for the logical completeness but also with regard to new features, such as the physical bounds on information transfer, processing and the errors provided by the full relativistic treatments.[1)–11)] There is also an important question whether Bell's inequality is always violated for entangled states for observers in different Lorentz frames. Violation of Bell's inequality is perhaps the most drastic feature distinguishing quantum theory from classical physics.[12)] Bell's proof that there are states of two-quantum-particle systems that do not satisfy Bell's inequality derived from Einstein's assumptions[13)] of the principle of local causes has changed our traditional viewpoint of Nature quite significantly. Specifically, it was shown that all non-product states, otherwise known as entangled states, always violate the Bell inequality when special relativity is not taken into account.[14)] Therefore, it is an interesting question to whether the above mentioned condition changes if one considers special relativity.

Under the Lorentz transformation, the Hilbert space vectors representing the quantum states undergo unitary transformations.[15)] On the other hand, the Pauli matrices are not Lorentz covariant, and therefore as a need to find relativistically invariant operators corresponding to the spin in order to investigate Bell's inequality within special relativity.[16)] Sometime ago, Fleming[17)] showed that a covariant spin-vector operator that reduces to the ordinary spin operator in the non-relativistic limit, can be derived from the Pauli-Lubanski pseudo vector, and Czachor[1)] showed

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that the degree of violation of Bell's inequality depends on the velocity of the pair of spin-$\frac{1}{2}$ particles with respect to the laboratory. A unitary transformation corresponding a Lorentz boost of the quantum states was not considered in those works.

In a previous work,[6)] we calculated Bell observables for entangled states in the rest frame with both momentum vector and the spin in the z-direction, as seen by an observer moving in the x-direction, and showed that the entangled states satisfy Bell's inequality when the boost speed approaches the speed of light. Also, we showed that average of the Bell observable for Lorentz transformed entangled states becomes

$$c(\vec{a},\vec{a}',\vec{b},\vec{b}') = \langle \hat{a}\otimes\hat{b}\rangle + \langle \hat{a}\otimes\hat{b}'\rangle + \langle \hat{a}'\otimes\hat{b}\rangle - \langle \hat{a}'\otimes\hat{b}'\rangle = \frac{2}{\sqrt{2-\beta^2}}(1+\sqrt{1-\beta^2}), \tag{1}$$

where $\hat{a}$ and $\hat{b}$ are the relativistic spin observables for Alice and Bob, respectively, related to the Pauli-Lubanski pseudo vector, and $\beta = \frac{v}{c}$ is the ratio of the boost speed and the speed of light.

In this paper, we derive the transformation for the relativistic entanglement of spin 1/2 particles with case the general momentum is not necessarily perpendicular to the boost direction. We also calculate the average of the Bell observable for the momentum-conserved entangled Bell states for spin-$\frac{1}{2}$ particles and show that the universal relation given by Eq. (1) still holds in general.

§2. Relativistic entanglements

A multi-particle state vector is denoted by

$$\Psi_{p_1\sigma_1;p_1\sigma_2;\ldots} = a^+(\vec{p}_1,\sigma_1)a^+(\vec{p}_2,\sigma_2)\ldots\Psi_0, \tag{2}$$

where p_i labels the four-momentum, σ_i is the spin z component, $a^+(\vec{p}_i,\sigma_i)$ is the creation operator, which adds a particle with momentum $\vec{p}_i$ and spin σ_i, and Ψ_0 is the Lorentz invariant vacuum state. The Lorentz transformation Λ induces a unitary transformation on vectors in the Hilbert space as

$$\Psi \to U(\Lambda)\Psi, \tag{3}$$

and the operators U satisfis the composition rule

$$U(\bar{\Lambda})U(\Lambda) = U(\bar{\Lambda}\Lambda), \tag{4}$$

while the creation operator has the transformation rule[15)]

$$U(\Lambda)a^+(\vec{p},\sigma)U(\Lambda)^{-1} = \sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\bar{\sigma}} D^{(j)}_{\bar{\sigma}\sigma}(W(\Lambda,p))a^+(\vec{p}_\Lambda,\bar{\sigma}). \tag{5}$$

Here, $W(\Lambda,p)$ is Wigner's little group element, given by

$$W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p), \tag{6}$$

with $D^{(j)}(W)$ the representation of W for spin $j, p^\mu = (\vec{p}, p^0)$, $(\Lambda p)^\mu = (\vec{p}_\Lambda, (\Lambda p)^0)$ with $\mu = 1, 2, 3, 0$, and $L(p)$ is the Lorentz transformation such that

$$p^\mu = L^\mu_v k^v, \tag{7}$$

where $k^v = (0, 0, 0, m)$ is the four-momentum taken in the particle's rest frame. One can also use the conventional ket-notation to represent the quantum states as

$$\begin{aligned} \Psi_{p,\sigma} &= a^+(\vec{p}, \sigma)\Phi_0 \\ &= |\vec{p}, \sigma\rangle \\ &= |\vec{p}\rangle \otimes |\sigma\rangle. \end{aligned} \tag{8}$$

The Wigner representation of the Lorentz group for spin-$\frac{1}{2}$ becomes:[6)]

$$\begin{aligned} &D^{(1/2)}(W(\Lambda, p)) \\ &= \frac{1}{[(p^0+m)((\Lambda p)^0+m)]^{1/2}} \left\{ (p^0+m)\cosh\frac{\alpha}{2} + (\vec{p}\cdot\hat{e})\sinh\frac{\alpha}{2} - i\sinh\frac{\alpha}{2}\vec{\sigma}\cdot(\vec{p}\times\hat{e}) \right\} \\ &= \cos\frac{\Omega_{\vec{p}}}{2} + i\sin\frac{\Omega_{\vec{p}}}{2}(\vec{\sigma}\cdot\hat{n}), \end{aligned} \tag{9}$$

with

$$\cos\frac{\Omega_{\vec{p}}}{2} = \frac{\cosh\frac{\alpha}{2}\cosh\frac{\delta}{2} + \sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}(\hat{e}\cdot\hat{p})}{\left[\frac{1}{2} + \frac{1}{2}\cosh\alpha\cosh\delta + \frac{1}{2}\sinh\alpha\sinh\delta(\hat{e}\cdot\hat{p})\right]^{1/2}} \tag{10}$$

and

$$\sin\frac{\Omega_{\vec{p}}}{2}\hat{n} = \frac{\sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}(\hat{e}\times\hat{p})}{\left[\frac{1}{2} + \frac{1}{2}\cosh\alpha\cosh\delta + \frac{1}{2}\sinh\alpha\sinh\delta(\hat{e}\cdot\hat{p})\right]^{1/2}}, \tag{11}$$

where $\cosh\delta = \frac{p^0}{m}$ We note that Eq. (9) indicates that the Lorentz group can be represented by a pure rotation about the axis $\hat{n} = \hat{e}\times\hat{p}$ for the two-component spinor.

We define the momentum-conserved entangled Bell states for spin-$\frac{1}{2}$ particles in the rest frame as

$$\Psi_{00} = \frac{1}{\sqrt{2}}\left\{ a^+\left(\vec{p}, \frac{1}{2}\right) a^+\left(-\vec{p}, \frac{1}{2}\right) + a^+\left(\vec{p}, -\frac{1}{2}\right) a^+\left(-\vec{p}, -\frac{1}{2}\right) \right\} \Psi_0, \tag{12a}$$

$$\Psi_{01} = \frac{1}{\sqrt{2}}\left\{ a^+\left(\vec{p}, \frac{1}{2}\right) a^+\left(-\vec{p}, \frac{1}{2}\right) - a^+\left(\vec{p}, -\frac{1}{2}\right) a^+\left(-\vec{p}, -\frac{1}{2}\right) \right\} \Psi_0, \tag{12b}$$

$$\Psi_{10} = \frac{1}{\sqrt{2}}\left\{ a^+\left(\vec{p}, \frac{1}{2}\right) a^+\left(-\vec{p}, -\frac{1}{2}\right) + a^+\left(\vec{p}, -\frac{1}{2}\right) a^+\left(-\vec{p}, \frac{1}{2}\right) \right\} \Psi_0, \tag{12c}$$

$$\Psi_{11} = \frac{1}{\sqrt{2}}\left\{ a^+\left(\vec{p}, \frac{1}{2}\right) a^+\left(-\vec{p}, -\frac{1}{2}\right) - a^+\left(\vec{p}, -\frac{1}{2}\right) a^+\left(-\vec{p}, \frac{1}{2}\right) \right\} \Psi_0, \tag{12d}$$

where Ψ_0 is the Lorentz invariant vacuum state.

For an observer in another reference frame S' described by an arbitrary boost Λ, the transformed Bell states are given by

$$\Psi_{ij} \to U(\Lambda)\Psi_{ij}. \tag{13}$$

For example, from Eqs. (5) and (12a), $U(\Lambda)\Psi_{00}$ becomes

$$\begin{aligned} U(\Lambda)\Psi_{00} &= \frac{1}{\sqrt{2}}\left\{U(\Lambda)a^+\left(\vec{p},\frac{1}{2}\right)U^{-1}(\Lambda)U(\Lambda)a^+\left(-\vec{p},\frac{1}{2}\right)U^{-1}(\Lambda)\right. \\ &\quad \left. + U(\Lambda)a^+\left(\vec{p},-\frac{1}{2}\right)U^{-1}(\Lambda)U(\Lambda)a^+\left(-\vec{p},-\frac{1}{2}\right)U^{-1}(\Lambda)\right\}U(\Lambda)\Psi_0 \\ &= \frac{1}{\sqrt{2}}\sum_{\sigma\sigma'}\left\{\sqrt{\frac{(\Lambda p)^0}{p^0}}D^{(\frac{1}{2})}_{\sigma\frac{1}{2}}(W(\Lambda,p))\right. \\ &\quad \times\sqrt{\frac{(\Lambda Pp)^0}{(Pp)^0}}D^{(\frac{1}{2})}_{\sigma'\frac{1}{2}}(W(\Lambda,Pp))a^+(\vec{p}_\Lambda,\sigma)a^+(-\vec{p}_\Lambda,\sigma') \\ &\quad + \sqrt{\frac{(\Lambda p)^0}{p^0}}D^{(\frac{1}{2})}_{\sigma-\frac{1}{2}}(W(\Lambda,p)) \\ &\quad \left. \times\sqrt{\frac{(\Lambda Pp)^0}{(Pp)^0}}D^{(\frac{1}{2})}_{\sigma'-\frac{1}{2}}(W(\Lambda,Pp))a^+(\vec{p}_\Lambda,\sigma)a^+(-\vec{p}_\Lambda,\sigma')\right\}\Psi_0 \end{aligned} \tag{14}$$

and so on.

2.1. *The case that momentum and boost vectors are in the same plane*

We assume that $\vec{p}$ is in the $x-z$ plane, $\vec{p} = (p\sin\theta, 0, p\cos\theta)$, and the boost Λ is in x-direction. In this case, we have

$$\cos\frac{\Omega_{\pm\vec{p}}}{2} = \frac{\cosh\frac{\alpha}{2}\cosh\frac{\delta}{2} \pm \sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}\sin\theta}{\left[\frac{1}{2}+\frac{1}{2}\cosh\alpha\cosh\delta \pm \sinh\alpha\sinh\delta\sin\theta\right]^{1/2}}, \tag{15}$$

$$\sin\frac{\Omega_{\pm\vec{p}}}{2}\hat{n}_\pm = \frac{(\mp\hat{y})\sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}\cos\theta}{\left[\frac{1}{2}+\frac{1}{2}\cosh\alpha\cosh\delta \pm \frac{1}{2}\sinh\alpha\sinh\delta\sin\theta\right]^{1/2}}, \tag{16}$$

and

$$\begin{aligned} D^{1/2}(W(\Lambda,p)) &= \cos\frac{\Omega_{\vec{p}}}{2} - i\sigma_y\sin\frac{\Omega_{\vec{p}}}{2} \\ &= \begin{pmatrix} \cos\frac{\Omega_{\vec{p}}}{2} & -\sin\frac{\Omega_{\vec{p}}}{2} \\ \sin\frac{\Omega_{\vec{p}}}{2} & \cos\frac{\Omega_{\vec{p}}}{2} \end{pmatrix}, \end{aligned} \tag{17}$$

$$D^{1/2}(W(\Lambda, Pp)) = \cos\frac{\Omega_{-\vec{p}}}{2} + i\sigma_y \sin\frac{\Omega_{-\vec{p}}}{2} = \begin{pmatrix} \cos\frac{\Omega_{-\vec{p}}}{2} & \sin\frac{\Omega_{-\vec{p}}}{2} \\ -\sin\frac{\Omega_{-\vec{p}}}{2} & \cos\frac{\Omega_{-\vec{p}}}{2} \end{pmatrix}, \tag{18}$$

where $\hat{n}_{\pm} = \mp\hat{y}$. Then from Eqs. (17), (18) and (14), we obtain

$$\begin{aligned} U(\Lambda)\Psi_{00} &= \frac{(\Lambda p)^0}{p^0}\cos\frac{\Omega_{\vec{p}}+\Omega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda,\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda,\frac{1}{2}\right)\right. \\ &\quad \left. + a^+\left(\vec{p}_\Lambda,-\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda,-\frac{1}{2}\right)\right\}\Psi_0 \\ &\quad - \frac{(\Lambda p)^0}{p^0}\sin\frac{\Omega_{\vec{p}}+\Omega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda,\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda,-\frac{1}{2}\right)\right. \\ &\quad \left. - a^+\left(\vec{p}_\Lambda,-\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda,\frac{1}{2}\right)\right\}\Psi_0 \\ &= \frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda,-\vec{p}_\Lambda\rangle \otimes \left\{\cos\frac{\Omega_{\vec{p}}+\Omega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|-\frac{1}{2},-\frac{1}{2}\right\rangle\right)\right\} \\ &\quad - \frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda,-\vec{p}_\Lambda\rangle \otimes \left\{\sin\frac{\Omega_{\vec{p}}+\Omega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle - \left|-\frac{1}{2},\frac{1}{2}\right\rangle\right)\right\} \\ &= \frac{(\Lambda p)^0}{p^0}\left\{\cos\frac{\Omega_{\vec{p}}+\Omega_{-\vec{p}}}{2}\Psi'_{00} - \sin\frac{\Omega_{\vec{p}}+\Omega_{-\vec{p}}}{2}\Psi'_{11}\right\}, \end{aligned} \tag{19a}$$

where Ψ'_{ij} represents the Bell states in the moving frame S' whose momenta are transformed as $\vec{p} \to \vec{p}_\Lambda$, $-\vec{p} \to -\vec{p}_\Lambda$. Similarly, we have

$$\begin{aligned} U(\Lambda)\Psi_{01} &= \frac{(\Lambda p)^0}{p^0}\cos\frac{\Omega_{\vec{p}}-\Omega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda,\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda,\frac{1}{2}\right)\right. \\ &\quad \left. - a^+\left(\vec{p}_\Lambda,-\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda,-\frac{1}{2}\right)\right\}\Psi_0 \\ &\quad + \frac{(\Lambda p)^0}{p^0}\sin\frac{\Omega_{\vec{p}}-\Omega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda,\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda,-\frac{1}{2}\right)\right. \\ &\quad \left. + a^+\left(\vec{p}_\Lambda,-\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda,\frac{1}{2}\right)\right\}\Psi_0 \\ &= \frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda,-\vec{p}_\Lambda\rangle \otimes \left\{\cos\frac{\Omega_{\vec{p}}-\Omega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle - \left|-\frac{1}{2},-\frac{1}{2}\right\rangle\right)\right\} \\ &\quad + \frac{(\Lambda)^0}{p^0}|\vec{p}_\Lambda,-\vec{p}_\Lambda\rangle \otimes \left\{\sin\frac{\Omega_{\vec{p}}-\Omega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|-\frac{1}{2},\frac{1}{2}\right\rangle\right)\right\} \\ &= \frac{(\Lambda p)^0}{p^0}\left\{\cos\frac{\Omega_{\vec{p}}-\Omega_{-\vec{p}}}{2}\Psi'_{01} + \sin\frac{\Omega_{\vec{p}}-\Omega_{-\vec{p}}}{2}\Psi'_{10}\right\}, \end{aligned} \tag{19b}$$

$$
\begin{aligned}
U(\varLambda)\varPsi_{10} &= \frac{(\varLambda p)^0}{p^0}\cos\frac{\varOmega_{\vec{p}}-\varOmega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\varLambda,\frac{1}{2}\right)a^+\left(-\vec{p}_\varLambda,\frac{1}{2}\right)\right.\\
&\quad\left.+\,a^+\left(\vec{p}_\varLambda,-\frac{1}{2}\right)a^+\left(-\vec{p}_\varLambda,-\frac{1}{2}\right)\right\}\varPsi_0\\
&\quad-\frac{(\varLambda p)^0}{p^0}\sin\frac{\varOmega_{\vec{p}}-\varOmega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\varLambda,\frac{1}{2}\right)a^+\left(-\vec{p}_\varLambda,-\frac{1}{2}\right)\right.\\
&\quad\left.-\,a^+\left(\vec{p}_\varLambda,-\frac{1}{2}\right)a^+\left(-\vec{p}_\varLambda,-\frac{1}{2}\right)\right\}\varPsi_0\\
&= \frac{(\varLambda)^0}{p^0}|\vec{p}_\varLambda,-\vec{p}_\varLambda\rangle\otimes\left\{\cos\frac{\varOmega_{\vec{p}}-\varOmega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle+\left|-\frac{1}{2},\frac{1}{2}\right\rangle\right)\right\}\\
&\quad-\frac{(\varLambda p)^0}{p^0}|\vec{p}_\varLambda,-\vec{p}_\varLambda\rangle\otimes\left\{\sin\frac{\varOmega_{\vec{p}}-\varOmega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle-\left|-\frac{1}{2},-\frac{1}{2}\right\rangle\right)\right\}\\
&= \frac{(\varLambda p)^0}{p^0}\left\{\cos\frac{\varOmega_{\vec{p}}-\varOmega_{-\vec{p}}}{2}\varPsi'_{10}-\sin\frac{\varOmega_{\vec{p}}-\varOmega_{-\vec{p}}}{2}\varPsi'_{01}\right\}
\end{aligned}
\tag{19c}
$$

and

$$
\begin{aligned}
U(\varLambda)\varPsi_{11} &= \frac{(\varLambda p)^0}{p^0}\cos\frac{\varOmega_{\vec{p}}+\varOmega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\varLambda,\frac{1}{2}\right)a^+\left(-\vec{p}_\varLambda,-\frac{1}{2}\right)\right.\\
&\quad\left.-\,a^+\left(\vec{p}_\varLambda,-\frac{1}{2}\right)a^+\left(-\vec{p}_\varLambda,\frac{1}{2}\right)\right\}\varPsi_0\\
&\quad+\frac{(\varLambda p)^0}{p^0}\sin\frac{\varOmega_{\vec{p}}+\varOmega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\varLambda,\frac{1}{2}\right)a^+\left(-\vec{p}_\varLambda,\frac{1}{2}\right)\right.\\
&\quad\left.+\,a^+\,(\vec{p}_\varLambda,-\frac{1}{2})a^+\left(-\vec{p}_\varLambda,-\frac{1}{2}\right)\right\}\varPsi_0\\
&= \frac{(\varLambda p)^0}{p^0}|\vec{p}_\varLambda,-\vec{p}_\varLambda\rangle\otimes\left\{\cos\frac{\varOmega_{\vec{p}}+\varOmega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle-\left|-\frac{1}{2},\frac{1}{2}\right\rangle\right)\right\}\\
&\quad+\frac{(\varLambda p)^0}{p^0}|\vec{p}_\varLambda,-\vec{p}_\varLambda\rangle\otimes\left\{\sin\frac{\varOmega_{\vec{p}}+\varOmega_{-\vec{p}}}{2}\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle+\left|-\frac{1}{2},-\frac{1}{2}\right\rangle\right)\right\}\\
&= \frac{(\varLambda p)^0}{p^0}\left\{\cos\frac{\varOmega_{\vec{p}}+\varOmega_{-\vec{p}}}{2}\varPsi'_{11}+\sin\frac{\varOmega_{\vec{p}}+\varOmega_{-\vec{p}}}{2}\varPsi'_{00}\right\},
\end{aligned}
\tag{19d}
$$

where

$$
\begin{aligned}
\cos\frac{\varOmega_{\vec{p}}+\varOmega_{-\vec{p}}}{2} &= \cos\frac{\varOmega_{\vec{p}}}{2}\cos\frac{\varOmega_{-\vec{p}}}{2}-\sin\frac{\varOmega_{\vec{p}}}{2}\sin\frac{\varOmega_{-\vec{p}}}{2}\\
&= \frac{\left(\cosh\frac{\alpha}{2}\cosh\frac{\delta}{2}\right)^2-\left(\sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}\right)^2}{\left[\left(\frac{1}{2}+\frac{1}{2}\cosh\alpha\cosh\delta\right)^2-\left(\frac{1}{2}\sinh\alpha\sinh\delta\sin\theta\right)^2\right]^{\frac{1}{2}}},
\end{aligned}
\tag{20a}
$$

$$\cos\frac{\Omega_{\vec{p}}-\Omega_{-\vec{p}}}{2} = \cos\frac{\Omega_{\vec{p}}}{2}\cos\frac{\Omega_{-\vec{p}}}{2}+\sin\frac{\Omega_{\vec{p}}}{2}\sin\frac{\Omega_{-\vec{p}}}{2}$$
$$= \frac{\left(\cosh\frac{\alpha}{2}\cosh\frac{\delta}{2}\right)^2+\left(\sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}\right)^2\cos 2\theta}{\left[\left(\frac{1}{2}+\frac{1}{2}\cosh\alpha\cosh\delta\right)^2-\left(\frac{1}{2}\sinh\alpha\sinh\delta\sin\theta\right)^2\right]^{\frac{1}{2}}}, \tag{20b}$$

$$\sin\frac{\Omega_{\vec{p}}+\Omega_{-\vec{p}}}{2} = \sin\frac{\Omega_{\vec{p}}}{2}\cos\frac{\Omega_{-\vec{p}}}{2}+\sin\frac{\Omega_{\vec{p}}}{2}\sin\frac{\Omega_{-\vec{p}}}{2}$$
$$= \frac{2\cosh\frac{\alpha}{2}\cosh\frac{\delta}{2}\sinh\frac{\delta}{2}\sinh\frac{\delta}{2}\cos\theta}{\left[\left(\frac{1}{2}+\frac{1}{2}\cosh\alpha\cosh\delta\right)^2-\left(\frac{1}{2}\sinh\alpha\sinh\delta\sin\theta\right)^2\right]^{\frac{1}{2}}}, \tag{20c}$$

$$\sin\frac{\Omega_{\vec{p}}-\Omega_{-\vec{p}}}{2} = \sin\frac{\Omega_{\vec{p}}}{2}\cos\frac{\Omega_{-\vec{p}}}{2}-\sin\frac{\Omega_{\vec{p}}}{2}\sin\frac{\Omega_{-\vec{p}}}{2}$$
$$= \frac{-\left(\sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}\right)^2\sin 2\theta}{\left[\left(\frac{1}{2}+\frac{1}{2}\cosh\alpha\cosh\delta\right)^2-\left(\frac{1}{2}\sinh\alpha\sinh\delta\sin\theta\right)^2\right]^{\frac{1}{2}}}. \tag{20d}$$

2.2. *The case that the momentum and boost vectors are not in the same plane*

We consider the general case of a momentum vector out of plane, $\vec{p}=(p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$, and a boost in the x-direction. In this case, we have

$$\cos\frac{\Omega_{\pm\vec{p}}}{2} = \frac{\cosh\frac{\alpha}{2}\cosh\frac{\delta}{2}\pm\sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}\sin\theta\cos\phi}{\left[\frac{1}{2}+\frac{1}{2}\cosh\alpha\cosh\delta\pm\frac{1}{2}\sinh\alpha\sinh\delta\sin\theta\cosh\phi\right]^{1/2}}, \tag{21}$$

$$\sin\frac{\Omega_{\pm\vec{p}}}{2}\hat{n}_\pm = \frac{r\hat{n}_\pm\sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}}{\left[\frac{1}{2}+\frac{1}{2}\cosh\alpha\cosh\delta\pm\frac{1}{2}\sinh\alpha\sinh\delta\sin\theta\cos\phi\right]^{1/2}}, \tag{22}$$

and

$$D^{1/2}(W(\Lambda,p)) = \cos\frac{\Omega_{\vec{p}}}{2}+i\sin\frac{\Omega_{\vec{p}}}{2}\sigma\cdot(-\hat{y}\cos\eta+\hat{z}\sin\eta)$$
$$= \begin{pmatrix} \cos\frac{\Omega_{\vec{p}}}{2}+i\sin\frac{\Omega_{\vec{p}}}{2}\sin\eta & -\sin\frac{\Omega_{\vec{p}}}{2}\cos\eta \\ \sin\frac{\Omega_{\vec{p}}}{2}\cos\eta & \cos\frac{\Omega_{\vec{p}}}{2}-i\sin\frac{\Omega_{\vec{p}}}{2}\sin\eta \end{pmatrix}, \tag{23}$$

$$D^{1/2}(W(\Lambda, Pp)) = \cos\frac{\Omega_{-\vec{p}}}{2} - i\sin\frac{\Omega_{-\vec{p}}}{2}\sigma\cdot(-\hat{y}\cos\eta + \hat{z}\sin\eta)$$
$$= \begin{pmatrix} \cos\frac{\Omega_{-\vec{p}}}{2} - i\sin\frac{\Omega_{-\vec{p}}}{2}\sin\eta & \sin\frac{\Omega_{-\vec{p}}}{2}\cos\eta \\ -\sin\frac{\Omega_{-\vec{p}}}{2}\cos\eta & \cos\frac{\Omega_{-\vec{p}}}{2} + i\sin\frac{\Omega_{-\vec{p}}}{2}\sin\eta \end{pmatrix}, \quad (24)$$

where $\hat{n}_\pm = \pm(-\hat{y}\cos\eta + \hat{z}\sin\eta)$, $\cos\eta = \frac{\cos\theta}{r}$, $\sin\eta = \frac{\sin\theta\sin\phi}{r}$, $r = \sqrt{\sin^2\theta\sin^2\phi + \cos^2\theta}$.

Let $\bar{\Omega}_{\vec{p}} = \frac{\Omega_{\vec{p}} + \Omega_{-\vec{p}}}{2}$, $\Delta\Omega_{\vec{p}} = \frac{\Omega_{\vec{p}} - \Omega_{-\vec{p}}}{2}$.

Then, from Eqs. (23), (24) and (14), we obtain

$$\begin{aligned}
U(\Lambda)\Psi_{00} = {} & \frac{(\Lambda p)^0}{p^0}(\cos\bar{\Omega}_{\vec{p}}\cos^2\eta + \cos\Delta\Omega_{\vec{p}}\sin^2\eta)\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\right. \\
& \left. + a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, -\frac{1}{2}\right)\right\}\Psi_0 \\
& - \frac{(\Lambda p)^0}{p^0}\sin\bar{\Omega}_{\vec{p}}\cos\eta\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, -\frac{1}{2}\right)\right. \\
& \left. - a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\right\}\Psi_0 \\
& + i\frac{(\Lambda p)^0}{p^0}\sin\Delta\Omega_{\vec{p}}\sin\eta\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\right. \\
& \left. - a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, -\frac{1}{2}\right)\right\}\Psi_0 \\
& - i\frac{(\Lambda p)^0}{p^0}(-\cos\bar{\Omega}_{\vec{p}} + \cos\Delta\Omega_{\vec{p}})\sin\eta\cos\eta\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, -\frac{1}{2}\right)\right. \\
& \left. + a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\right\}\Psi_0 \\
= {} & \frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left\{(\cos\bar{\Omega}_{\vec{p}}\cos^2\eta + \cos\Delta\Omega_{\vec{p}}\sin^2\eta)\right. \\
& \left. \times \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, \frac{1}{2}\right\rangle + \left|-\frac{1}{2}, -\frac{1}{2}\right\rangle\right)\right\} \\
& - \frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left\{\sin\bar{\Omega}_{\vec{p}}\cos\eta\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle - \left|-\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right\} \\
& + i\frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left\{\sin\Delta\Omega_{\vec{p}}\sin\eta\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, \frac{1}{2}\right\rangle - \left|-\frac{1}{2}, -\frac{1}{2}\right\rangle\right)\right\} \\
& - i\frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left\{(-\cos\bar{\Omega}_{\vec{p}} + \cos\Delta\Omega_{\vec{p}})\right. \\
& \left. \times \sin\eta\cos\eta\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \left|-\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right\}
\end{aligned}$$

$$= \frac{(\Lambda p)^0}{p^0}\{(\cos\bar{\Omega}_{\vec{p}}\cos^2\eta + \cos\Delta\Omega_{\vec{p}}\sin^2\eta)\Psi'_{00} - \sin\bar{\Omega}_{\vec{p}}\cos\eta\Psi'_{11}$$
$$+ i\sin\Delta\Omega_{\vec{p}}\sin\eta\Psi'_{01} - i(-\cos\bar{\Omega}_{\vec{p}} + \cos\Delta\Omega_{\vec{p}})\sin\eta\cos\eta\Psi'_{10}\}, \tag{25a}$$

where Ψ'_{ij} represents the Bell states in the moving frame S' whose momenta are transformed as $\vec{p} \to \vec{p}_\Lambda$, $-\vec{p} \to -\vec{p}_\Lambda$. Similarly, we have

$$U(\Lambda)\Psi_{01} = \frac{(\Lambda p)^0}{p^0}\cos\Delta\Omega_{\vec{p}}\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\right.$$
$$\left. - a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, -\frac{1}{2}\right)\right\}\Psi_0$$
$$+ \frac{(\Lambda p)^0}{p^0}\sin\Delta\Omega_{\vec{p}}\cos\eta\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, -\frac{1}{2}\right)\right.$$
$$\left. + a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\right\}\Psi_0$$
$$+ i\frac{(\Lambda p)^0}{p^0}\sin\Delta\Omega_{\vec{p}}\sin\eta\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\right.$$
$$\left. + a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right)a^+\left(\vec{p}-_\Lambda, -\frac{1}{2}\right)\right\}\Psi_0$$
$$= \frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left\{\cos\Delta\Omega_{\vec{p}}\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, \frac{1}{2}\right\rangle - \left|-\frac{1}{2}, -\frac{1}{2}\right\rangle\right)\right\}$$
$$+ \frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left\{\sin\Delta\Omega_{\vec{p}}\cos\eta\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \left|-\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right\}$$
$$+ i\frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left\{\sin\Delta\Omega_{\vec{p}}\sin\eta\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, \frac{1}{2}\right\rangle + \left|-\frac{1}{2}, -\frac{1}{2}\right\rangle\right)\right\}$$
$$= \frac{(\Lambda p)^0}{p^0}\{\cos\Delta\Omega_{\vec{p}}\Psi'_{01} + \sin\Delta\Omega_{\vec{p}}\cos\eta\Psi'_{10} + i\sin\Delta\Omega_{\vec{p}}\sin\eta\Psi'_{00}\}, \tag{25b}$$

$$U(\Lambda)\Psi_{10} = \frac{(\Lambda p)^0}{p^0}(\cos\bar{\Omega}_{\vec{p}}\sin^2\eta + \cos\Delta\Omega_{\vec{p}}\cos^2\eta)\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, -\frac{1}{2}\right)\right.$$
$$\left. + a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\right\}\Psi_0$$
$$- \frac{(\Lambda p)^0}{p^0}\sin\Delta\Omega_{\vec{p}}\cos\eta\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\right.$$
$$\left. - a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, -\frac{1}{2}\right)\right\}\Psi_0$$
$$+ i\frac{(\Lambda p)^0}{p^0}\sin\bar{\Omega}_{\vec{p}}\sin\eta\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, -\frac{1}{2}\right)\right.$$
$$\left. - a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\right\}\Psi_0$$
$$- i\frac{(\Lambda p)^0}{p^0}(\cos\bar{\Omega}_{\vec{p}} - \cos\Delta\Omega_{\vec{p}})\sin\eta\cos\eta\frac{1}{\sqrt{2}}\left\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right)a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\right.$$

$$
\begin{aligned}
&+ a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right) a^+\left(-\vec{p}_\Lambda, -\frac{1}{2}\right)\Bigg\}\Psi_0\\
&= \frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \Bigg\{(\cos\bar{\Omega}_{\vec{p}}\sin^2\eta + \cos\Delta\Omega_{\vec{p}}\cos^2\eta)\\
&\quad \times \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \left|-\frac{1}{2}, \frac{1}{2}\right\rangle\right)\Bigg\}\\
&\quad - \frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left\{\sin\Delta\Omega_{\vec{p}}\cos\eta\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, \frac{1}{2}\right\rangle - \left|-\frac{1}{2}, -\frac{1}{2}\right\rangle\right)\right\}\\
&\quad + i\frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left\{\sin\bar{\Omega}_{\vec{p}}\sin\eta\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle - \left|-\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right\}\\
&\quad - i\frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \Bigg\{(\cos\bar{\Omega}_{\vec{p}} - \cos\Delta\Omega_{\vec{p}})\\
&\quad \times \cos\eta\sin\eta\ \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, \frac{1}{2}\right\rangle + \left|-\frac{1}{2}, -\frac{1}{2}\right\rangle\right)\Bigg\}\\
&= \frac{(\Lambda p)^0}{p^0}\{(\cos\bar{\Omega}_{\vec{p}}\sin^2\eta + \cos\Delta\Omega_{\vec{p}}\cos^2\eta)\Psi'_{10} - \sin\Delta\Omega_{\vec{p}}\cos\eta\Psi'_{01}\\
&\quad + i\sin\bar{\Omega}_{\vec{p}}\sin\eta\Psi'_{11} - i(\cos\bar{\Omega}_{\vec{p}} - \cos\Delta\Omega_{\vec{p}})\cos\eta\sin\eta\Psi'_{00}\},
\end{aligned}
\tag{25c}
$$

and

$$
\begin{aligned}
U(\Lambda)\Psi_{11} &= \frac{(\Lambda p)^0}{p^0}\cos\bar{\Omega}_{\vec{p}}\frac{1}{\sqrt{2}}\Bigg\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right) a^+\left(-\vec{p}_\Lambda, -\frac{1}{2}\right)\\
&\quad - a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right) a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\Bigg\}\Psi_0\\
&\quad + \frac{(\Lambda p)^0}{p^0}\sin\bar{\Omega}_{\vec{p}}\cos\eta\frac{1}{\sqrt{2}}\Bigg\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right) a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\\
&\quad + a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right) a^+\left(-\vec{p}_\Lambda, -\frac{1}{2}\right)\Bigg\}\Psi_0\\
&\quad + i\frac{(\Lambda p)^0}{p^0}\sin\bar{\Omega}_{\vec{p}}\sin\eta\frac{1}{\sqrt{2}}\Bigg\{a^+\left(\vec{p}_\Lambda, \frac{1}{2}\right) a^+\left(-\vec{p}_\Lambda, -\frac{1}{2}\right)\\
&\quad + a^+\left(\vec{p}_\Lambda, -\frac{1}{2}\right) a^+\left(-\vec{p}_\Lambda, \frac{1}{2}\right)\Bigg\}\Psi_0\\
&= \frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left\{\cos\bar{\Omega}_{\vec{p}}\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle - \left|-\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right\}\\
&\quad + \frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left\{\sin\bar{\Omega}_{\vec{p}}\cos\eta\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, \frac{1}{2}\right\rangle + \left|-\frac{1}{2}, -\frac{1}{2}\right\rangle\right)\right\}\\
&\quad + i\frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left\{\sin\bar{\Omega}_{\vec{p}}\sin\eta\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \left|-\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right\}\\
&= \frac{(\Lambda p)^0}{p^0}\{\cos\bar{\Omega}_{\vec{p}}\Psi'_{11} + \sin\bar{\Omega}_{\vec{p}}\cos\eta\Psi'_{00} + i\sin\bar{\Omega}_{\vec{p}}\sin\eta\Psi'_{10}\}.
\end{aligned}
\tag{25d}
$$

If we regard Ψ'_{ij} representing Bell states in the moving frame S', then to an observer in S', the effects of the Lorentz transformation on entangled Bell states among themselves should appear as rotations of Bell states in the frame S'.

§3. Bell's inequality

We are now ready to check whether the Lorentz transformed Bell states always violate Bell's inequality in special relativity.

One of the most essential features of quantum mechanics that distinguishes it from classical physics is that the expectation value, or the quantum correlation of the measurement of the observables $\vec{\alpha}_1 \cdot \vec{\sigma}_1$ and $\vec{\alpha}_2 \cdot \vec{\sigma}_2$ for a two-particle system, where $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are the Pauli spin matrices pertaining to the two particles and some unit vectors $\vec{\alpha}_1$ and $\vec{\alpha}_2$ given by[18)]

$$\langle \vec{\alpha}_1 \cdot \vec{\sigma}_1 \vec{\alpha}_2 \cdot \vec{\sigma}_2 \rangle = -\vec{\alpha}_1 \cdot \vec{\alpha}_2 \tag{26}$$

for the singlet state, is always stronger than the classical correlations. The original Bell's inequality was derived for any physical system with dichotomic observables, whose possible values are ± 1. Because any Hermitian operator defines observables, one could extend Bell's inequality to the relativistic regime for any normalized relativistic observables.

It is known[16)] that neither the rest frame spin $\vec{\sigma}$ nor the Dirac spin operator $\vec{\Sigma}$, which is associated with the spin of a moving particle as seen by a stationary observer, can be a relativistic spin operator. Another plausible candidate is the Pauli-Lubanski pseudovector W^μ, which itself is a Casimir operator satisfying $W^\mu W_\mu = m^2 s(s+1)$, where m and s are the mass and spin of the particle, respectively, and $W^\mu = (p^0(\vec{e}\cdot\vec{s})\vec{e} + mc(\vec{s} - (\vec{e}\cdot\vec{s})\vec{e}), p^0\vec{v}\cdot\vec{s}/c^2)$ for the observer in the moving frame with boost velocity $\vec{v}$.[1),17),19)] Here $\vec{s}$ is the spin vector in the rest frame, $\vec{e}$ is the unit vector in the Lorentz boost direction, and $\beta = v/c$ the ratio of the boost speed and the speed of light.

In the non-relativistic case, the measurement of the spin in the direction of the unit vector $\vec{a}$ is represented by the observable $\vec{a}\cdot\vec{s}$, and if we extend this definition of the observable to the relativistic case as $\vec{a}\cdot\vec{s}_\Lambda$, then we have

$$\vec{a}\cdot\vec{s}_\Lambda = [\sqrt{1-\beta^2}(\vec{a} - \vec{e}(\vec{a}\cdot\vec{e})) + \vec{e}(\vec{a}\cdot\vec{e})]\cdot\vec{s}. \tag{27}$$

Here we postulate the relativistic spin to be[1),6)]

$$\vec{s}_\Lambda = \frac{mc}{p^0}\vec{s} + \left(1 - \frac{mc}{p^0}\right)(\vec{e}\cdot\vec{s})\vec{e} = \sqrt{1-\beta^2}(\vec{s} - \vec{e}(\vec{s}\cdot\vec{e})) + \vec{e}(\vec{s}\cdot\vec{e}) = \vec{W}/p^0 \tag{28}$$

and the normalized relativistic spin observable to be given by[1),6)]

$$\hat{a} = \frac{[\sqrt{1-\beta^2}(\vec{a} - \vec{e}(\vec{a}\cdot\vec{e})) + \vec{e}(\vec{a}\cdot\vec{e})]}{\sqrt{1+\beta^2[(\hat{e}\cdot\vec{a})^2 - 1]}}\cdot\vec{\sigma}, \tag{29}$$

where we have normalized the relativisitic spin observable by the absolute value of its eigenvalue. Here $\vec{a}$ and $\vec{s}_\Lambda$ are a unit direction vector and the relativistic spin operator as seen by the moving observer. We can specify a more clear physical meaning of Eqs. (28) and (29) by invoking principle of the special relativity. If we consider $\vec{a}_\Lambda$ to be the Lorentz transformation (now as seen in the rest frame) of the direction vector (of the moving frame), then from Eqs. (28) and (29), we obtain

$$\frac{\vec{a}_\Lambda \cdot \vec{s}}{|\lambda(\vec{a}_\Lambda \cdot \vec{s})|} = \frac{\vec{a} \cdot \vec{s}_\Lambda}{|\lambda(\vec{a} \cdot \vec{s}_\Lambda)|}, \tag{30}$$

which is consistent with the principle of special relativity which asserts the physics does not change across frames. As a result, we can interpret $\hat{a}$ as the correct normalized relativistic observable for the observer in the moving frame. Here $\lambda(\hat{O})$ denotes the eigenvalue of an operator $\hat{O}$.

It is straightforward to calculate the classical correlation $\langle \hat{a}\hat{b} \rangle_{\text{classical}}$ when the moving observer is receding (approaching) from (to) the rest frame with the speed of light. In this case, we have

$$\langle \hat{a}\hat{b} \rangle_{\text{classical}} = \frac{\vec{a} \cdot \vec{e}}{|\vec{a} \cdot \vec{e}|} \cdot \frac{\vec{b} \cdot \vec{e}}{|\vec{b} \cdot \vec{e}|} = \pm 1, \tag{31}$$

and it should be noted that information in the direction perpendicular to the unit boost vector $\vec{e}$ is lost, as both spins are toward the boost axis as a result of the Lorentz transformation.

The normalized relativistic spin observables $\vec{a}$ and $\vec{b}$ are given by[6)]

$$\hat{a} = \frac{(\sqrt{1-\beta^2}\vec{a}_\perp + \vec{a}_\parallel) \cdot \vec{\sigma}}{\sqrt{1+\beta^2[(\hat{e}\cdot\vec{a})^2 - 1]}} \tag{32}$$

and

$$\hat{b} = \frac{(\sqrt{1-\beta^2}\vec{b}_\perp + \vec{b}_\parallel) \cdot \vec{\sigma}}{\sqrt{1+\beta^2[(\hat{e}\cdot\vec{b})^2 - 1]}}, \tag{33}$$

where the subscripts $\perp$ and $\parallel$ denote the components of $\vec{a}$ and $\vec{b}$ that are perpendicular and parallel to the boost direction, respectively. Moreover, we have $|\vec{a}| = |\vec{b}| = 1$.

3.1. *The case that the momentum and boost vectors are in the same plane*

Case I: $\Psi_{00} \to U(\Lambda)\Psi_{00}$

From Eq. (19a), we have

$$\begin{aligned} U(\Lambda)\Psi_{00} = \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \Bigg[\frac{1}{\sqrt{2}} \cos\frac{\Omega_{\vec{p}} + \Omega_{-\vec{p}}}{2} \left(\left|\frac{1}{2}, \frac{1}{2}\right\rangle + \left|-\frac{1}{2}, -\frac{1}{2}\right\rangle \right) \\ - \frac{1}{\sqrt{2}} \sin\frac{\Omega_{\vec{p}} + \Omega_{\vec{p}}}{2} \left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle - \left|-\frac{1}{2}, \frac{1}{2}\right\rangle \right) \Bigg]. \end{aligned} \tag{34}$$

Then, after some mathematical manipulations, we get

$$\hat{a}\otimes\hat{b}\left|\frac{1}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}}\left\{(1-\beta^2)a_z b_z\left|\frac{1}{2},\frac{1}{2}\right\rangle \right.$$
$$+\sqrt{1-\beta^2}a_z(b_x+ib_y\sqrt{1-\beta^2})\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$+\sqrt{1-\beta^2}b_z(a_x+ia_y\sqrt{1-\beta^2})\left|-\frac{1}{2},\frac{1}{2}\right\rangle$$
$$\left. +(a_x+ia_y\sqrt{1-\beta^2})(b_x+ib_y\sqrt{1-\beta^2})\left|-\frac{1}{2},-\frac{1}{2}\right\rangle\right\}, \tag{35a}$$

$$\hat{a}\otimes\hat{b}\left|-\frac{1}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}}$$
$$\times\left\{(a_x-ia_y\sqrt{1-\beta^2})(b_x-ib_y\sqrt{1-\beta^2})\left|\frac{1}{2},\frac{1}{2}\right\rangle\right.$$
$$-\sqrt{1-\beta^2}b_z(a_x-ia_y\sqrt{1-\beta^2})\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$-\sqrt{1-\beta^2}a_z(b_x-ib_y\sqrt{1-\beta^2})\left|-\frac{1}{2},\frac{1}{2}\right\rangle$$
$$\left.+(1-\beta^2)a_z b_z\left|-\frac{1}{2},-\frac{1}{2}\right\rangle\right\}, \tag{35b}$$

$$\hat{a}\otimes\hat{b}\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}}$$
$$\times\left\{\sqrt{1-\beta^2}a_z(b_x-ib_y\sqrt{1-\beta^2})\left|\frac{1}{2},\frac{1}{2}\right\rangle\right.$$
$$-(1-\beta^2)a_z b_z\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$+(a_x+ia_y\sqrt{1-\beta^2})(b_x-ib_y\sqrt{1-\beta^2})\left|-\frac{1}{2},\frac{1}{2}\right\rangle$$
$$\left.-\sqrt{1-\beta^2}b_z(a_x+ia_y\sqrt{1-\beta^2})\left|-\frac{1}{2},-\frac{1}{2}\right\rangle\right\}, \tag{35c}$$

$$\hat{a}\otimes\hat{b}|-\frac{1}{2},\frac{1}{2}\rangle = \frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}}$$
$$\times\left\{\sqrt{1-\beta^2}b_z(a_x-ia_y\sqrt{1-\beta^2})\left|\frac{1}{2},\frac{1}{2}\right\rangle\right.$$
$$+(a_x-ia_y\sqrt{1-\beta^2})(b_x+ib_y\sqrt{1-\beta^2})\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$-(1-\beta^2)a_z b_z\left|-\frac{1}{2},\frac{1}{2}\right\rangle$$
$$\left.-\sqrt{1-\beta^2}a_z(b_x+ib_y\sqrt{1-\beta^2})\left|-\frac{1}{2},-\frac{1}{2}\right\rangle\right\} \tag{35d}$$

for a boost in the x-direction. The calculation of $\langle \hat{a} \otimes \hat{b} \rangle$ is straightforward and it yields,

$$\langle \hat{a} \otimes \hat{b} \rangle = \frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}}\{[a_x b_x + (1-\beta^2)a_z b_z]\cos(\Omega_{\vec{p}} + \Omega_{-\vec{p}}) - (1-\beta^2)a_y b_y - \sqrt{1-\beta^2}(a_z b_x - b_z a_x)\sin(\Omega_{\vec{p}} + \Omega_{-\vec{p}})\}. \tag{36}$$

It is interesting that in the ultra-relativistic limit, $\beta \to 1$, Eq. (36) becomes

$$\langle \hat{a} \otimes \hat{b} \rangle \to \frac{a_x}{|a_x|} \cdot \frac{b_x}{|b_x|} \cos(\Omega_{\vec{p}} + \Omega_{-\vec{p}}), \tag{37}$$

implying that joint measurements are not correlated. As a result, one might suspect that the entangled state satisfies Bell's inequality. We now consider the vectors $\vec{a} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$, $\vec{a}' = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), \vec{b} = (0,1,0), \vec{b}' = (1,0,0)$, which lead to the maximum violation of Bell's inequality in the non-relativistic domain, $\Omega_{\vec{p}} = \Omega_{-\vec{p}} = 0$ and $\beta = 0$. Then the Bell observable for the four relevant joint measurements becomes

$$\langle \hat{a} \otimes \hat{b} \rangle + \langle \hat{a} \otimes \hat{b}' \rangle + \langle \hat{a}' \otimes \hat{b} \rangle - \langle \hat{a}' \otimes \hat{b}' \rangle = \frac{2}{\sqrt{2-\beta^2}}(\sqrt{1-\beta^2} + \cos(\Omega_{\vec{p}} + \Omega_{-\vec{p}})). \tag{38}$$

In the ultra-relativistic limit, where $\beta = 1$, Eq. (38) gives the maximum value of 2 satisfying the Bell's inequality, as expected.

Case II: $\Psi_{01} \to U(\Lambda)\Psi_{01}$

From Eq. (19b), we have

$$U(\Lambda)\Psi_{01} = \frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \left[\frac{1}{\sqrt{2}}\cos\frac{\Omega_{\vec{p}} - \Omega_{-\vec{p}}}{2}\left(\left|\frac{1}{2}, \frac{1}{2}\right\rangle - \left|-\frac{1}{2}, -\frac{1}{2}\right\rangle\right) + \frac{1}{\sqrt{2}}\sin\frac{\Omega_{\vec{p}} - \Omega_{-\vec{p}}}{2}\left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \left|-\frac{1}{2}, \frac{1}{2}\right\rangle\right)\right]. \tag{39}$$

From Eqs. (35a) to (35d), we obtain

$$\langle \hat{a} \otimes \hat{b} \rangle = \frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}}\{[-a_x b_x + (1-\beta^2)a_z b_z]\cos(\Omega_{\vec{p}} - \Omega_{-\vec{p}}) + (1-\beta^2)a_y b_y + \sqrt{1-\beta^2}(a_z b + b_z a_x)\sin(\Omega_{\vec{p}} - \Omega_{-\vec{p}})\}. \tag{40}$$

Then, in the ultra-relativistic limit, $\beta \to 1$, we have

$$\langle \hat{a} \otimes \hat{b} \rangle \to -\frac{a_x}{|a_x|} \cdot \frac{b_x}{|b_x|} \cos(\Omega_{\vec{p}} - \Omega_{-\vec{p}}), \tag{41}$$

again, indicating that the joint measurements become uncorrelated in this limit. We consider the vectors , $\vec{a} = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $\vec{a}' = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $\vec{b} = (0,1,0)$, $\vec{b}' = (1,0,0)$ which lead to the maximum violation of Bell's inequality in the non-relativistic regime. Then the Bell observable for the four relevant joint measurements becomes

$$\begin{aligned} &\langle \hat{a} \otimes \hat{b} \rangle + \langle \hat{a} \otimes \hat{b}' \rangle + \langle \hat{a}' \otimes \hat{b} \rangle - \langle \hat{a}' \otimes \hat{b}' \rangle \\ &\quad = \frac{2}{\sqrt{2-\beta^2}} (\sqrt{1-\beta^2} + \cos(\Omega_{\vec{p}} - \Omega_{-\vec{p}})), \end{aligned} \tag{42}$$

thus giving the same maximum value as in Case I. It can also be shown that one can obtain the same value for the Bell observables given by Eq. (42) for $U(\Lambda)\Psi_{10}$ and $U(\Lambda)\Psi_{11}$. This implies that Eq. (42) is a universal result.

3.2. *The case that momentum and boost vectors are not in the same plane*

Case I: $\Psi_{00} \to U(\Lambda)\Psi_{00}$

From Eq. (25a), we have

$$\begin{aligned} U(\Lambda)\Psi_{00} = &\frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \{(\cos \bar{\Omega}_{\vec{p}} \cos^2\eta + \cos \Delta\Omega_{\vec{p}} \sin^2\eta) \\ &\times \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}, \frac{1}{2} \right\rangle + \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle \right) \\ &- \sin \bar{\Omega}_{\vec{p}} \cos\eta \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}, -\frac{1}{2} \right\rangle - \left| -\frac{1}{2}, \frac{1}{2} \right\rangle \right) \\ &+ i \sin \Delta\Omega_{\vec{p}} \sin\eta \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}, \frac{1}{2} \right\rangle - \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle \right) \\ &- i(-\cos \bar{\Omega}_{\vec{p}} + \cos \Delta\Omega_{\vec{p}}) \sin\eta \cos\eta \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}, -\frac{1}{2} \right\rangle + \left| -\frac{1}{2}, \frac{1}{2} \right\rangle \right) \Big\}. \end{aligned} \tag{43}$$

Then, from Eqs. (35a) to (35d), we obtain

$$\begin{aligned} \langle \hat{a} \otimes \hat{b} \rangle = &\frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}} \left\{ \frac{A_+ + A_-}{2} X^2 + \frac{C_+ + C_-}{2} Y^2 \right. \\ &+ \frac{E_+ + E_-}{2} Z^2 + \frac{G_+ + G_-}{2} W^2 - 2 \left[\frac{B_+ + B_-}{2} XY + \frac{A_+ - A_-}{2i} XZ \right. \\ &\left.\left. + \frac{B_+ - B_-}{2i} XW - \frac{D_+ - D_-}{2i} YZ + \frac{C_+ - C_-}{2i} YW + \frac{F_+ + F_-}{2} ZW \right] \right\}, \end{aligned} \tag{44}$$

where

$$\begin{aligned} X &= \cos \bar{\Omega}_{\vec{p}} \cos^2\eta + \cos \Delta\Omega_{\vec{p}} \sin^2\eta, \\ Y &= \sin \bar{\Omega}_{\vec{p}} \cos\eta, \\ Z &= \sin \Delta\Omega_{\vec{p}} \sin\eta, \\ W &= -\cos \bar{\Omega}_{\vec{p}} \cos\eta \sin\eta + \cos \Delta\Omega_{\vec{p}} \sin\eta \cos\eta, \end{aligned}$$

$$\begin{aligned}
\frac{A_+ + A_-}{2} &= a_x b_x - (1-\beta^2) a_y b_y + (1-\beta^2) a_z b_z, \\
\frac{C_+ + C_-}{2} &= -a_x b_x - (1-\beta^2) a_y b_y - (1-\beta^2) a_z b_z, \\
\frac{E_+ + E_-}{2} &= -a_x b_x + (1-\beta^2) a_y b_y + (1-\beta^2) a_z b_z, \\
\frac{G_+ + G_-}{2} &= a_x b_x + (1-\beta^2) a_y b_y - (1-\beta^2) a_z b_z, \\
\frac{B_+ + B_-}{2} &= \frac{D_+ + D_-}{2} = \sqrt{1-\beta^2}(a_z b_x - b_z a_x), \\
\frac{F_+ + F_-}{2} &= \frac{H_+ + H_-}{2} = \sqrt{1-\beta^2}(a_z b_x + b_z a_x), \\
\frac{A_+ - A_-}{2i} &= \frac{E_+ - E_-}{2i} = \sqrt{1-\beta^2}(a_x b_y + b_x a_y), \\
\frac{C_+ - C_-}{2i} &= \frac{G_+ - G_-}{2i} = \sqrt{1-\beta^2}(a_x b_y - b_x a_y), \\
\frac{B_+ - B_-}{2i} &= \frac{H_+ - H_-}{2i} = (1-\beta^2)(a_z b_y + b_z a_y), \\
\frac{D_+ - D_-}{2i} &= \frac{F_+ - F_-}{2i} = (1-\beta^2)(a_z b_y - b_z a_y).
\end{aligned} \tag{45}$$

It is interesting that in the ultra-relativistic limit, $\beta \to 1$, Eq. (44) becomes

$$\langle \hat{a} \otimes \hat{b} \rangle \to \frac{a_x}{|a_x|} \cdot \frac{b_x}{|b_x|}(X^2 - Y^2 - Z^2 + W^2), \tag{46}$$

implying that joint measurements are not correlated. As a result, one might suspect that the entangled state satisfies Bell's inequality. We now consider the vectors $\vec{a} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$, $\vec{a}' = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$, $\vec{b} = (0,1,0)$, $\vec{b}' = (1,0,0)$, which lead to the maximum violation of Bell's inequality in the non-relativistic domain, $\Omega_{\vec{p}} = \Omega_{-\vec{p}} = 0$ and $\beta = 0$. Then, the Bell observable for the four relevant joint measurements becomes

$$\begin{aligned}
&\langle \hat{a} \otimes \hat{b} \rangle + \langle \hat{a} \otimes \hat{b}' \rangle + \langle \hat{a}' \otimes \hat{b} \rangle - \langle \hat{a}' \otimes \hat{b}' \rangle \\
&\quad = \frac{2}{\sqrt{2-\beta^2}}\{(X^2 - Y^2 - Z^2 + W^2) + (X^2 + Y^2 - Z^2 - W^2)\sqrt{1-\beta^2}\}.
\end{aligned} \tag{47}$$

In the ultra-relativistic limit where $\beta = 1$, Eq. (47) gives the maximum value of 2 satisfying Bell's inequality as expected (see the appendix).

Case II: $\Psi_{01} \to U(\Lambda)\Psi_{01}$

From Eq. (25b), we have

$$\begin{aligned}
U(\Lambda)\Psi_{01} = \frac{(\Lambda p)^0}{p^0} |\vec{p}_\Lambda, -\vec{p}_\Lambda\rangle \otimes \Bigg\{ &\cos \Delta\Omega_{\vec{p}} \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}, \frac{1}{2} \right\rangle - \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle \right) \\
&- \sin \Delta\Omega_{\vec{p}} \cos \eta \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}, -\frac{1}{2} \right\rangle + \left| -\frac{1}{2}, \frac{1}{2} \right\rangle \right) \\
&+ i \sin \Delta\Omega_{\vec{p}} \sin \eta \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}, \frac{1}{2} \right\rangle + \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle \right) \Bigg\}.
\end{aligned} \tag{48}$$

Also from Eqs. (35a) to (35d), we obtain

$$\begin{aligned}\langle \hat{a}\otimes\hat{b}\rangle =& \frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}}\left\{\frac{G_+ + G_-}{2}X'^2 + \frac{A_+ + A_-}{2}Y'^2\right.\\ &\left.+\frac{E_+ + E_-}{2}Z'^2 + 2\left[\frac{F_+ + F_-}{2}X'Z' + \frac{A_+ - A_-}{2i}Y'Z' - \frac{B_+ - B_-}{2i}X'Y'\right]\right\},\end{aligned} \tag{49}$$

where

$$\begin{aligned}X' &= \sin\Delta\Omega_{\vec{p}}\cos\eta,\\ Y' &= \sin\Delta\Omega_{\vec{p}}\sin\eta,\\ Z' &= \cos\Delta\Omega_{\vec{p}}.\end{aligned} \tag{50}$$

Then, in the ultra-relativistic limit, $\beta\to 1$, we have

$$\langle \hat{a}\otimes\hat{b}\rangle \to -\frac{a_x}{|a_x|}\cdot\frac{b_x}{|b_x|}(-X'^2 - Y'^2 + Z'^2) = -\frac{a_x}{|a_x|}\cdot\frac{b_x}{|b_x|}\cos 2\Delta\Omega_{\vec{p}}, \tag{51}$$

again, indicating that joint measurements become uncorrelated in this limit. We consider the vectors $\vec{a} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\vec{a}' = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\vec{b} = (0,1,0)$, $\vec{b}' = (1,0,0)$, which lead to the maximum violation of the Bell's inequality in the non-relativistic regime. Then the Bell observable for the four relevant joint measurements becomes

$$\begin{aligned}&\langle \hat{a}\otimes\hat{b}\rangle + \langle \hat{a}\otimes\hat{b}'\rangle + \langle \hat{a}'\otimes\hat{b}\rangle - \langle \hat{a}'\otimes\hat{b}'\rangle\\ &\quad= \frac{2}{\sqrt{2-\beta^2}}\{(-X'^2 - Y'^2 + Z'^2) + (X'^2 - Y'^2 + Z'^2)\sqrt{1-\beta^2}\}\\ &\quad= \frac{2}{\sqrt{2-\beta^2}}\{\cos 2\Delta\Omega_{\vec{p}} + (\cos^2\eta + \sin^2\eta\cos 2\Delta\Omega_{\vec{p}})\sqrt{1-\beta^2}\},\end{aligned} \tag{52}$$

thus giving same maximum value as in case I. It can also be shown that one can obtain the same value for the Bell observables given by Eq. (52) for $U(\Lambda)\Psi_{10}$ and $U(\Lambda)\Psi_{11}$. This implies that Eq. (52) is a universal result.

The results above agree with our previous results,[6)] which did not take into account the general momentum. It can also be shown that similar results would be obtained for the case[20)] in which one observer is in the rest frame and the other observer is in the moving frame and do joint measurements of spins. Now, one can see that the quantum correlation approaches the classical correlation when the speed of the moving observer reaches the speed of light, and in both cases, the information in the direction perpendicular to the boost axis is lost. This is somewhat analogous to the cases of β-decay of nuclei and high energy electrons and positrons emitted in the decay of muons, for which emitted electrons and positrons are polarized such that their spins tend to lie in the same direction as the motion and the projections of their spins in the direction of the motion become ± 1 for relativistic particles.[21)]

It should be noted that if one simply rotates the spin directions instead of using relativistic spin observables, then the entanglement between the spins of the Bell states is not changed, and the results of the spin measurements would be exactly same as if they were done in the rest frame. Thus, they give the maximum violation of Bell inequality. It is interesting that the entanglement remains, though it is degraded, when Bell's inequality is satisfied. The most plausible reason for this is that the quantum correlations in the direction perpendicular to the boost are lost and become classical. Therefore, we can conclude that Bell's inequality is not always violated for entangled state in special relativity.

§4. Summary

In this work, we studied the Lorentz transformed entangled Bell states and the Bell observables in the case of general momentum to investigate whether Bell's inequality is always violated in special relativity. We calculated the Bell observable for joint four measurements and found that the results are universal for all entangled states:

$$\begin{aligned} c(\vec{a},\vec{a}',\vec{b},\vec{b}') &= \langle \hat{a}\otimes\hat{b}\rangle + \langle \hat{a}\otimes\hat{b}'\rangle + \langle \hat{a}'\otimes\hat{b}\rangle - \langle \hat{a}'\otimes\hat{b}'\rangle \\ &= \frac{2}{\sqrt{2-\beta^2}}(1+\sqrt{1-\beta^2}). \end{aligned}$$

Here $\hat{a}$ and $\hat{b}$ are the relativistic spin observables derived from the Pauli-Lubanski pseudo vector. It turns out that the Bell observable is a monotonically decreasing function of β and approaches the limiting value of 2 as $\beta \to 1$. This indicates that Bell's inequality is not always violated in the ultra-relativistic limit. We also showed that quantum information, along the direction perpendicular to the boost is eventually lost, and Bell's inequality is not always violated for entangled states in special relativity. This could impose restrictions on certain quantum information processing, such as quantum cryptography using massive types of particle. In particular, unless both the sender and receiver measure along the boost direction, there will be information loss.

Acknowledgements

This work was supported by the Korean Science and Engineering Foundation and the Korean Ministry of Science and Technology through the Creative Research Initiatives Program under Contact No. M1-0116-00-0008. We are indebted to M. Czachor for valuable discussions.

Appendix

We have, $\cosh\alpha = \frac{1}{\sqrt{1-\beta^2}}$ and $\cosh\delta = \frac{p^0}{m}$ from Eqs. (10) and (11), and we have defined, $\cos\eta = \frac{\cos\theta}{r}$, $\sin\eta = \frac{\sin\theta\sin\phi}{r}$, $r = \sqrt{\sin^2\theta\sin^2\phi + \cos^2\theta}$, and $\bar{\Omega}_{\vec{p}} = \frac{\Omega_{\vec{p}}+\Omega_{-\vec{p}}}{2}$, $\Delta\Omega_{\vec{p}} = \frac{\Omega_{\vec{p}}-\Omega_{-\vec{p}}}{2}$. From Eqs. (21) and (22), we obtain

$$\cos\bar{\Omega}_{\vec{p}} = \cos\frac{\Omega_{\vec{p}}}{2}\cos\frac{\Omega_{-\vec{p}}}{2} - \sin\frac{\Omega_{\vec{p}}}{2}\sin\frac{\Omega_{-\vec{p}}}{2}$$

$$
\begin{aligned}
&= \frac{\cosh^2\dfrac{\alpha}{2}\cosh^2\dfrac{\delta}{2} - \sinh^2\dfrac{\alpha}{2}\sinh^2\dfrac{\delta}{2}}{\left[\left(\dfrac{1}{2}+\dfrac{1}{2}\cosh\alpha\cosh\delta\right)^2 - \left(\dfrac{1}{2}\sinh\alpha\sinh\delta\sin\theta\cos\phi\right)^2\right]^{\frac{1}{2}}} \\
&= \frac{\dfrac{\cosh^2\dfrac{\alpha}{2}\cosh^2\dfrac{\delta}{2}}{\sinh^2\dfrac{\alpha}{2}\sinh^2\dfrac{\delta}{2}} - 1}{\left[\left(\dfrac{1+\cosh\alpha\cosh\delta}{2\sinh^2\dfrac{\alpha}{2}\sinh^2\dfrac{\delta}{2}}\right)^2 - \left(\dfrac{\sinh\alpha\sinh\delta}{2\sinh^2\dfrac{\alpha}{2}\sinh^2\dfrac{\delta}{2}}\right)^2(1-r^2)\right]^{\frac{1}{2}}} \\
&= \frac{\coth^2\dfrac{\alpha}{2}\coth^2\dfrac{\delta}{2} - 1}{\left[\left(\coth^2\dfrac{\alpha}{2}\coth^2\dfrac{\delta}{2}+1\right)^2 - 4\coth^2\dfrac{\alpha}{2}\coth^2\dfrac{\delta}{2}(1-r^2)\right]^{\frac{1}{2}}} \\
&= \frac{\coth^2\dfrac{\alpha}{2}\coth^2\dfrac{\delta}{2} - 1}{\left[\left(\coth^2\dfrac{\alpha}{2}\coth^2\dfrac{\delta}{2}-1\right)^2 + 4\coth^2\dfrac{\alpha}{2}\coth^2\dfrac{\delta}{2}r^2\right]^{\frac{1}{2}}} \\
&= \frac{t-1}{[(t-1)^2+4\mathrm{tr}^2]^{\frac{1}{2}}},
\end{aligned}
\tag{A·1}
$$

where $t = \coth^2\frac{\alpha}{2}\coth^2\frac{\delta}{2}, 1 \le t$, and

$$
\begin{aligned}
\cos\mathit{\Delta}\mathit{\Omega}_{\vec{p}} &= \cos\frac{\mathit{\Omega}_{\vec{p}}}{2}\cos\frac{\mathit{\Omega}_{-\vec{p}}}{2} + \sin\frac{\mathit{\Omega}_{\vec{p}}}{2}\sin\frac{\mathit{\Omega}_{-\vec{p}}}{2} \\
&= \frac{\cosh^2\dfrac{\alpha}{2}\cosh^2\dfrac{\delta}{2} - \sinh^2\dfrac{\alpha}{2}\sinh^2\dfrac{\delta}{2}(\sin^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi - \cos^2\theta)}{\left[\left(\dfrac{1}{2}+\dfrac{1}{2}\cosh\alpha\cosh\delta\right)^2 - \left(\dfrac{1}{2}\sinh\alpha\sinh\delta\sin\theta\cos\phi\right)^2\right]^{\frac{1}{2}}} \\
&= \frac{\dfrac{\cosh^2\dfrac{\alpha}{2}\cosh^2\dfrac{\delta}{2}}{\sinh^2\dfrac{\alpha}{2}\sinh^2\dfrac{\delta}{2}} - (1-2r^2)}{\left[\left(\dfrac{1+\cosh\alpha\cosh\delta}{2\sinh^2\dfrac{\alpha}{2}\sinh^2\dfrac{\delta}{2}}\right)^2 - \left(\dfrac{\sinh\alpha\sinh\delta}{2\sinh^2\dfrac{\alpha}{2}\sinh^2\dfrac{\delta}{2}}\right)^2(1-r^2)\right]^{\frac{1}{2}}}
\end{aligned}
$$

$$= \frac{\coth^2\frac{\alpha}{2}\coth^2\frac{\delta}{2} - (1-2r^2)}{\left[\left(\coth^2\frac{\alpha}{2}\coth^2\frac{\delta}{2}+1\right)^2 - 4\coth^2\frac{\alpha}{2}\coth^2\frac{\delta}{2}(1-r^2)\right]^{\frac{1}{2}}}$$

$$= \frac{\left(\coth^2\frac{\alpha}{2}\coth^2\frac{\delta}{2}-1\right)+2r^2}{\left[\left(\coth^2\frac{\alpha}{2}\coth^2\frac{\delta}{2}-1\right)^2 + 4\coth^2\frac{\alpha}{2}\coth^2\frac{\delta}{2}r^2\right]^{\frac{1}{2}}}$$

$$= \frac{(t-1)+2r^2}{[(t-1)^2+4\mathrm{tr}^2]^{\frac{1}{2}}}. \tag{A·2}$$

From Eqs. (45), (A·1) and (A·2), we get

$$\begin{aligned}
X^2 - Y^2 - Z^2 + W^2 &= \cos^2\bar{\mathit{\Omega}}_{\vec{p}}\cos^2\eta + \cos^2\mathit{\Delta\Omega}_{\vec{p}}\sin^2\eta \\
&\quad - \sin^2\bar{\mathit{\Omega}}_{\vec{p}}\cos^2\eta - \sin^2\mathit{\Delta\Omega}_{\vec{p}}\sin^2\eta \\
&= 2(\cos^2\bar{\mathit{\Omega}}_{\vec{p}}\cos^2\eta + \cos^2\mathit{\Delta\Omega}_{\vec{p}}\sin^2\eta) - 1 \\
&= 2\frac{(t-1)^2\cos^2\eta + (t-1+2r^2)^2\sin^2\eta}{(t-1)^2+4\mathrm{tr}^2} - 1 \\
&= 1 - 8r^2\cos^2\eta\frac{t+(1-r^2)\tan^2\eta}{(t-1)^2+4\mathrm{tr}^2}
\end{aligned} \tag{A·3}$$

and

$$\begin{aligned}
X^2 + Y^2 - Z^2 - W^2 &= \cos^2\bar{\mathit{\Omega}}_{\vec{p}}\cos^2\eta\cos 2\eta - \cos^2\mathit{\Delta\Omega}_{\vec{p}}\sin^2\eta\cos 2\eta + \sin^2\bar{\mathit{\Omega}}_{\vec{p}}\cos^2\eta \\
&\quad - \sin^2\mathit{\Delta\Omega}_{\vec{p}}\sin^2\eta + 4\cos\bar{\mathit{\Omega}}_{\vec{p}}\cos\mathit{\Delta\Omega}_{\vec{p}}\sin^2\eta\cos^2\eta \\
&= \cos 2\eta + (1-\cos 2\eta)\{\cos^2\mathit{\Delta\Omega}_{\vec{p}} - \cos^2\eta(\cos\bar{\mathit{\Omega}}_{\vec{p}} - \cos\mathit{\Delta\Omega}_{\vec{p}})^2\} \\
&= \cos 2\eta + (1-\cos 2\eta)\left\{\frac{(t-1+2r^2)^2 - \cos^2\eta(-2r^2)^2}{(t-1)^2+4\mathrm{tr}^2}\right\} \\
&= 1 - 8r^2\sin^2\eta\frac{1-r^2\sin^2\eta}{(t-1)^2+4\mathrm{tr}^2}.
\end{aligned} \tag{A·4}$$

From (A·3), we define

$$f(t) = \frac{t+a}{(t-1)^2+4r^2t}. \tag{A·5}$$

Then, we have

$$\frac{dr(t)}{dt} = -\frac{(t-1)(t+2a+1)+4r^2a}{\{(t-1)^2+4r^2t\}^2} < 0, \text{ for } t \geq 1, \forall\theta \text{ and } \forall\phi, \tag{A·6}$$

where $a = (1-r^2)\tan^2\eta \geq 0$. From Eqs. (A·5) and (A·6), we obtain

$$0 = f(\infty) \leq f(t) \leq f(1) = \frac{1+a}{4r^2} \tag{A·7}$$

and

$$
\begin{aligned}
1 - 8r^2\cos^2\eta f(1) &= 1 - 8r^2\cos^2\eta\frac{1 + (1 - r^2)\tan^2\eta}{4r^2} \\
&= 2\sin^2\theta\sin^2\phi - 1, \qquad \text{(A·8)}
\end{aligned}
$$

and therefore

$$2\sin^2\theta\sin^2\phi - 1 \le X^2 - Y^2 - Z^2 + W^2 \le 1. \qquad \text{(A·9)}$$

From (A·4), we define

$$g(t) = \frac{b}{(t-1)^2 + 4r^2 t}. \qquad \text{(A·10)}$$

Then we have

$$\frac{dg(t)}{dt} = -2b\frac{(t-1) + 2r^2}{\{(t-1)^2 + 4r^2 t\}^2} \le 0, \text{ for } t \ge 1, \forall\theta \text{ and } \forall\phi, \qquad \text{(A·11)}$$

where $b = 1 - r^2\sin^2\eta \ge 0$. From Eqs. (A10) and (A11), we have

$$0 = g(\infty) \le g(t) \le g(1) = \frac{b}{4r^2} \qquad \text{(A·12)}$$

and

$$
\begin{aligned}
1 - 8r^2\sin^2\eta g(1) &= 1 - 8r^2\sin^2\eta\frac{1 - r^2\sin^2\eta}{4r^2} \\
&= 1 - 2\sin^2\eta(1 - r^2\sin^2\eta) \\
&= \cos 2\eta + \frac{r^2}{2}(1 - \cos 2\eta)^2 \\
&\ge \cos 2\eta. \qquad \text{(A·13)}
\end{aligned}
$$

Therefore, we obtain

$$\cos 2\eta \le X^2 + Y^2 - Z^2 - W^2 \le 1. \qquad \text{(A·14)}$$

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