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Relativistic Lamé functions: the special case $g = 2$

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Abstract. We study a class of eigenfunctions of an analytic difference operator generalizing the special Lamé operator $-\mathrm{d}^2/\mathrm{d}x^2 + 2\wp(x)$, paying particular attention to quantum-mechanical aspects. We show that in a suitable scaling limit the pertinent eigenfunctions lead to the eigenfunctions of the operator $-\mathrm{d}^2/\mathrm{d}x^2 + 2c\delta(x)$ in a finite volume. We establish various orthogonality and non-orthogonality results by direct calculations, generalize the 'one-gap picture' associated with the above Lamé operator, and obtain duality properties for the hyperbolic, trigonometric and rational specializations.

Contents

1. Introduction

In two recent papers [1, 2] we introduced and studied eigenfunctions of an analytic difference operator that generalizes the Lamé operator

$$H_{\mathrm{nr}}(g) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + g(g-1)\wp(x) \tag{1.1}$$

where $\wp$ is the Weierstrass $\wp$ function. This analytic difference operator (hereafter abbreviated to AΔO) reads

$$H_{\mathrm{rel}}(g) = \left(\frac{\sigma(x - \mathrm{i}\beta g)}{\sigma(x)}\right)^{1/2} T_{\mathrm{i}\beta} \left(\frac{\sigma(x + \mathrm{i}\beta g)}{\sigma(x)}\right)^{1/2} + (\mathrm{i} \to -\mathrm{i}) \tag{1.2}$$

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where σ is the Weierstrass σ function, and where T_α denotes translation over α:

$$(T_\alpha f)(x) = f(x - \alpha) \qquad \alpha \in \mathbb{C}. \tag{1.3}$$

The subscripts 'nr' and 'rel' in these formulae stand for 'non-relativistic' and 'relativistic'. Indeed, one clearly has the limiting relation

$$H_{\text{rel}}(g) = 2 + \beta^2 H_{\text{nr}}(g) + \text{O}(\beta^4) \qquad \beta \to 0. \tag{1.4}$$

Accordingly, the parameter β may be viewed as $1/c$, with c the speed of light. Admittedly, this interpretation may seem unconvincing without further explanation, but it is beyond the scope of this paper to supply the necessary background. Instead, we refer the reader to our survey [3] and lecture notes [4] concerning non-relativistic N-particle Calogero–Moser systems and their relativistic generalizations. (The operators (1.1) and (1.2) are the (reduced) $N = 2$ versions of the quantum dynamics defining these integrable systems.)

The present paper is concerned with the special choice $g = 2$ in the above operators and their hyperbolic, trigonometric, and rational specializations. Though the $g = 2$ case is covered by previous papers, both in the elliptic regime [1] and in the hyperbolic and trigonometric regimes [2], it has special features allowing a simpler and more explicit treatment. Indeed, this paper is largely self-contained.

On the other hand, it is illuminating to compare a number of formulae and results with their general counterparts in [1] and [2]. (We refer to equations in the latter papers through prefixes I and II, respectively.) In particular, our focusing on functions that are not only eigenfunctions of $H_{\text{rel}}(2)$ (1.2), but also of a second independent AΔO commuting with $H_{\text{rel}}(2)$, might appear unmotivated without some acquaintance with the symmetries exhibited by this AΔO pair in the general case (cf equation I(1.12)).

Apart from its transparency and accessibility, an important reason for a separate study of the $g = 2$ case is its remarkable connection to the (reduced) two-particle sector of the quantized nonlinear Schrödinger model, also known as the delta-function gas. In [3] we already mentioned that the $g = 2$ hyperbolic relativistic eigenfunction transforms lead to the eigenfunction transform of the delta-function boson gas on the line in a certain scaling limit. Here, we not only supply the details of this assertion, but also prove that the relation persists at the elliptic level. Specifically, the *finite* elliptic real period corresponds to the *finite-volume* Lieb–Liniger eigenfunctions [5].

A second reason for zooming in on the $g = 2$ case is that the 'one-gap picture' associated with the differential operator $H_{\text{nr}}(2)$ (1.1) admits a rather complete generalization to our analytic difference operator $H_{\text{rel}}(2)$ (1.2). (The 'band problem' was not addressed in I, since the constraint system for the general case only yields explicit information concerning eigenvalues and eigenfunctions in the unbounded spectral interval.)

Last but not least, the surprising *duality* properties of the eigenfunctions can be more readily understood for $g = 2$. More precisely, these properties emerge in the hyperbolic, trigonometric and rational regimes. Thus far, no useful elliptic generalization of these symmetries has been found. The $N = 2, g = 2$ setting studied here might provide the simplest starting point for a search.

Before sketching the organization of this paper, we summarize some notation, conventions and operator pairs that play a crucial role below. First of all, we do not work with the Weierstrass σ function occurring in (1.2), but rather with its close relative

$$s(r, a; z) \equiv \sigma\left(z; \frac{\pi}{2r}, \frac{\text{i}a}{2}\right) \exp(-\eta z^2 r/\pi). \tag{1.5}$$

(Compare with Whittaker and Watson [6] for the elliptic notation and results used here and below.) Various salient features of $s(z)$ can be read off from the two product representations:

$$s(r,a;z) = \exp(-rz^2/a)\frac{\mathrm{sh}(\pi z/a)}{\pi/a}\prod_{k=1}^{\infty}\frac{(1-\exp[-2\pi^2k/ar+2\pi z/a])(z\to -z)}{(1-\exp[-2\pi^2k/ar])^2} \tag{1.6}$$

$$s(r,a;z) = \frac{\sin(rz)}{r}\prod_{k=1}^{\infty}\frac{(1-\exp[-2kar+2\mathrm{i}rz])(z\to -z)}{(1-\exp[-2kar])^2}. \tag{1.7}$$

In particular, one reads off that $s(z)$ is an entire odd function with simple zeros in the elliptic lattice points $\mathbb{Z}\pi/r + \mathrm{i}\mathbb{Z}a$. Moreover, it is clear from these formulae that one has the limiting relations

$$\lim_{r\to 0} s(r,a;z) = \frac{\mathrm{sh}\,\pi z/a}{\pi/a} \qquad \text{(uniformly on compacts)} \tag{1.8}$$

$$\lim_{a\to\infty} s(r,a;z) = \frac{\sin rz}{r} \qquad \text{(uniformly on compacts)}. \tag{1.9}$$

(These limits enable us to pass from the elliptic to the hyperbolic and trigonometric levels without the need for renormalizations.) Finally, from (1.7) one sees that $s(z)$ is π/r-antiperiodic, and from (1.6) one infers that $s(z)$ obeys the analytic difference equation (hereafter abbreviated to AΔE)

$$\frac{s(z+\mathrm{i}a/2)}{s(z-\mathrm{i}a/2)} = -\exp(-2\mathrm{i}rz). \tag{1.10}$$

The iterated version of this AΔE, viz.,

$$\frac{s(r,a;z+\mathrm{i}La)}{s(r,a;z)} = (-)^L\exp(arL^2 - 2\mathrm{i}rLz) \qquad L\in\mathbb{Z} \tag{1.11}$$

will frequently be used below.

Though this is not necessary for some of our results, we assume from now on that the numbers r and a satisfy $r \in [0,\infty)$, $a \in (0,\infty]$. Similarly, we take $\beta \in (0,\infty)$. With these conventions in force, our starting-point elliptic Hamiltonian

$$H_- \equiv \exp(-2\beta r)\left(\frac{s(x-2\mathrm{i}\beta)}{s(x)}\right)^{1/2} T_{\mathrm{i}\beta}\left(\frac{s(x+2\mathrm{i}\beta)}{s(x)}\right)^{1/2} + (\mathrm{i}\to -\mathrm{i}) \tag{1.12}$$

and its various specializations are formally self-adjoint.

In view of (1.5), the Hamiltonian H_- is a positive multiple of $H_{\mathrm{rel}}(2)$ (1.2). The prefactor chosen in H_- and in the second Hamiltonian

$$H_+ \equiv \exp(2\beta r - 2ar)(T_{\mathrm{i}a} + T_{-\mathrm{i}a}) \tag{1.13}$$

guarantees certain invariance properties that will emerge below. These AΔOs are the $g = 2$ (more precisely, $b = 2a_+$) specializations of the commuting AΔO pair H_δ I(1.12). (Recall that we prefix equations from our previous papers [1] and [2] by I and II, respectively; the parameters a_-, a_+ employed there equal the parameters a, β of the present paper.) The H_--eigenfunctions studied below are in fact H_+-eigenfunctions as well, and it is this extra property that singles them out from the infinite number of linearly independent H_--eigenfunctions. (We return to this crucial uniqueness property in the main text, see the end of subsection 2.1.)

We proceed by introducing the auxiliary weight function

$$\hat{w}(x) \equiv 1/s(x+\mathrm{i}\beta)s(x-\mathrm{i}\beta) \tag{1.14}$$

and the auxiliary AΔOs

$$B_\delta \equiv \hat{w}(x)^{-1/2}\,H_\delta\,\hat{w}(x)^{1/2} \qquad \delta = +,-. \tag{1.15}$$

Thus we obtain (using (1.11) with $L = 1$)

$$B_- = \mathrm{e}^{-2\beta r}\frac{s(x+\mathrm{i}\beta)}{s(x)}T_{\mathrm{i}\beta} + (\mathrm{i} \to -\mathrm{i}) \tag{1.16}$$

$$B_+ = -\mathrm{e}^{2\beta r - 3ar}(\mathrm{e}^{-2\mathrm{i}rx}T_{\mathrm{i}a} + (\mathrm{i} \to -\mathrm{i})). \tag{1.17}$$

Note that B_- and B_+ may be viewed as commuting operators on the space of meromorphic functions. Below, we exhibit meromorphic (in fact, entire) joint B_δ-eigenfunctions $\mathcal{H}(\pm x)$, giving rise to joint H_δ-eigenfunctions $\hat{w}(x)^{1/2}\mathcal{H}(\pm x)$.

We mention at the outset that there exists one representation of the eigenfunctions and their specializations that is common to all cases considered: we always have

$$\mathcal{H}(x) = s(x+z)\mathrm{e}^{x\Sigma} \tag{1.18}$$

where z and Σ are complex numbers, related in general via a transcendental constraint. As will become clear, this structure is deceptively simple, inasmuch as in several instances a considerable effort appears inevitable in arriving at the desired results. In particular, the duality features to be uncovered in the hyperbolic, trigonometric and rational cases are very far from obvious when the representation (1.18) is employed.

We continue by sketching the plan of the paper and some of its results. Section 2 is concerned with the elliptic case $r \in (0, \infty), a \in (0, \infty)$, section 3 with the hyperbolic case $r = 0, a \in (0, \infty)$, section 4 with the trigonometric case $r \in (0, \infty), a = \infty$, and section 5 with the rational case $r = 0, a = \infty$. In section 6 we study the non-relativistic limit $\beta \downarrow 0$. We have isolated various distinct features of the elliptic eigenfunctions in several subsections.

Subsection 2.1 deals with algebraic (as opposed to functional-analytic/quantum-mechanical) aspects of the pertinent joint eigenfunctions. The choices

$$2\beta \in a\mathbb{N}^* \tag{1.19}$$

give rise to an AΔO H_- (1.12) with x-independent coefficients (just as H_+), so they can be quite easily handled. For the β-intervals

$$2\beta \in a(k, k+1) \qquad k \in \mathbb{N} \tag{1.20}$$

we view (1.18) as an ansatz for a B_--eigenfunction, which yields the constraint

$$\frac{s(z-\mathrm{i}\beta)}{s(z+\mathrm{i}\beta)} = \mathrm{e}^{2\mathrm{i}\beta\Sigma}. \tag{1.21}$$

We study this constraint in considerable detail, establishing in particular that some properties of the eigenfunctions and associated eigenvalues depend on the choice of interval (1.20). We also analyse the limits as β approaches the upper and lower boundary points. As it turns out, the limits $\beta \uparrow Ma$ and $\beta \downarrow Ma$, $M \in \mathbb{N}^*$, do not coincide, which reveals that a continuous interpolation to arbitrary $\beta \in (0, \infty)$ does not exist without further restrictions. (To understand why such interpolation ambiguities may occur *a priori*, it is crucial to be aware of the occurrence of infinite-dimensional joint eigenspaces whenever β/a is a rational number.)

Subsection 2.2 is devoted to orthogonality properties of the odd linear combination $\mathcal{H}(x) - \mathcal{H}(-x)$ for suitably discretized $\Sigma, z \in \mathrm{i}(0, \infty)$. Here, orthogonality refers to the Hilbert space

$$\mathcal{H}_{\hat{w}} \equiv L^2((0, \pi/r), \hat{w}(x)\,\mathrm{d}x). \tag{1.22}$$

Not surprisingly, the 'free' cases (1.19) are easily seen to give rise to orthogonal bases for $\mathcal{H}_{\hat{w}}$, but orthogonality is violated in the strongest possible way when β satisfies (1.20) with $k > 1$. We demonstrate orthogonality for $k = 0, 1$, but we have no proof that the pertinent functions are complete in $\mathcal{H}_{\hat{w}}$. (We conjecture that this is the case.)

A highlight of this paper is subsection 2.3, where we show how the Lieb–Liniger delta-function eigenfunctions emerge by fixing $c > 0$ (the repulsive delta-function coupling constant), choosing

$$\beta(c, a) \equiv a - a^2 c/\pi \tag{1.23}$$

and letting $a \downarrow 0$. (Thus β converges to the upper limit of the $k = 1$ interval (1.20).) As will be seen, the constraint (1.21) gives rise to the Bethe ansatz constraint occurring for the (finite-volume, $N = 2$) delta-function eigenfunctions [5].

Of course, the obvious conjecture is that the relation will continue to hold for $N > 2$. In the absence of suitable results on the elliptic relativistic $N > 2$ case, this conjecture cannot be tested, however. On the other hand, it may point the way towards finding at least the $g = 2$ elliptic relativistic $N > 2$ eigenfunctions. In particular, one may expect that the Bethe ansatz equations from [5] are mirrored in more general constraint equations for the elliptic eigenfunctions. This scenario is also plausible in view of the $N > 2$ results on the elliptic *non-relativistic* integer g eigenfunctions obtained by Dittrich and Inozemtsev [7, 8], and by Felder and Varchenko [9, 10].

In subsection 2.4 we go a long way towards extending the 'one-gap picture' associated with $H_{\rm nr}(2)$ (1.1) to our relativistic generalization $H_{\rm rel}(2)$ (1.2). To put the results in context, let us begin by recalling that the orthogonality results obtained in subsection 2.2 have a bearing on the problem of turning the AΔOs H_δ into *bona fide* self-adjoint operators on the Hilbert space

$$\mathcal{H} \equiv L^2((0, \pi/r), {\rm d}x). \tag{1.24}$$

Taking the ordinary differential operator $H_{\rm nr}(2)$ as a paradigm, this re-interpretation consists in viewing $H_{\rm nr}(2)$ as an operator that is essentially self-adjoint on the dense subspace $C_0^\infty((0, \pi/r))$ of $\mathcal{H}$. But this is not the only way to associate self-adjoint operators on $\mathcal{H}$ to $H_{\rm nr}(2)$: we may shift x over ${\rm i}a/2$, so as to obtain a Schrödinger operator with a real-analytic π/r-periodic potential $2\wp(x + {\rm i}a/2)$. This leads in a well known way to the consideration of Floquet/Bloch eigenfunctions, whose π/r-multipliers $\exp({\rm i}\theta), \theta \in (-\pi, \pi]$, may be fixed to obtain orthogonal bases for $\mathcal{H}$, see, e.g., [11, section XIII.16]. In this case one is dealing with a one-gap potential (and actually with essentially the only one having this property).

In subsection 2.4 we similarly shift the H_δ-eigenfunctions with $\Sigma, z \in {\rm i}(0, \infty)$ over ${\rm i}a/2$ and fix their π/r-multiplier $\exp({\rm i}\theta)$. Then the first question to answer is whether these functions are once more orthogonal in $\mathcal{H}$. We prove that for $\beta \in (0, a/2)$ each pertinent pair of eigenfunctions *is* orthogonal, whereas for β satisfying (1.20) with $k > 0$ it is *non-orthogonal*. Moreover, there exists a unique extra eigenfunction with $\Sigma \in {\rm i}(-r, r]$ and $z - \pi/2r \in {\rm i}(-a/2, a/2]$, which has the relevant multiplier and real eigenvalues E_-, E_+ in spectral bands. The additional eigenfunction also belongs to $\mathcal{H}$, and it is orthogonal to all of the previous eigenfunctions for $\beta \in (0, a/2)$. Just as in subsection 2.2, we cannot prove that the pertinent eigenfunctions are *complete* in $\mathcal{H}$, but we do expect that this is true. (For $H_{\rm nr}(2)$ completeness follows from Floquet theory, cf [11], but no such theory exists for AΔOs with periodic coefficients at the present time. Conceivably, the 'finite-gap integration' picture of the integer g eigenfunctions can be used to shed light on this issue, cf the paper by Krichever and Zabrodin [12] where this picture is expounded.)

In section 3 we study the hyperbolic ($r = 0$) specialization. At face value, the parameters z and Σ in the constraint (1.21) still seem to be on a quite different footing when $s(x)$ is replaced by ${\rm sh}(\pi x/a)$. But in fact the hyperbolic constraint is essentially (i.e., up to scaling) symmetric under interchange of z and Σ. This property quickly leads to the main novel feature of the hyperbolic regime (as compared with the elliptic regime): the (suitably renormalized) eigenfunctions are *symmetric* under interchange of x and a spectral variable p. Moreover, the B_δ-eigenvalues take the quite simple form $2\,\mathrm{ch}(\pi p/a)$ and $2\,\mathrm{ch}(\pi p/\beta)$ for $\delta = -$ and $+$, respectively. (As suggested by the latter result, the hyperbolic regime is also symmetric under $a \leftrightarrow \beta$—a property that does remain intact for the elliptic generalization, see our previous papers I and II. Since we are fixing g, the latter symmetry is not visible in the present paper, however.)

Physically speaking, the shift $x \to x + \mathrm{i}a/2$ in the hyperbolic setting amounts to changing one of the two particles into an antiparticle: the repulsive interaction turns into an attractive one. The band eigenfunctions from subsection 2.4 all converge to the unique particle–antiparticle bound state occurring for $g = 2$. It is an amazing fact that the repulsive (Bose) delta-function potential eigenfunctions on the line can be obtained not only as a scaling limit of the particle-particle eigenfunctions (this amounts to the specialization of subsection 2.3), but also in two distinct ways from the particle–antiparticle eigenfunctions. This state of affairs is detailed at the end of section 3.

The trigonometric ($a = \infty$) specialization studied in section 4 leads in particular to orthogonal polynomials that are basically q-Gegenbauer polynomials, cf II. This regime is related by analytic continuation to the hyperbolic one, so that duality properties can be easily obtained from the $x \leftrightarrow p$ symmetry of the latter regime. In particular, the three-term recurrence of the polynomials may be viewed as a consequence of the fact that the trigonometric eigenfunctions are also eigenfunctions of an AΔO acting on the spectral variable.

Section 5 contains the specialization to the rational case $r = 0, a = \infty$. The duality property now consists in the pertinent eigenfunctions being also eigenfunctions of the Schrödinger operator $H_{\mathrm{nr}}(2)$ (1.1), acting on the spectral variable and with $\wp(x)$ replaced by $\beta^2/\,\mathrm{sh}^2(\beta x)$. This result can also be obtained from a consideration of the non-relativistic limit, the subject of section 6.

Section 6 gives rise to operators and eigenfunctions that have been known and studied for a very long time. Nevertheless, the novel perspective on these quantities provided by their generalizations in sections 2–4 is illuminating, and accordingly we spell out the relevant $\beta \downarrow 0$ limits in some detail.

2. The elliptic case

2.1. Eigenfunctions: algebraic aspects

It is readily verified that a function $\mathcal{H}(x)$ of the form (1.18) is an eigenfunction of the AΔO B_+ (1.17), irrespective of the choice of β, z and Σ. Indeed, it follows from the s-AΔE (1.11) that $\mathcal{H}(x)$ is an eigenfunction of *each* of the two (commuting) summands of B_+. (Take $L = -1$ and $L = 1$ in (1.11), respectively.) By the same token, for the special β-values

$$\beta = Ma \qquad M \in \mathbb{N}^* \tag{2.1}$$

all functions of the form (1.18) are B_--eigenfunctions (with B_- given by (1.16)).

For the β-values

$$\beta = (M + 1/2)a \qquad M \in \mathbb{N} \tag{2.2}$$

this is no longer true, however. Nevertheless, they are also easily understood. (Note that just as for the β-values (2.1) the Hamiltonian H_- (1.12) amounts to an AΔO with constant coefficients.) In view of (1.14), an obvious choice to obtain joint eigenfunctions of the form (1.18) is to take $z = \mathrm{i}\beta$ and $\Sigma \in \mathbb{C}$. But this is not the only choice: using (1.11), one sees that

$$z = \pi/2r + \mathrm{i}\gamma \qquad \Sigma = 2\mathrm{i}r\gamma/a \tag{2.3}$$

yields a joint eigenfunction, too. (The z-parametrization used here may seem strange, but it will be convenient shortly.)

Let us next require that β belong to one of the β-intervals (1.20). (This amounts to choosing parameters in the set $\mathcal{D}$ of I, see I(3.33)–I(3.35).) Consider now the quotient $(B_-\mathcal{H})(x)/\mathcal{H}(x)$. It reads

$$\frac{e^{-2\beta r}}{s(x)s(x+z)}\left(s(x+i\beta)s(x-i\beta+z)e^{-i\beta\Sigma}+(i\to -i)\right)\equiv E(x). \tag{2.4}$$

Clearly, the function $E(x)$ is not x-independent in general. However, it is elliptic with periods π/r, ia, so it reduces to a constant whenever it has no poles. Choosing z not congruent to 0 (modulo the period lattice), each of the two terms has simple poles at $x \equiv 0$ and $x \equiv -z$. But the residues can be made to cancel by imposing the constraint (1.21): whenever it is fulfilled, we obtain a joint B_δ-eigenfunction.

As a matter of fact, it is expedient to write Σ as

$$\Sigma = 2ir + iy \tag{2.5}$$

and work with the spectral parameter y. Accordingly, we introduce the joint eigenfunctions

$$\mathcal{H}(x) = s(x+z)e^{2irx+ixy} \tag{2.6}$$

where z and y are related by

$$\frac{s(z-i\beta)}{s(z+i\beta)} = e^{-4\beta r-2\beta y}. \tag{2.7}$$

Since we may take $x = i\beta$ in (2.4), the associated B_δ-eigenvalues can now be written

$$E_- = \frac{s(2i\beta)}{s(i\beta)}\frac{s(z)}{s(z+i\beta)}e^{\beta y} \tag{2.8}$$

$$E_+ = e^{2\beta r}(e^{2izr+ay}+e^{-2izr-4ar-ay}). \tag{2.9}$$

Next, we observe that equations (2.7)–(2.9) are invariant under the transformation group generated by

$$z, y \to z+ia, y+2r \tag{2.10}$$

$$z, y \to -z, -y-4r \tag{2.11}$$

$$z, y \to z+\pi/r, y. \tag{2.12}$$

Clearly, $\mathcal{H}(x)$ (2.6) transforms as

$$\mathcal{H}(x) \to -\exp(ar-2izr)\mathcal{H}(x) \tag{2.13}$$

$$\mathcal{H}(x) \to -\mathcal{H}(-x) \tag{2.14}$$

$$\mathcal{H}(x) \to -\mathcal{H}(x) \tag{2.15}$$

under (2.10)–(2.12), respectively. Now we are primarily interested in real y, since this gives rise to real eigenvalues and turns out to suffice for the Hilbert space aspects dealt with in subsections 2.2–2.4. As we shall now detail, for any real y there are always (at least) two linearly independent joint eigenfunctions $\mathcal{H}(x)$ (2.6), corresponding to choices of z that are incongruent (modulo the period lattice).

The first case arises by choosing $z \in \pi/2r + i\mathbb{R}$ satisfying (2.7). (The corresponding 'band eigenfunctions' play no role in subsections 2.2 and 2.3, but they are crucial in subsection 2.4.) More generally, we assert that for a given $y \in \mathbb{R}$ and *all* $\beta > 0$ a number z of the form $\pi/2r + i\gamma$, $\gamma \in \mathbb{R}$, exists such that the constraint (2.7) holds; we assert in addition that such a solution is uniquely determined.

In order to prove this, we begin by noting that the product representation (1.7) entails

$$s(r, a; \pi/2r + \mathrm{i}\lambda) = \frac{\mathrm{ch}(r\lambda)}{r} \prod_{k=1}^{\infty} \frac{1 + \exp(-4kar) + 2\exp(-2kar)\,\mathrm{ch}(2r\lambda)}{(1 - \exp(-2kar))^2}. \tag{2.16}$$

From this we read off first of all that the right-hand side is an even function of λ, which is positive for real λ. To exploit this, we choose $z = \pi/2r + \mathrm{i}\gamma$, $\gamma \in \mathbb{R}$, in the constraint (2.7). Then the left-hand side is positive, so we obtain a uniquely determined $y = f(\gamma) \in \mathbb{R}$. Now for $2\beta/a$ integer we can use (1.11) to deduce

$$f(\gamma) = 2r(\gamma/a - 1) \qquad \beta = ka/2 \quad k \in \mathbb{N}^*. \tag{2.17}$$

(Note this amounts to (2.3) for k odd.) Thus, for these special β-values $f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and onto $\mathbb{R}$.

More generally, from (2.16) one readily infers that for all $\beta > 0$ the function

$$m_\beta : \mathbb{R} \to (0, \infty) \qquad \gamma \mapsto s(\pi/2r + \mathrm{i}\gamma - \mathrm{i}\beta)/s(\pi/2r + \mathrm{i}\gamma + \mathrm{i}\beta) \tag{2.18}$$

is monotonically decreasing. (Consider $\partial_\gamma \ln m_\beta(\gamma)$ to verify this.) Therefore, the function $f(\gamma)$ is monotonically increasing, and in view of (2.10) it maps $\mathbb{R}$ onto $\mathbb{R}$. Hence $f(\gamma)$ has a single-valued real-analytic inverse $\gamma(y)$ mapping $\mathbb{R}$ onto $\mathbb{R}$ for all $\beta > 0$, and so the above existence and uniqueness assertions follow.

Next, we observe that the transformation property (2.13) entails that we may as well restrict attention to $\gamma \in [-a/2, a/2]$, with the endpoints giving rise to the same function $\mathcal{H}(x)$ (2.6). Clearly, we have

$$f(-a/2) = -3r \qquad f(0) = -2r \qquad f(a/2) = -r \tag{2.19}$$

so that y varies over $[-3r, -r]$. Accordingly, we define the joint eigenfunctions

$$\mathcal{H}^b(x, y) = s(x + \pi/2r + \mathrm{i}\gamma(y))\mathrm{e}^{2\mathrm{i}rx + \mathrm{i}xy} \qquad y \in [-3r, -r]. \tag{2.20}$$

(Here, the superscript b stands for 'band', cf subsection 2.4.) For later use we record the β-independent functions

$$\mathcal{H}_1(x) \equiv \mathcal{H}^b(x, -2r) = s(x + \pi/2r) \tag{2.21}$$

$$\mathcal{H}_2(x) \equiv \mathcal{H}^b(x, -3r) = \mathcal{H}^b(x, -r) = s(x + \pi/2r + \mathrm{i}a/2)\mathrm{e}^{\mathrm{i}rx} \tag{2.22}$$

which correspond to (2.19).

We postpone a study of the eigenvalues E_- (2.8) and E_+ (2.9) associated with $\mathcal{H}^b(x, y)$ (2.20) to subsection 2.4, and proceed with the second case: it arises by taking suitable $z \in \mathrm{i}(0, \infty)$. This choice is not as easily understood as the 'band choice' $z \in \pi/2r + \mathrm{i}\mathbb{R}$ just treated. It will occupy us for the remainder of this subsection.

Let us begin by insisting once again on the β-restriction (1.20). It entails that for real y near ∞ the constraint (2.7) can be solved by a unique $z(y)$ near $\mathrm{i}\beta$, located on the imaginary axis above/below $\mathrm{i}\beta$ for k even/odd. (Observe that for $x \in \mathrm{i}\mathbb{R}$ the function $s(r, a; x)/\,\mathrm{sh}(\pi x/a)$ is positive, cf equation (1.6).) The question now arises whether $z(y)$ extends to a single-valued real-analytic solution for arbitrary $y \in \mathbb{R}$.

As will become clear shortly, this is a quite delicate matter, which depends on the choice of β-interval. In our previous paper I, we restricted y to an interval (K, ∞), with K satisfying a number of restrictions, including real-analyticity of $z(y)$ on (K, ∞). Thus we could view $\mathcal{H}(x)$ (2.6) (and its $g \neq 2$ generalizations) as a well-defined real-analytic function $\mathcal{H}(x, y)$ on (K, ∞). Here, we shall analyse the more general choice $y \in \mathbb{R}$, indicating once more the y-dependence explicitly. As will transpire, however, this may give rise to multi-valuedness both for $\mathcal{H}(x, y)$ and for $E_\delta(y)$. (This feature depends on the choice of β.)

We are also aiming to clarify what happens with the eigenfunctions and eigenvalues as β converges to the endpoints of the intervals (1.20). The obvious choice of joint B_δ-eigenfunctions $\mathcal{H}(x, y)$ for these endpoints reads

$$\mathcal{H}(x, y) \equiv s(x + \mathrm{i}\beta)\mathrm{e}^{2\mathrm{i}rx+\mathrm{i}xy} \qquad 2\beta \in a\mathbb{N}^* \tag{2.23}$$

cf the beginning of this section. (This choice is not only natural for continuity reasons, but also for quantum-mechanical purposes, cf subsection 2.2.) Using the AΔE (1.11), the associated eigenvalues $E_\delta(y)$ are readily determined. (Note that the right-hand side of (2.8) is ill defined for $2\beta/a$ integer and $z = \mathrm{i}\beta$.)

Returning to β-values in the intervals (1.20), we begin our analysis by studying the function

$$y(z) = -2r - \frac{1}{2\beta}\ln\left(\frac{s(z - \mathrm{i}\beta)}{s(z + \mathrm{i}\beta)}\right) \tag{2.24}$$

resulting from (2.7). Taking k even and letting z ascend the imaginary axis from $\mathrm{i}\beta$ to $\mathrm{i}(k + 1)a - \mathrm{i}\beta$, we read off that $y(z)$ varies from ∞ to $-\infty$; halfway the z-interval we obtain

$$y(\mathrm{i}(k + 1)a/2) = -r + kr \qquad (k \text{ even}). \tag{2.25}$$

Similarly, taking k odd and letting z descend the imaginary axis from $\mathrm{i}\beta$ to $\mathrm{i}ka - \mathrm{i}\beta$, the function $y(z)$ varies from ∞ to $-\infty$, with

$$y(\mathrm{i}ka/2) = -2r + kr \qquad (k \text{ odd}). \tag{2.26}$$

For later use we note that both (2.25) and (2.26) yield a joint eigenfunction proportional to

$$\mathcal{H}_3(x) \equiv s(x + \mathrm{i}a/2)\mathrm{e}^{\mathrm{i}rx}. \tag{2.27}$$

Writing the eigenvalues (2.8)–(2.9) in the more informative form

$$E_- = (-)^k \frac{\mathrm{i}s(2\mathrm{i}\beta)}{s(\mathrm{i}\beta)^2}\mathrm{e}^{-2\beta r}\left(\frac{1}{\wp(z) - \wp(\mathrm{i}\beta)}\right)^{1/2} \tag{2.28}$$

$$E_+ = 2\mathrm{e}^{2\beta r - 2ar}\,\mathrm{ch}(2\mathrm{i}zr + ay(z) + 2ar) \tag{2.29}$$

we read off invariance under $z \to -z + \mathrm{i}(k + 1)a$ for k even and $z \to -z + \mathrm{i}ka$ for k odd. At the symmetry points (2.25)/(2.26) the functions $\mathcal{H}(x, y(z))$ and $\mathcal{H}(-x, y(z))$ are no longer linearly independent, whereas they are independent otherwise. (This follows by inspection of zeros.) Thus we may and will restrict attention to z varying over the open intervals

$$I_k \equiv \mathrm{i}(\beta, (k + 1)a/2) \qquad k \text{ even} \tag{2.30}$$

$$I_k \equiv \mathrm{i}(ka/2, \beta) \qquad k \text{ odd} \tag{2.31}$$

cf also the paragraph containing (2.10).

From (2.28) we now read off that E_- decreases monotonically from ∞ to a minimum value when z goes from $\mathrm{i}\beta$ to the other endpoint of I_k. But from (2.29) this conclusion cannot be drawn; it is only evident that E_+ increases to ∞ as z goes to $\mathrm{i}\beta$ and that the ch argument vanishes at the other endpoint (2.25)/(2.26). To establish whether E_+ is monotonic on I_k, too, we clearly need more information on $y(z)$.

As it turns out, the behaviour of $y(z)$ depends on k, and the resulting case by case analysis on which we now embark will also enable us to derive information on the inverse function $z(y)$ and on the state of affairs for the limiting β-values $a/2, a, 3a/2, \ldots$, cf equation (1.20). Taking first $k = 0$, the interval between $z - \mathrm{i}\beta$ and $z + \mathrm{i}\beta$ is a subset of $\mathrm{i}(0, a)$. Using (1.5) we infer

$$\partial_z^2 \ln(s(z)) = -\wp(z) - 2\eta r/\pi \tag{2.32}$$

so we may write

$$-2\beta y'(z) = \frac{s'(z-\mathrm{i}\beta)}{s(z-\mathrm{i}\beta)} - \frac{s'(z+\mathrm{i}\beta)}{s(z+\mathrm{i}\beta)} = \int_{z+\mathrm{i}\beta}^{z-\mathrm{i}\beta} \mathrm{d}t\,(-\wp(t) - 2\eta r/\pi) \qquad (k=0). \tag{2.33}$$

Now for $t \in \mathrm{i}(0,a)$ the integrand has a minimum at $t = \mathrm{i}a/2$, and via the product representation (1.6) one can obtain the identity

$$-\wp\left(\frac{\mathrm{i}a}{2}; \frac{\pi}{2r}, \frac{\mathrm{i}a}{2}\right) - \frac{2\eta r}{\pi} = 4r^2 \sum_{n=1}^{\infty} \frac{n}{\mathrm{sh}\, nar}. \tag{2.34}$$

(Compare with, e.g., [13, equations (2.93)–(2.98)] for details.) Therefore, the integrand is positive, and so $y'(z)$ does not vanish for z between $\mathrm{i}\beta$ and $\mathrm{i}a - \mathrm{i}\beta$.

As a consequence, both $y(z)$ and $\mathrm{i}z$ decrease as z goes from $\mathrm{i}\beta$ to $\mathrm{i}a/2$, so that E_+ (2.29) is monotonic on I_0. Moreover, the inverse function $z(y)$ is well defined and real-analytic for real y.

Now for the $g = 2$ case at issue, the parameter K used in our previous paper I may be defined as the smallest number for which three requirements hold true: (i) the function $z(y)$ is real-analytic on (K,∞); (ii) the eigenvalues $E_\delta(y)$ separate points on (K,∞); (iii) the functions $\mathcal{H}(x,y)$ and $\mathcal{H}(-x,y)$ are linearly independent on (K,∞). Now linear independence holds true for $y > -r$, but not for $y = -r$ (cf the paragraph containing (2.10)); also, as we have just seen, the eigenvalues $E_\delta(y)$ are monotonic on $(-r,\infty)$. Thus we have

$$K = -r \qquad \beta \in (0, a/2). \tag{2.35}$$

Next, we choose $k = 1$ in (1.20). For $z \in I_1$ (2.31) we now have $z - \mathrm{i}\beta \in \mathrm{i}(-a, 0)$ and $z + \mathrm{i}\beta \in \mathrm{i}(a, 2a)$, so that (2.33) can no longer be used. But from (1.10) we deduce

$$\frac{s'(z+\mathrm{i}a/2)}{s(z+\mathrm{i}a/2)} - \frac{s'(z-\mathrm{i}a/2)}{s(z-\mathrm{i}a/2)} = -2\mathrm{i}r \tag{2.36}$$

so we may write

$$-2\beta y'(z) = \int_{z+\mathrm{i}\beta-2\mathrm{i}a}^{z-\mathrm{i}\beta} \mathrm{d}t\,(-\wp(t) - 2\eta r/\pi) + 4\mathrm{i}r \qquad (k=1). \tag{2.37}$$

In view of (2.34), the integral yields a number in $\mathrm{i}(0,\infty)$, so that $y'(z) \neq 0$ for z between $\mathrm{i}\beta$ and $\mathrm{i}a - \mathrm{i}\beta$. Thus, $z(y)$ is well defined and real-analytic on $\mathbb{R}$, and so we have

$$K = -r \qquad \beta \in (a/2, a). \tag{2.38}$$

In this case, however, $y(z)$ decreases and $\mathrm{i}z$ increases as z goes from $\mathrm{i}\beta$ to $\mathrm{i}a/2$, so that it is not clear from (2.29) whether E_+ is monotonic on I_1.

This is actually true, however. Indeed, using (2.37) the pertinent derivative can be written

$$2\mathrm{i}r + ay'(z) = -\frac{a}{2\beta}\int_{z+\mathrm{i}\beta-2\mathrm{i}a}^{z-\mathrm{i}\beta} \mathrm{d}t\,(-\wp(t) - 2\eta r/\pi) + [2\mathrm{i}r - 2\mathrm{i}ar/\beta] \qquad (k=1). \tag{2.39}$$

Since $\beta \in (a/2, a)$, the term in square brackets yields a number in $\mathrm{i}(-\infty, 0)$, just as the first term on the right-hand side. Thus the derivative is non-zero, so E_+ decreases as z goes from $\mathrm{i}\beta$ to $\mathrm{i}a/2$.

Next, we determine what happens when β converges to the excluded values $a/2$ and a. Fixing $y \in \mathbb{R}$ and letting $\beta \to a/2$, it is clear from the above that $z(y) \to \mathrm{i}a/2$. The resulting limit functions

$$\mathcal{H}(a/2; x, y) = s(x + \mathrm{i}a/2)\mathrm{e}^{2\mathrm{i}rx+\mathrm{i}xy} \qquad y \in \mathbb{R} \tag{2.40}$$

coincide with the functions (2.23) for $\beta = a/2$. They are obviously joint eigenfunctions of B_- and B_+ with eigenvalues

$$E_- = 2e^{-ar/2}\,\mathrm{ch}(a(y+r)/2) \qquad E_+ = 2e^{-ar}\,\mathrm{ch}\,a(y+r). \tag{2.41}$$

Note also that

$$B_+ = B_-^2 - 2e^{-ar} \qquad \beta = a/2. \tag{2.42}$$

We now fix $y \in [0,\infty)$ and let $\beta \uparrow a$. Then we deduce from (2.7) that $z(y)$ converges to ia. The limit functions

$$\mathcal{H}(a;x,y) = s(x+\mathrm{i}a)e^{2\mathrm{i}rx+\mathrm{i}xy} \qquad y \in [0,\infty) \tag{2.43}$$

coincide with (2.23) for $\beta = a$ and yield eigenvalues

$$E_- = E_+ = 2\,\mathrm{ch}\,ay \qquad y \in [0,\infty) \tag{2.44}$$

in agreement with

$$B_- = B_+ = -e^{-ar}(e^{-2\mathrm{i}rx}T_{\mathrm{i}a} + (\mathrm{i} \to -\mathrm{i})) \qquad \beta = a. \tag{2.45}$$

Next, we fix $y \in (-\infty,-2r]$, yielding $z(y) \downarrow 0$ for $\beta \uparrow a$. Hence we get limit functions

$$\mathcal{H}(a;x,y) = s(x)e^{2\mathrm{i}rx+\mathrm{i}xy} \qquad y \in (-\infty,-2r] \tag{2.46}$$

which are *different* from (2.23), with eigenvalue

$$E_- = E_+ = 2\,\mathrm{ch}\,a(y+2r) \qquad y \in (-\infty,-2r]. \tag{2.47}$$

Finally, equation (2.7) entails

$$y \in (-2r,0),\ \beta \uparrow a \Rightarrow z \to \mathrm{i}a + \mathrm{i}ay/2r \tag{2.48}$$

yielding limit functions

$$\mathcal{H}(a;x,y) = s(x+\mathrm{i}a+\mathrm{i}ay/2r)e^{2\mathrm{i}rx+\mathrm{i}xy} \qquad y \in (-2r,0) \tag{2.49}$$

with eigenvalue

$$E_- = E_+ = 2 \qquad y \in (-2r,0). \tag{2.50}$$

After this study of the β-interval $(0,a]$, we continue by choosing $\beta \in (a,3a/2)$. Proceeding as before (cf equation (2.37)), we once again obtain

$$-2\beta y'(z) = \int_{z+\mathrm{i}\beta-2\mathrm{i}a}^{z-\mathrm{i}\beta} \mathrm{d}t\,(-\wp(t) - 2\eta r/\pi) + 4\mathrm{i}r \qquad (k=2). \tag{2.51}$$

But now the integral yields a number in $\mathrm{i}(-\infty,0)$, which can be made as small as we please by choosing β close to a and z near $3\mathrm{i}a/2$. On the other hand, the z-derivative of the right-hand side is positive on I_2 (2.30), so we deduce that $y'(z)$ has a unique zero z_0 in I_2, provided β is sufficiently close to a. Accordingly, the function $y(z)$ decreases from ∞ to $y(z_0) = r - d_0$, $d_0 > 0$, and then increases to r as z ascends the imaginary axis from iβ to $3\mathrm{i}a/2$.

As a consequence, the inverse function $z(y)$ extends to a real-analytic monotonic function on $(r-d_0,\infty)$. More generally, it continues to a *multi-valued* function on $\mathbb{R}$, namely, triple-valued for $y \in (r-d_0, r+d_0)$, double-valued for the turning points $r \pm d_0$, single-valued otherwise. Now linear independence of $\mathcal{H}(x,y)$ and $\mathcal{H}(-x,y)$ for $y \in (r-d_0,\infty)$ is readily checked (from a comparison of zeros), so we have

$$K = r - d_0 \qquad \beta \in (a,3a/2). \tag{2.52}$$

Taking $\beta \downarrow a$, one readily obtains $d_0 \to r$ and three overlapping y-intervals with limits

$$y \in [0, \infty) : \quad \mathcal{H}_1(x, y) = s(x + \mathrm{i}a)\mathrm{e}^{2\mathrm{i}rx+\mathrm{i}xy} \qquad E_+ = E_- = 2\,\mathrm{ch}\, ay \tag{2.53}$$

$$y \in (0, 2r) : \quad \mathcal{H}_2(x, y) = s(x + \mathrm{i}a + \mathrm{i}ay/2r)\mathrm{e}^{2\mathrm{i}rx+\mathrm{i}xy} \qquad E_+ = E_- = 2 \tag{2.54}$$

$$y \in (-\infty, 2r] : \quad \mathcal{H}_3(x, y) = s(x + 2\mathrm{i}a)\mathrm{e}^{2\mathrm{i}rx+\mathrm{i}xy} \qquad E_+ = E_- = 2\,\mathrm{ch}\, a(y - 2r). \tag{2.55}$$

These limits should be compared with the $\beta \uparrow a$ limits (2.43)–(2.50). Specifically, it should be noted that the function $\mathcal{H}(a; x, y)$ equals $\mathcal{H}_1(x, y)$ (2.53) for $y \in [0, \infty)$, whereas for $y \in (-2r, 0)$ and $y \in (-\infty, -2r]$ it equals a multiple of $\mathcal{H}_2(x, y + 2r)$ and $\mathcal{H}_3(x, y + 4r)$, respectively.

Taking next β sufficiently close to $3a/2$, it is clear from (2.51) that no zeros occur. Thus the function $z(y)$ is a (single-valued) real-analytic function on $\mathbb{R}$, and we deduce

$$d_0 = 0 \qquad \beta \in [\beta_0, 3a/2) \qquad \beta_0 \in (a, 3a/2). \tag{2.56}$$

The derivative (2.39) is easily seen to be non-zero for *all* $\beta \in (a, 3a/2)$, so that E_+ (2.29) is monotonic on I_2, just as E_- (2.28). Noting $z(y) \to 3\mathrm{i}a/2$ as $\beta \uparrow 3a/2$, we obtain limit functions

$$\mathcal{H}(3a/2; x, y) = s(x + 3\mathrm{i}a/2)\mathrm{e}^{2\mathrm{i}rx+\mathrm{i}xy} \qquad y \in \mathbb{R} \tag{2.57}$$

coinciding with (2.23) for $\beta = 3a/2$, with eigenvalues

$$E_- = 2\mathrm{e}^{3ar/2}\,\mathrm{ch}(3a(y - r)/2) \qquad E_+ = 2\mathrm{e}^{ar}\,\mathrm{ch}\, a(y - r). \tag{2.58}$$

Proceeding with the choice $\beta \in (3a/2, 2a)$, we obtain $z - \mathrm{i}\beta \in \mathrm{i}a(-1, 0)$ and $z + \mathrm{i}\beta \in \mathrm{i}a(3, 4)$, so we have

$$-2\beta y'(z) = \int_{z+\mathrm{i}\beta-4\mathrm{i}a}^{z-\mathrm{i}\beta} \mathrm{d}t\,(-\wp(t) - 2\eta r/\pi) + 6\mathrm{i}r \qquad (k = 3). \tag{2.59}$$

Thus $y'(z)$ is non-zero on I_3 (2.31) and $z(y)$ is well defined and real-analytic on $\mathbb{R}$. Correspondingly, we obtain

$$K = r \qquad \beta \in (3a/2, 2a). \tag{2.60}$$

In contrast to previous cases, E_+ is readily seen *not* to be monotonic on I_3 when β is close to $2a$. (The derivative $2\mathrm{i}r + ay'(z)$ changes sign near $z = 3\mathrm{i}a/2$.)

Fixing $y \in \mathbb{R}$, we now let $\beta \uparrow 2a$. Then we obtain limit functions

$$\mathcal{H}(2a; x, y) = \mathrm{e}^{2\mathrm{i}rx+\mathrm{i}xy} \cdot \begin{cases} s(x + 2\mathrm{i}a) & y \in [2r, \infty) \\ s(x + \mathrm{i}a + \mathrm{i}ay/2r) & y \in (0, 2r) \\ s(x + \mathrm{i}a) & y \in (-\infty, 0] \end{cases} \tag{2.61}$$

with eigenvalues

$$E_- = 2\mathrm{e}^{4ar} \cdot \begin{cases} \mathrm{ch}\, 2a(y - 2r) & y \in [2r, \infty) \\ 1 & y \in (0, 2r) \\ \mathrm{ch}\, 2ay & y \in (-\infty, 0] \end{cases} \tag{2.62}$$

$$E_+ = 2\mathrm{e}^{2ar} \cdot \begin{cases} \mathrm{ch}\, a(y - 2r) & y \in [2r, \infty) \\ 1 & y \in (0, 2r) \\ \mathrm{ch}\, ay & y \in (-\infty, 0]. \end{cases} \tag{2.63}$$

It will be clear by now how this analysis can be extended to $\beta \in (2a, \infty)$, so we omit further details. In particular, defining

$$\mathcal{B}_M \equiv a(M, M + 1/2) \cup a(M + 1/2, M + 1) \qquad M \in \mathbb{N} \tag{2.64}$$

and fixing $\beta \in \mathcal{B}_M$, one readily deduces that $z(y)$ extends to a monotonic real-analytic function on $((2M-1)r, \infty)$, entailing

$$K \leqslant (2M-1)r \qquad \beta \in \mathcal{B}_M. \tag{2.65}$$

This information suffices for the orthogonality analysis on which we embark shortly.

Before doing so, we conclude this subsection by commenting on the ambiguities revealed above. They show by example that there exists no joint B_δ-eigenfunction $\mathcal{H}(x, y)$ that is single-valued and real-analytic for all $\beta > 0$ and $y \in \mathbb{R}$. (By contrast, for the hyperbolic case such functions do exist, cf section 3.)

In this connection it is important to recall the uniqueness result obtained in appendix B of I. It says that for β/a irrational and $y \in (L, \infty)$ (with $L \geqslant K$), the joint B_δ-eigenspace corresponding to eigenvalues $E_\delta(y)$ is spanned by the functions $\mathcal{H}(x, y)$ and $\mathcal{H}(-x, y)$. (Here, the AΔOs B_δ are viewed as operators on the space of meromorphic functions.) Since such β-values are dense, continuous interpolations are uniquely determined.

2.2. Eigenfunctions: orthogonality for real x

In this subsection we study the orthogonality properties of suitable linear combinations of the joint H_δ-eigenfunctions

$$\mathcal{F}(\pm x, y) \equiv \hat{w}(x)^{1/2}\mathcal{H}(\pm x, y) \qquad x \in (0, \pi/r) \quad y \in \mathbb{R} \tag{2.66}$$

in the Hilbert space $\mathcal{H}$ (1.24). Here, we take the positive square root, so we may as well work with $\mathcal{H}(\pm x, y)$ and the Hilbert space $\mathcal{H}_{\hat{w}}$ (1.22).

First of all, it should be emphasized that we need no restriction on y to ensure square-integrability. Indeed, for β satisfying (1.20) the functions $\mathcal{F}(\pm x, y)$ clearly extend to real-analytic functions on $\mathbb{R}$; for the β-values (1.19) one readily verifies (using equations (2.23), (1.14) and (1.11)) that

$$\mathcal{F}(x, y) = \mathrm{i}^k \exp \mathrm{i}x(y + 2r - k) \qquad x \in (0, \pi/r) \quad y \in \mathbb{R} \quad \beta = ka/2 \quad k \in \mathbb{N}^* \tag{2.67}$$

so square-integrability is plain, too.

Since the AΔOs H_δ are formally self-adjoint, with real eigenvalues on the functions $\mathcal{F}(\pm x, y)$, one might be inclined to expect that the standard boundary conditions giving rise to orthogonal bases for the free cases (2.67) will also give rise to orthogonal bases when β satisfies (1.20). As we will see, this expectation is not borne out by the facts, however.

Taking the differential operator $H_{\mathrm{nr}}(2)$ (1.1) as a lead, we recall it is already essentially self-adjoint on $C_0^\infty((0, \pi/r))$; its eigenfunctions are generically *not* square-integrable over $(0, \pi/r)$ and one must restrict attention to linear combinations that vanish at $x = 0$ and at $x = \pi/r$ (Dirichlet conditions). But the well-developed self-adjointness theory for differential operators has no analog for analytic difference operators, and so we opt for a pragmatic approach: we impose Dirichlet conditions in the relativistic case, too, and prove that orthogonality holds true for $k = 0, 1$ in (1.20), and that it breaks down for $k > 1$. At the end of this subsection we briefly return to other boundary conditions.

Correspondingly, our principal aim in this subsection is to investigate orthogonality properties of the functions

$$\psi_n(x) = s(x + z_n)\mathrm{e}^{\mathrm{i}(n+2)rx} + s(x - z_n)\mathrm{e}^{-\mathrm{i}(n+2)rx} \qquad z_n \equiv z(nr) \tag{2.68}$$

where we take

$$\beta \in \mathcal{B}_M \qquad n \in \mathbb{N} \qquad n \geqslant 2M \tag{2.69}$$

(so that z_n is a well-defined number between $\mathrm{i}\beta$ and $\mathrm{i}(M + 1/2)a$, cf equation (2.65)), in the Hilbert space $\mathcal{H}_{\hat{w}}$ (1.22).

To begin with, we read off from (2.68) that we have

$$\psi_n(\pi/r - x) = (-)^n \psi_n(x) \tag{2.70}$$

whereas $\hat{w}(x)$ is clearly invariant under $x \to \pi/r - x$. Thus we deduce that the inner product

$$(\psi_n, \psi_m) = \int_0^{\pi/r} \frac{\overline{\psi_n(x)}\psi_m(x)}{s(x+\mathrm{i}\beta)s(x-\mathrm{i}\beta)}\,\mathrm{d}x \tag{2.71}$$

vanishes when $n - m$ is odd, independently of the choice of β.

More generally, we are going to prove

$$(\psi_n, \psi_m) = 0 \qquad \beta \in \mathcal{B}_0 \qquad n > m \geqslant 0 \tag{2.72}$$

whereas

$$(\psi_n, \psi_m) \neq 0 \qquad \beta \in \mathcal{B}_M \quad M > 0 \quad n > m \geqslant 2M \quad n - m \text{ even}. \tag{2.73}$$

We establish these results by direct calculation (as opposed to our arguments in section 4 of I, where we exploited the eigenfunction property).

To prove (2.72), we need to calculate the integral

$$I_l(b, c, d) \equiv \int_0^{\pi/r} \mathrm{d}x\, \frac{s(x+c)s(x-d)}{s(x+\mathrm{i}b)s(x-\mathrm{i}b)} \mathrm{e}^{2\mathrm{i}lrx} \qquad l \in \mathbb{Z} \quad c, d \in \mathbb{C} \tag{2.74}$$

with $b \in \mathcal{B}_0$. Denoting the integrand by $F(x)$, one easily checks $F(x + \mathrm{i}a) = \mu F(x)$, where the multiplier reads

$$\mu \equiv \exp(-2\mathrm{i}r[c - d - ila]). \tag{2.75}$$

Moreover, $F(x)$ is π/r-periodic, so we can evaluate the integral via an elementary contour integration. This yields

$$I_l(b, c, d) = \frac{2\mathrm{i}\pi}{s(2\mathrm{i}b)} \frac{1}{(1-\mu)} \big[s(\mathrm{i}b + c)s(\mathrm{i}b - d)\mathrm{e}^{-2lbr} - \mu s(\mathrm{i}b - c)s(\mathrm{i}b + d)\mathrm{e}^{2lbr} \big] \qquad b \in \mathcal{B}_0 \tag{2.76}$$

for $\mu \neq 1$, whereas for $\mu = 1$ one obtains the $\mu \to 1$ limit of the right-hand side. (Note the term in square brackets vanishes when μ (2.75) equals 1, as should be the case, of course. Observe also that (2.74) is manifestly invariant under $b \to -b$, whereas the (analytic continuation of the) right-hand side of (2.76) is not an even function of b.)

Since (2.72) holds true for $n - m$ odd, we fix a pair $n \neq m$ with $n - m$ even. Then, using (2.71), (2.68) and (2.74), we obtain

$$(\psi_n, \psi_m) = 2(I_{(m-n)/2}(\beta, z_m, z_n) + I_{2+(m+n)/2}(\beta, z_m, -z_n)). \tag{2.77}$$

Substituting (2.76) and (2.75), and using

$$s(\mathrm{i}\beta - z_j)/s(\mathrm{i}\beta + z_j) = -\mathrm{e}^{-2(j+2)\beta r} \tag{2.78}$$

one now verifies the announced pairwise orthogonality (2.72).

Before turning to the proof of (2.73), we obtain an explicit norm formula for $\beta \in \mathcal{B}_0$. Indeed, taking $l = 0$ and letting $d \to c$ in (2.76) and (2.75), one arrives at

$$I_0(b, c, c) = \frac{\pi}{rs(2\mathrm{i}b)} \big[2\mathrm{i}rs(\mathrm{i}b - c)s(\mathrm{i}b + c) + s(\mathrm{i}b + c)s'(\mathrm{i}b - c) + s(\mathrm{i}b - c)s'(\mathrm{i}b + c) \big] \qquad b \in \mathcal{B}_0 \tag{2.79}$$

and using this result one readily deduces

$$(\psi_n, \psi_n) = \frac{2\pi}{rs(2\mathrm{i}\beta)} \big[s(\mathrm{i}\beta + z_n)s'(\mathrm{i}\beta - z_n) + s(\mathrm{i}\beta - z_n)s'(\mathrm{i}\beta + z_n) \big] \qquad \beta \in \mathcal{B}_0 \quad n \geqslant 0. \tag{2.80}$$

(Note this converges to $2\pi/r$ for $n \to \infty$.)

In order to prove (2.73), we first observe that the integral (2.74) with $b \in \mathcal{B}_M$ is well defined, and can be reduced to the case $M = 0$ by exploiting (1.11). Specifically, a routine calculation yields

$$I_l(b, c, d) = \frac{2\mathrm{i}\pi}{s(2\mathrm{i}b)} \frac{1}{(1-\mu)} \left(\mu^{-M} s(\mathrm{i}b + c) s(\mathrm{i}b - d) \mathrm{e}^{-2lbr} - \mu^{M+1} s(\mathrm{i}b - c) s(\mathrm{i}b + d) \mathrm{e}^{2lbr}\right) \tag{2.81}$$

where $b \in \mathcal{B}_M$. (Note this entails that analytic continuation of (2.76) yields a wrong answer, just as it does for negative b.) Now (2.77) follows for $M > 0$, too, so we can invoke (2.81) to obtain (also using (2.78))

$$(\psi_n, \psi_m) = \frac{4\mathrm{i}\pi}{s(2\mathrm{i}\beta)} s(\mathrm{i}\beta + z_n) s(\mathrm{i}\beta + z_m) \mathrm{e}^{-(n+m+4)\beta r} D_{nm} \tag{2.82}$$

where

$$D_{nm} \equiv Q_M(t_+) - Q_M(t_-) \qquad t_\pm \equiv \mathrm{i}(z_m \pm z_n)/a + (1 + m/2) \pm (1 + n/2) \tag{2.83}$$

with

$$Q_M(t) \equiv \mathrm{sh}((2M+1)art)/\,\mathrm{sh}(art). \tag{2.84}$$

It is not difficult to see that this implies (2.73). Indeed, it is straightforward to verify that $Q_M(t)$ is increasing on $(0, \infty)$, so the difference D_{nm} can only vanish when t_+ equals t_- or $-t_-$. This yields $z_n = \mathrm{i}(1 + n/2)a$ or $z_m = \mathrm{i}(1 + m/2)a$, respectively. But since z_j, $j \geqslant 2M$, is a number between $\mathrm{i}\beta$ and $\mathrm{i}(M + 1/2)a$, it cannot equal $\mathrm{i}(1 + j/2)a$, and so (2.73) follows.

Let us now consider the special β-values $ka/2$, $k \in \mathbb{N}^*$. Choosing $k = 1$, we can use (2.23) to obtain

$$\begin{aligned} \psi_n(a/2; x) &= s(x + \mathrm{i}a/2)\mathrm{e}^{\mathrm{i}(n+2)rx} + s(x - \mathrm{i}a/2)\mathrm{e}^{-\mathrm{i}(n+2)rx} \\ &= 2\mathrm{i}s(x + \mathrm{i}a/2)\mathrm{e}^{\mathrm{i}rx} \sin(n+1)rx \qquad n \in \mathbb{N} \quad n \geqslant 0. \end{aligned} \tag{2.85}$$

Obviously, these functions give rise to an orthogonal base for the Hilbert space

$$\mathcal{H}_{1/2} \equiv L^2((0, \pi/r), \hat{w}_{1/2}(x)\,\mathrm{d}x) \qquad \hat{w}_{1/2}(x) \equiv 1/s(x + \mathrm{i}a/2)s(x - \mathrm{i}a/2) \tag{2.86}$$

cf equation (1.14).

Choosing next $\beta = (M + 1/2)a$, $M \in \mathbb{N}^*$, the function $\hat{w}(x)$ (1.14) reduces to a positive multiple of $\hat{w}_{1/2}$. Moreover, we have (cf equation (2.23))

$$\begin{aligned} \psi_n((M+1/2)a; x) &= s(x + \mathrm{i}(M+1/2)a)\mathrm{e}^{\mathrm{i}(n+2)rx} + s(x - \mathrm{i}(M+1/2)a)\mathrm{e}^{-\mathrm{i}(n+2)rx} \\ &= c_M \psi_{n-2M}(a/2; x) \qquad n \in \mathbb{N} \quad n \geqslant 2M \end{aligned} \tag{2.87}$$

so that these functions yield an orthogonal base for $\mathcal{H}_{1/2}$ as well.

Similarly, the choices $\beta = Ma$ yield orthogonal bases (cf equation (2.23))

$$\begin{aligned} \psi_n(Ma; x) &= s(x + \mathrm{i}Ma)\mathrm{e}^{\mathrm{i}(n+2)rx} + s(x - \mathrm{i}Ma)\mathrm{e}^{-\mathrm{i}(n+2)rx} \\ &= d_M s(x) \cos(n - 2M + 2)rx \qquad n \in \mathbb{N} \quad n \geqslant 2M - 2 \end{aligned} \tag{2.88}$$

for the Hilbert space

$$\mathcal{H}_0 \equiv L^2((0, \pi/r), s(x)^{-2}\,\mathrm{d}x). \tag{2.89}$$

Of course, these special cases are easily understood in terms of the AΔOs H_δ (1.12)–(1.13): they reduce to 'free' AΔOs and the sine- and cosine-bases serve to define associated self-adjoint operators on the Hilbert space $\mathcal{H}$ (1.24).

By contrast, it should be mentioned that the above orthogonality results for $\beta \in \mathcal{B}_0$ do *not* suffice to rigorously conclude that the H_δ-eigenfunctions $\hat{w}(x)^{1/2}\psi_n(x)$ correspond to commuting self-adjoint operators H_δ on $\mathcal{H}$. This is because we have not proved that the functions $\{\psi_n\}_{n=0}^{\infty}$ are *complete* in $\mathcal{H}_{\hat{w}}$. Note also that in view of our non-orthogonality results, the *formal* self-adjointness of the AΔOs H_δ is quite misleading. For further information on this circle of problems we refer to section 4 of I and to [4, subsection 4.1].

We conclude this subsection by ruling out orthogonality for other choices of boundary conditions and *all* β satisfying (1.20). First, consider the even functions

$$e_n(x) \equiv \hat{w}(x)^{1/2}[\mathcal{H}(x, nr) + \mathcal{H}(-x, nr)] \tag{2.90}$$

with (2.69) in effect. (They correspond to Neumann conditions.) We have

$$e_n(\pi/r - x) = (-)^{n+1} e_n(x) \tag{2.91}$$

so e_n and e_m are orthogonal as vectors in $\mathcal{H}$ whenever $n - m$ is odd. For $n - m$ even, however, the above calculations are easily modified to yield

$$(e_n, e_m) = \frac{4\mathrm{i}\pi}{s(2\mathrm{i}\beta)} s(\mathrm{i}\beta + z_n)s(\mathrm{i}\beta + z_m)\mathrm{e}^{-(n+m+4)\beta r}(-Q_M(t_+) - Q_M(t_-)) \tag{2.92}$$

cf equations (2.82)–(2.84). Thus we conclude

$$(e_n, e_m) \neq 0 \qquad \beta \in \mathcal{B}_M \quad M \geqslant 0 \quad n > m \geqslant 2M \quad n - m \text{ even}. \tag{2.93}$$

By contrast, for $\beta > 0$ satisfying the complementary restriction (1.19), the vectors e_n give rise to orthogonal bases for $\mathcal{H}$. (One need only modify (2.85)–(2.88) in an obvious fashion to check this.) Likewise, we can construct orthogonal bases $\{\mathcal{F}(x, y)\}$ for $\mathcal{H}$ by taking all y in the set $r\theta/\pi + 2r\mathbb{Z}$ with $\theta \in (-\pi, \pi]$, cf equation (2.67). Since we have $\mathcal{F}(x, y) \in \mathcal{H}$ for all $\beta > 0$ and $y \in \mathbb{R}$, and since H_δ takes real eigenvalues on $\mathcal{F}(x, y)$, one might guess that the same boundary conditions give rise to orthogonality at least for $\beta \in (0, a/2)$.

With the above integrals at our disposal, it is quite easy to see that this is not the case. Indeed, for $\beta \in \mathcal{B}_0$ (2.64) we may fix $y \in \mathbb{R}$ and $l \in \mathbb{Z}$ to obtain

$$(\mathcal{F}(\cdot, y), \mathcal{F}(\cdot, y + 2lr)) = I_l(\beta, z(y + 2lr), z(y)). \tag{2.94}$$

Now we have $|z(y + 2lr) - z(y)| < a$, so that μ (2.75) is a positive number not equal to 1 for $l \neq 0$. From (2.76) we then have

$$(\mathcal{F}(\cdot, y), \mathcal{F}(\cdot, y + 2lr)) = \frac{2\mathrm{i}\pi}{s(2\mathrm{i}\beta)} \frac{s(\mathrm{i}\beta + z(y + 2lr))s(\mathrm{i}\beta - z(y))}{1 - \mu} d_l(y) \qquad l \in \mathbb{Z}^* \tag{2.95}$$

$$d_l(y) \equiv 1 - \mu \exp(4l\beta r) \frac{s(\mathrm{i}\beta - z(y + 2lr))}{s(\mathrm{i}\beta + z(y + 2lr))} \frac{s(\mathrm{i}\beta + z(y))}{s(\mathrm{i}\beta - z(y))}. \tag{2.96}$$

But using the constraint (2.7) one obtains $d_l(y) = 1 - \mu$, so that

$$(\mathcal{F}(\cdot, y), \mathcal{F}(\cdot, y + 2lr)) \neq 0 \qquad l \in \mathbb{Z}^* \quad \beta \in \mathcal{B}_0 \tag{2.97}$$

as announced.

Likewise, one may study $\beta \in \mathcal{B}_M, M > 0$, taking $y > (2M - 1)r$ and $l \in \mathbb{N}^*$ (say) to avoid eventual multi-valuedness. Then (2.94) is still valid, and now one can use (2.81) to deduce that the pertinent vectors are not orthogonal in $\mathcal{H}$. Thus the 'Floquet/Bloch' boundary conditions to hand violate orthogonality for all β satisfying (1.20) when we insist on keeping x real, as we have done throughout this subsection. Letting $x - \mathrm{i}a/2 \in (0, \pi/r)$, however, the state of affairs is different, cf subsection 2.4.

2.3. The relation to the delta-function gas in finite volume

As announced in the introduction, the above eigenfunctions can be tied in with the well known eigenfunctions of the ($N = 2$, centre-of-mass, finite-volume) repulsive delta-function Bose gas. We proceed by supplying the details of the pertinent limiting transition. To this end we fix $c > 0$ (the delta-function coupling constant), and introduce the functions [5]

$$D_n(x) \equiv \mathrm{i}\left(\frac{c + \mathrm{i}k_n}{c - \mathrm{i}k_n}\right)^{1/2} \mathrm{e}^{\mathrm{i}xk_n} + \text{c.c.} \qquad n \in \mathbb{N}. \tag{2.98}$$

Here, $k_n \in (0, \infty)$ is the unique solution of the equation

$$(n+1)r = k_n + \frac{2r}{\pi}\,\mathrm{Arctg}(k_n/c) \qquad n \in \mathbb{N} \tag{2.99}$$

and the square-root sign is fixed by requiring continuity for $c \in (0, \infty)$ and convergence to 1 for $c \to \infty$. It is straightforward to verify that these functions are pairwise orthogonal in $\mathcal{H}$ (1.24), with norms given by

$$\int_0^{\pi/r} \mathrm{d}x\, |D_n(x)|^2 = \frac{2\pi}{r} + \frac{4c}{c^2 + k_n^2} \qquad n \in \mathbb{N}. \tag{2.100}$$

Next, choosing $a < \pi/2c$ from now on, we define $\beta(c, a)$ by (1.23), so that $2\beta \in (a, 2a)$. Then the functions

$$\Phi_n(a; x) \equiv \left[s(r, a; x + \mathrm{i}\beta)s(r, a; x - \mathrm{i}\beta)\right]^{-1/2} \psi_n(a, \beta; x) \qquad n \in \mathbb{N} \tag{2.101}$$

(with $\beta \equiv \beta(c, a)$) are pairwise orthogonal in $\mathcal{H}$, as we have proved above. In the following theorem we state the relevant limit for the functions $\Phi_n(a; x)$, but we find it convenient to prove a more general result.

Theorem 2.1. *Fixing $n \in \mathbb{N}$, one has*

$$\Phi_n(a; x) = D_n(x) + \mathrm{O}(a) \qquad a \downarrow 0 \quad x \in (0, \pi/r) \tag{2.102}$$

where the bound is uniform on compact subsets of $(0, \pi/r)$.

Proof. Since we have $2\beta(c, a) \in (a, 2a)$, the numbers $z(nr)$ in the definition (2.68) of $\psi_n(x)$ lie on the line segment between $\mathrm{i}\beta$ and $\mathrm{i}a/2$. More generally, we have shown above that for a fixed $y \in \mathbb{R}$ the equation (2.7) can be solved by a unique $z = z(a, \beta, y)$ on the line segment between $\mathrm{i}\beta$ and $\mathrm{i}a - \mathrm{i}\beta$ (cf the paragraph containing (2.37)).

We now fix $y \in \mathbb{R}$ and prove

$$\frac{s(x+z)}{|s(x+\mathrm{i}\beta)|}\mathrm{e}^{2\mathrm{i}rx + \mathrm{i}xy} = \mathrm{i}\left(\frac{c + \mathrm{i}k}{c - \mathrm{i}k}\right)^{1/2} \mathrm{e}^{\mathrm{i}xk} + \mathrm{O}(a) \qquad x \in (0, \pi/r) \quad a \downarrow 0. \tag{2.103}$$

Here, we have $z = z(a, \beta, y)$ and $\beta = \beta(c, a)$, and $k \in \mathbb{R}$ is the unique solution of the equation

$$y = -r + k + \frac{2r}{\pi}\,\mathrm{Arctg}(k/c). \tag{2.104}$$

Moreover, the bound is uniform on compacts in $(0, \pi/r)$. (Clearly, the assertion of the theorem is a consequence of this more general result.)

In order to prove (2.103), it is expedient to reparametrize z as

$$z = \mathrm{i}\beta - \mathrm{i}af. \tag{2.105}$$

Thus we have (cf equation (2.24))

$$y = -2r - \frac{1}{2\beta}\ln\left(\frac{s(-\mathrm{i}af)}{s(2\mathrm{i}\beta - \mathrm{i}af)}\right) \tag{2.106}$$

and as y varies over $\mathbb{R}$, f varies over $(0, 1-2ac/\pi)$. Now we may view the right-hand side of (2.106) as a function $F(a, f)$ defined for $a \in (0, \pi/2c)$ and $f \in (0, 1-2ac/\pi)$. Doing so, we assert that we have

$$F(a, f) = F(0, f) + \mathrm{O}(a) \qquad a \downarrow 0. \tag{2.107}$$

Here we have introduced

$$F(0, f) \equiv -2rf + c\cot(\pi f) \qquad f \in (0, 1) \tag{2.108}$$

and the bound is uniform on compacts in $(0, 1)$.

Taking the assertion just made for granted, we obtain a function $F(a, f)$ that is jointly continuous for $a \in [0, \pi/2c)$, $f \in (0, 1-2ac/\pi)$. Moreover, we have

$$(\partial_f F)(a, f) < 0 \qquad a \in [0, \pi/2c) \quad f \in (0, 1-2ac/\pi) \tag{2.109}$$

see also the paragraph containing (2.37). From this it readily follows that for a given $y \in \mathbb{R}$ the equation $y = F(a, f)$ has a unique solution $f = f(a, y)$, which is continuous in a for $a \in [0, \pi/2c)$.

We now prove our assertion (2.107). To this end we exploit the product representation (1.6). Recalling (1.23), it entails

$$\begin{aligned}\frac{s(2\mathrm{i}\beta - \mathrm{i}af)}{s(-\mathrm{i}af)} &= \exp(4r[\beta^2 + \mathrm{i}af]/a)\frac{\mathrm{sh}(2\pi\mathrm{i}\beta/a - \mathrm{i}\pi f)}{\mathrm{sh}(-\mathrm{i}\pi f)}\left(1 + \mathrm{O}\left(\exp\left(-\frac{2\pi^2}{ar}\right)\right)\right)\\ &= 1 + 4ra(1-f) + 2ac\cot(\pi f) + \mathrm{O}(a^2) \qquad a \downarrow 0\end{aligned} \tag{2.110}$$

where the bounds are uniform on compact subsets of the f-interval $(0, 1)$. Thus from (2.106) we have

$$F(a, f) = -2r + \frac{1}{2a}[4ra(1-f) + 2ac\cot(\pi f)] + \mathrm{O}(a) \qquad a \downarrow 0 \tag{2.111}$$

so that our assertion (2.107) follows.

To proceed, we use (1.6) once more to deduce

$$\frac{s(x + \mathrm{i}\beta - \mathrm{i}af)}{|s(x + \mathrm{i}\beta)|} = -\exp(-2\mathrm{i}rx - \mathrm{i}\pi f + 2\mathrm{i}frx) + \mathrm{O}(a) \qquad a \downarrow 0 \tag{2.112}$$

where the bound is uniform for (x, f) in compact subsets of $(0, \pi/r) \times (0, 1)$. In particular, choosing f equal to the above solution $f(a, y)$, we obtain

$$\begin{aligned}&\frac{s(x + \mathrm{i}\beta - \mathrm{i}af(a, y))}{|s(x + \mathrm{i}\beta)|}\exp(2\mathrm{i}rx + \mathrm{i}xy)\\ &\quad = -\exp[-\mathrm{i}\pi f(0, y) + 2\mathrm{i}f(0, y)rx + \mathrm{i}xy] + \mathrm{O}(a) \qquad a \downarrow 0.\end{aligned} \tag{2.113}$$

Here, the bound is uniform for x in a compact subset of $(0, \pi/r)$, and $f = f(0, y)$ is the unique solution of

$$y = -2rf + c\cot(\pi f). \tag{2.114}$$

To conclude the proof, we now rewrite f as

$$f \equiv \frac{1}{\pi}\mathrm{Arccot}(k/c) = \frac{1}{2} - \frac{1}{\pi}\mathrm{Arctg}(k/c) \qquad k \in \mathbb{R}. \tag{2.115}$$

Then one easily checks that (2.113) amounts to (2.103), whilst relation (2.114) turns into (2.104). □

It should be pointed out that the limit relation just proved is not strong enough to rigorously conclude that $\Phi_n(a; x)$ converges to $D_n(x)$ in the Hilbert space $\mathcal{H}$ (1.24). Indeed, it does not exclude that $|\Phi_n(a;x)|$ diverges as $a \downarrow 0$ and $x \downarrow 0$ or $x \uparrow \pi/r$. To see that the behaviour at the endpoints is a quite subtle matter, notice first of all that $\Phi_n(a;x)$ vanishes at $x = 0$ and $x = \pi/r$, since $\psi_n(x)$ (2.68) does. By contrast, $D_n(x)$ is non-zero for $x = 0$ and $x = \pi/r$, so that (2.102) is *false* for $x = 0$ and $x = \pi/r$. Similarly, (2.103) is false for $x = 0$ and $x = \pi/r$; for these x-values the left-hand side actually *diverges* as $a \downarrow 0$.

As a matter of fact, we do not know whether $|\Phi_n(a;x)|$ remains bounded on $[0, \pi/r]$ for $a \downarrow 0$. Even so, it can be seen that $\Phi_n(a;\cdot)$ does converge to $D_n(\cdot)$ in the $\mathcal{H}$-topology. Indeed, a reader familiar with Hilbert space estimates will have little difficulty verifying that for L^2-convergence to result from theorem 2.1, it is necessary and sufficient that one have

$$\lim_{a\downarrow 0} \int_0^{\pi/r} \mathrm{d}x\, |\Phi_n(a;x)|^2 = \int_0^{\pi/r} \mathrm{d}x\, |D_n(x)|^2. \tag{2.116}$$

Now both integrals are explicitly known from the norm formulae (2.80) and (2.100). A third application of the product representation (1.6) then shows that (2.116) holds true. As a consequence, one deduces Hilbert space convergence of the $g = 2$ eigenfunctions $\Phi_n(a;\cdot)$ to the delta-function eigenfunctions $D_n(\cdot)$.

To conclude this subsection, we would like to mention that (in contrast to the functions $\{\Phi_n(a;\cdot)\}_{n=0}^{\infty}$) the functions $\{D_n(\cdot)\}_{n=0}^{\infty}$ are known to be *complete* in $\mathcal{H}$. Indeed, this follows from a paper by Dorlas [14]; he actually proves completeness of the Bethe ansatz eigenfunctions for arbitrary N.

2.4. The one-gap picture

Taking $x \to x + \mathrm{i}a/2$ in the AΔO H_- (1.12) and using the AΔE (1.10), we obtain the formally self-adjoint operator

$$\tilde{H}_- = \left(\frac{s(x + \mathrm{i}a/2 - 2\mathrm{i}\beta)}{s(x + \mathrm{i}a/2)}\right)^{1/2} T_{\mathrm{i}\beta} \left(\frac{s(x - \mathrm{i}a/2 + 2\mathrm{i}\beta)}{s(x - \mathrm{i}a/2)}\right)^{1/2} + (\mathrm{i} \to -\mathrm{i}). \tag{2.117}$$

(Of course, the AΔO H_+ is invariant under $x \to x + \mathrm{i}a/2$.) Fixing β satisfying (1.20), the H_--eigenfunctions $\mathcal{F}(x, y)$ (2.66) give rise to $\tilde{H}_-$-eigenfunctions by shifting x to $x + \mathrm{i}a/2$. Omitting an irrelevant multiplicative constant, they can be written

$$\tilde{\mathcal{F}}(x, y) \equiv \tilde{w}(x)^{1/2} s(x - \mathrm{i}a/2 + z(y)) \exp(\mathrm{i}rx + \mathrm{i}yx) \tag{2.118}$$

$$\tilde{w}(x) \equiv 1/s(x - \mathrm{i}a/2 + \mathrm{i}\beta) s(x + \mathrm{i}a/2 - \mathrm{i}\beta). \tag{2.119}$$

Obviously, the functions $\tilde{\mathcal{F}}(x, y)$ belong to the Hilbert space $\mathcal{H}$ (1.24) for all real y. They satisfy

$$\tilde{\mathcal{F}}(x + \pi/r, y) = \exp(\mathrm{i}\pi y/r) \tilde{\mathcal{F}}(x, y) \tag{2.120}$$

so we obtain eigenfunctions with the same π/r-multiplier $\exp \mathrm{i}\theta$ by requiring

$$y \in r\theta/\pi + 2r\mathbb{Z} \qquad \theta \in (-\pi, \pi]. \tag{2.121}$$

Now we first choose $\beta \in (0, a/2)$. Then $z(y)$ is a single-valued real-analytic function $\mathbb{R} \to \mathrm{i}(\beta, a - \beta)$ (as we have shown in subsection 2.1), so we may introduce

$$\phi_j(\theta) \equiv \tilde{\mathcal{F}}(\cdot, r\theta/\pi + 2jr) \qquad \theta \in (-\pi, \pi] \quad j \in \mathbb{Z} \tag{2.122}$$

where the right-hand side is viewed as a vector in $\mathcal{H}$. Using (2.118) and (2.74), we now obtain the inner product

$$(\phi_j(\theta), \phi_k(\theta)) = I_{k-j}(a/2 - \beta, \mathrm{i}a/2 - z(r\theta/\pi + 2jr), \mathrm{i}a/2 - z(r\theta/\pi + 2kr)). \tag{2.123}$$

Taking $k \neq j$, one sees that the quantity μ (2.75) is not equal to 1. Using equations (2.76), (1.11) and (2.7), it can then straightforwardly be verified that

$$(\phi_j(\theta), \phi_k(\theta)) = 0 \qquad \theta \in (-\pi, \pi] \quad j \neq k \quad \beta \in (0, a/2). \tag{2.124}$$

(This result should be compared with (2.94)–(2.97).)

In words, we have just proved that the $\tilde{H}_-$-eigenvectors $\phi_k(\theta)$, $k \in \mathbb{Z}$, are pairwise orthogonal when $\beta \in (0, a/2)$. But as we shall show shortly, they are not complete in $\mathcal{H}$. (This is true in spite of the fact that their limits for $\beta \uparrow a/2$ are manifestly complete; recall $z(y) \to \mathrm{i}a/2$ for $\beta \to a/2$.) Before doing so, we study the choices (1.20) with $k > 0$, however.

Taking first $\beta \in (a/2, a)$, the function $z(y)$ is still a single-valued real-analytic function $\mathbb{R} \to \mathrm{i}(a - \beta, \beta)$. Thus the vectors $\phi_j(\theta)$ (2.122) are again well defined, and we obtain

$$(\phi_j(\theta), \phi_k(\theta)) = I_{k-j}(\beta - a/2, \mathrm{i}a/2 - z(r\theta/\pi + 2jr), \mathrm{i}a/2 - z(r\theta/\pi + 2kr)). \tag{2.125}$$

But when we now use the explicit formula (2.76), we find that the right-hand side is a non-zero multiple of $1 - \mu^2$. Thus we deduce

$$(\phi_j(\theta), \phi_k(\theta)) \neq 0 \qquad \theta \in (-\pi, \pi] \quad j \neq k \quad \beta \in (a/2, a). \tag{2.126}$$

Turning next to the β-interval $(a, 3a/2)$, we recall that $z(y)$ is not single-valued on $\mathbb{R}$ for β close to a and $y \in [r - d_0, r + d_0]$. For $j \in \mathbb{Z}$ such that $r\theta/\pi + 2jr$ does not belong to this interval, we may and shall define $\phi_j(\theta)$ by (2.122). For such integers we again obtain (2.125), and so we once again deduce non-orthogonality for $j \neq k$.

Now fixing $y \in (r - d_0, r + d_0)$, we obtain three distinct z-values, each of which defines a distinct vector in $\mathcal{H}$. Their inner products with vectors corresponding to the same θ, but y-values outside the critical interval, do not vanish, as can readily be established via the above calculation. Likewise, the three vectors are not pairwise orthogonal, the two vectors corresponding to $y = r - d_0$ are not orthogonal, and neither are the two $(r + d_0)$-vectors.

Of course, for β close to $3a/2$ we have $d_0 = 0$. Hence equation (2.122) is unambiguously defined for arbitrary $\theta \in (-\pi, \pi]$ and $j \in \mathbb{Z}$, yielding pairwise non-orthogonal vectors. Similarly, for $\beta \in (3a/2, 2a)$ no ambiguity occurs, and we can now use (2.81) with $M = 1$ to deduce

$$(\phi_j(\theta), \phi_k(\theta)) \neq 0 \qquad \theta \in (-\pi, \pi] \quad j \neq k \quad \beta \in (3a/2, 2a). \tag{2.127}$$

Clearly, this analysis can be extended to larger β-values. In particular, it is not hard to check that one has

$$(\tilde{\mathcal{F}}(\cdot, y), \tilde{\mathcal{F}}(\cdot, y + 2lr)) \neq 0 \qquad \beta \in \mathcal{B}_M \quad M > 0 \quad y > (2M - 1)r \quad l \in \mathbb{N}^*. \tag{2.128}$$

Thus far, we have only taken the eigenfunctions $\mathcal{H}(x, y)$, $y \in \mathbb{R}$, into account. We continue by studying the role of the band functions $\mathcal{H}^b(x, y)$ (2.20), keeping x real at first. Let us begin by comparing the ranges of the eigenvalues E_δ for the two choices of eigenfunctions. With (1.20) in force, we may use equations (2.28)–(2.29) when y varies over $\mathbb{R}$ in $\mathcal{H}(x, y)$. Omitting the positive prefactors, the resulting positive quantities

$$\epsilon_- \equiv (\wp(z) - \wp(\mathrm{i}\beta))^{-1/2} \tag{2.129}$$

$$\epsilon_+ \equiv \mathrm{ch}(2\mathrm{i}zr + ay + 2ar) \tag{2.130}$$

vary over $[(e_3 - \wp(\mathrm{i}\beta))^{-1/2}, \infty)$ and $[1, \infty)$, respectively.

Next, letting y vary over $[-3r, -r]$ in $\mathcal{H}^b(x, y)$, we still may use (2.29). But in (2.28) we should omit the factor $(-)^k$ when we take the positive square root (as we do throughout this paper). Indeed, from (2.8) one reads off that the band energies E_- flip sign as β passes the numbers $(M + 1/2)a$, $M \in \mathbb{N}$. (Recall $s(x) > 0$ for $x = \pi/2r + \mathrm{i}\lambda$ with λ real, cf equation (2.16).) Keeping this change in mind, we are again reduced to finding the ranges of ϵ_- and ϵ_+ for

$$(z, y) = (\pi/2r + i\gamma, f(\gamma)) \qquad \gamma \in [-a/2, a/2]. \tag{2.131}$$

The range of ϵ_- is plain: it is given by $[(e_1 - \wp(i\beta))^{-1/2}, (e_2 - \wp(i\beta))^{-1/2}]$. For ϵ_+ we obtain the range of the function

$$\epsilon_+(\gamma) \equiv -\operatorname{ch}(-2\gamma r + af(\gamma) + 2ar) \tag{2.132}$$

with γ varying over $[-a/2, a/2]$. Now we have already seen that $f(\gamma)$ is monotonically increasing. Recalling (2.19) and noting that $f(\gamma)$ is not linear in γ (since we require (1.20)), we obtain a range $[-\kappa, -1]$ for ϵ_+, where $\kappa > 1$ depends on β.

The upshot is that $E_\delta(y)$ varies over an unbounded interval $[E_\delta^{(3)}, \infty)$ for $\mathcal{H}(x, y)$ and over a band $[E_\delta^{(1)}, E_\delta^{(2)}]$ for $\mathcal{H}^b(x, y)$, with

$$0 < E_-^{(1)} < E_-^{(2)} < E_-^{(3)} \qquad \beta \in a(l, l + 1/2) \quad l \in \mathbb{N} \tag{2.133}$$

$$E_-^{(1)} < E_-^{(2)} < 0 < E_-^{(3)} \qquad \beta \in a(l + 1/2, l + 1) \quad l \in \mathbb{N} \tag{2.134}$$

$$E_+^{(1)} < E_+^{(2)} < 0 < E_+^{(3)} \qquad 2\beta \in a(k, k + 1) \quad k \in \mathbb{N}. \tag{2.135}$$

Turning to the excluded β-values, we can use (2.29) and (2.17) to deduce that the $\delta = +$ band shrinks to a point. To be specific, we obtain

$$E_+^{(1)} = E_+^{(2)} = -2\exp[2r(\beta - a)] \qquad E_+^{(3)} = 2\exp[2r(\beta - a)] \qquad \beta = ka/2 \quad k \in \mathbb{N}^*. \tag{2.136}$$

Similarly, for $\beta - a/2$ an integer multiple of a, we read off from (2.8) that the $\delta = -$ band shrinks to 0; the B_--eigenvalue on $\mathcal{H}_3(x)$ (2.27) yields the limit of $E_-^{(3)}$:

$$E_-^{(1)} = E_-^{(2)} = 0 \qquad E_-^{(3)} = 2\exp[2r\beta(\beta - a)/a] \qquad \beta = (M + 1/2)a \quad M \in \mathbb{N}. \tag{2.137}$$

Finally, when β is an integer multiple of a, we likewise obtain

$$\left.\begin{aligned} &E_-^{(1)} = E_-^{(2)} = 2(-)^M \exp[2r\beta(\beta - a)/a] \\ &E_-^{(3)} = 2\exp[2r\beta(\beta - a)/a] \end{aligned}\right\} \qquad \beta = Ma \quad M \in \mathbb{N}^*. \tag{2.138}$$

Having disposed of the algebra, we can proceed with analysis. Fixing $\beta \in (0, a/2)$, we recall that we have already proved that the vectors $\phi_j(\theta)$ (2.122) are pairwise orthogonal in $\mathcal{H}$ (1.24), cf equation (2.124). We now show that they do not yield a base for $\mathcal{H}$.

To this end we define the $\tilde{H}_-$-eigenfunctions (cf equations (2.118)–(2.119))

$$\tilde{\mathcal{F}}^b(x, y) \equiv \tilde{w}^{1/2}(x) s(x - ia/2 + \pi/2r + i\gamma(y)) \exp(irx + iyx) \qquad y \in (-3r, -r]. \tag{2.139}$$

Of these there is a unique function that has the π/r-multiplier $\exp(i\theta), \theta \in (-\pi, \pi]$. Specifically, this function reads

$$\phi^b(\theta) \equiv \tilde{\mathcal{F}}^b(x, r\theta/\pi - 2r) \qquad \theta \in (-\pi, \pi]. \tag{2.140}$$

The point is now that we have

$$(\phi^b(\theta), \phi_k(\theta)) = 0 \qquad \theta \in (-\pi, \pi] \qquad k \in \mathbb{Z} \qquad \beta \in (0, a/2). \tag{2.141}$$

Taking this for granted, it is plain that the vectors $\{\phi_k(\theta)\}_{k\in\mathbb{Z}}$ are not complete in $\mathcal{H}$, as asserted.

To substantiate (2.141), we invoke the integral (2.74) to write

$$(\phi^b(\theta), \phi_k(\theta)) = I_{k+1}(a/2 - \beta, ia/2 + \pi/2r - i\gamma(r\theta/\pi - 2r), ia/2 - z(r\theta/\pi + 2kr)). \tag{2.142}$$

The quantity μ (2.75) is negative in this case, so we can use equation (2.76) to calculate the right-hand side. Using equation (2.7), it is now routine to check that it vanishes, proving (2.141).

We would like to point out that the $\beta \uparrow a/2$ limits of the functions $\phi^b(\theta)$ exist, but do not belong to $\mathcal{H}$. (Indeed, for $x \in (0, \pi/r)$ one has $\tilde{w}(x)^{1/2} \to 1/s(x)$ as $\beta \uparrow a/2$, cf equation (2.119).) We leave a study of non-orthogonality properties of $\phi^b(\theta)$ for $\beta > a/2$ to the interested reader, and require $\beta \in (0, a/2)$ for the remainder of this subsection.

We conjecture that the vectors

$$\{\phi_k(\theta)\}_{k\in\mathbb{Z}} \qquad \phi^b(\theta) \qquad \theta \in (-\pi, \pi] \qquad \beta \in (0, a/2) \tag{2.143}$$

are complete in $\mathcal{H}$, and hence give rise to an orthogonal base. Taking this for granted, it follows in a well known way (cf [11, section XIII.16]) that we may also view $\tilde{H}_-$ as a self-adjoint operator on $L^2(\mathbb{R})$, with purely absolutely continuous spectrum

$$[E_-^{(1)}, E_-^{(2)}] \cup [E_-^{(3)}, \infty) \qquad E_-^{(j)} = \frac{\mathrm{i}s(2\mathrm{i}\beta)}{s(\mathrm{i}\beta)^2} \mathrm{e}^{-2\beta r}(e_j - \wp(\mathrm{i}\beta))^{-1/2} \quad j = 1, 2, 3 \tag{2.144}$$

of multiplicity two.

On account of equations (2.21), (2.22) and (2.27), the functions at the spectral boundary points may be taken to be

$$\tilde{\mathcal{F}}_1(x) \equiv \tilde{w}(x)^{1/2} s(x + \pi/2r + \mathrm{i}a/2) \exp \mathrm{i}rx \tag{2.145}$$

$$\tilde{\mathcal{F}}_2(x) \equiv \tilde{w}(x)^{1/2} s(x + \pi/2r) \tag{2.146}$$

$$\tilde{\mathcal{F}}_3(x) \equiv \tilde{w}(x)^{1/2} s(x). \tag{2.147}$$

Note that $\tilde{\mathcal{F}}_1$ is a π/r-periodic function, whereas $\tilde{\mathcal{F}}_2$ and $\tilde{\mathcal{F}}_3$ are π/r-antiperiodic; furthermore, $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ are even, while $\tilde{\mathcal{F}}_3$ is odd.

3. The hyperbolic case

We continue by studying the hyperbolic specialization. Thus $s(x)$ equals $a\pi^{-1}\,\mathrm{sh}(\pi x/a)$, cf equation (1.8). In this case it is convenient to employ the variables a_+ and a_- from I instead of β and a. We also use the notation

$$s_\delta(x) = \mathrm{sh}(\pi x/a_\delta) \qquad c_\delta(x) = \mathrm{ch}(\pi x/a_\delta) \qquad e_\delta(x) = \exp(\pi x/a_\delta) \qquad \delta = +, -. \tag{3.1}$$

To bring out the remarkable self-duality property of this limiting case, we switch to the new spectral variable

$$p \equiv a_+ a_- y/\pi. \tag{3.2}$$

Accordingly, B_- (1.16) and B_+ (1.17) turn into

$$B_- = \frac{s_-(x + \mathrm{i}a_+)}{s_-(x)} T_{\mathrm{i}a_+} + (\mathrm{i} \to -\mathrm{i}) \tag{3.3}$$

$$B_+ = -(T_{\mathrm{i}a_-} + (\mathrm{i} \to -\mathrm{i})). \tag{3.4}$$

Similarly, taking first $a_+ \in \mathcal{A}_M$, with

$$\mathcal{A}_M \equiv a_-(M, M + 1/2) \cup a_-(M + 1/2, M + 1) \qquad M \in \mathbb{N} \tag{3.5}$$

the joint eigenfunction (2.6) becomes

$$\mathcal{H}(x) = \pi^{-1} a_- s_-(x + z) \exp(\mathrm{i}\pi x p/a_+ a_-) \tag{3.6}$$

where z and p are related via

$$\frac{s_-(z - ia_+)}{s_-(z + ia_+)} = e_-(-2p). \tag{3.7}$$

This entails

$$p'(z) = -\frac{1}{2}\left(\operatorname{cth}\frac{\pi}{a_-}(z - ia_+) - \operatorname{cth}\frac{\pi}{a_-}(z + ia_+)\right)$$
$$= -\frac{i\sin(2\pi a_+/a_-)}{2s_-(z - ia_+)s_-(z + ia_+)}. \tag{3.8}$$

Hence, $p'(z)$ is non-zero and $p(z)$ decreases from ∞ to 0 as z goes from ia_+ to $i(M + 1/2)a_-$, cf also equations (2.10) and (2.11) with $r = 0$, $a = a_-$.

In the trigonometric case we will arrive at a relation similar to (3.7). We now digress to derive useful consequences of this type of relation, employing a standard form that is not cluttered by scale factors and reality restrictions.

Specifically, we start from a relation of the form

$$\frac{\operatorname{sh}(\alpha - t)}{\operatorname{sh}(\alpha + t)} = e^{-2\gamma} \qquad \alpha, \gamma, t \in \mathbb{C} \quad \alpha, \gamma, t \neq i\pi k/2 \quad k \in \mathbb{Z}. \tag{3.9}$$

In this formula α and γ appear to play different roles, but in fact (3.9) is equivalent to

$$\frac{\operatorname{sh}(\gamma - t)}{\operatorname{sh}(\gamma + t)} = e^{-2\alpha}. \tag{3.10}$$

Indeed, writing the left-hand side of (3.9) as $(\operatorname{th}\alpha \operatorname{ch} t - \operatorname{sh} t)/(\operatorname{th}\alpha \operatorname{ch} t + \operatorname{sh} t)$ and solving for $\operatorname{th}\alpha$, one obtains

$$\operatorname{th}\alpha \ \operatorname{th}\gamma = \operatorname{th} t. \tag{3.11}$$

Conversely, equation (3.11) entails (3.9) and (by symmetry) (3.10).

Yet another relation equivalent to (3.9) reads

$$\frac{2\operatorname{sh}\alpha \ \operatorname{ch} t}{\operatorname{sh}(\alpha + t)} = 1 + e^{-2\gamma}. \tag{3.12}$$

This will be used to get rid of the parameter z in eigenvalues. To get rid of z in eigenfunctions, we use the following consequence of (3.9):

$$2\operatorname{sh}(\eta + \alpha) = [\operatorname{sh}(\gamma + t)\operatorname{sh}(\gamma - t)]^{-1/2}(\operatorname{ch}(\eta + \gamma)e^{t} - \operatorname{ch}(\eta - \gamma)e^{-t}) \qquad \eta \in \mathbb{C}. \tag{3.13}$$

(This equation can be verified by writing $2\operatorname{sh}(\eta + \alpha) = e^{\eta}e^{\alpha} - e^{-\eta}e^{-\alpha}$, and then using (3.10) to write $e^{\pm\alpha}$ in terms of γ and t.) In the applications below the term in square brackets is positive and it is readily verified that the positive square root is needed.

Returning now to relation (3.7), we note that it is of the form (3.9), with $\alpha = \pi z/a_-$, $\gamma = \pi p/a_-$ and $t = i\pi a_+/a_-$. Invoking (3.13) with $\eta = \pi x/a_-$, we can write

$$2s_-(x + z) = [s_-(p + ia_+)s_-(p - ia_+)]^{-1/2}(qc_-(x + p) - \overline{q}c_-(x - p)) \tag{3.14}$$

where we have introduced the phase factor

$$q \equiv \exp(i\pi a_+/a_-). \tag{3.15}$$

Combining this with (3.6), it follows that the functions

$$K(x, p) \equiv 2[qc_-(x + p) - \overline{q}c_-(x - p)]\exp(i\pi xp/a_+a_-) \tag{3.16}$$

are B_δ-eigenfunctions, too. (These functions coincide with the functions $K_1(x, p)$ given by II(1.15).)

The function $K(x, p)$ is manifestly symmetric under $x \leftrightarrow p$ (*self-duality*), so it is also an eigenfunction of the AΔOs

$$\hat{B}_- = \frac{s_-(p + \mathrm{i}a_+)}{s_-(p)}\hat{T}_{\mathrm{i}a_+} + (\mathrm{i} \to -\mathrm{i}) \tag{3.17}$$

$$\hat{B}_+ = -(\hat{T}_{\mathrm{i}a_-} + (\mathrm{i} \to -\mathrm{i})) \tag{3.18}$$

where $\hat{T}_\alpha$ acts on functions of p by

$$(\hat{T}_\alpha F)(p) \equiv F(p - \alpha) \qquad \alpha \in \mathbb{C}. \tag{3.19}$$

The four eigenvalues involved can moreover be written

$$E_\delta = 2c_\delta(p) \qquad \hat{E}_\delta = 2c_\delta(x) \qquad \delta = +, -. \tag{3.20}$$

To substantiate the last assertion, we note that by symmetry one need only check $E_\delta = 2c_\delta(p)$. Now for $\delta = +$ this is evident from (3.18) and (3.16), whereas the $\delta = -$ result is not immediate, but can be obtained directly from (3.17) and (3.16). A quicker way, however, is to note that the elliptic formula (2.8) specializes to

$$E_- = 2\frac{s_-(z)}{s_-(z + \mathrm{i}a_+)}c_-(\mathrm{i}a_+)e_-(p). \tag{3.21}$$

Recalling (3.7) and the equivalence of (3.9) and (3.12), one deduces $E_- = e_-(p) + e_-(-p)$, as asserted.

With the constraint relation (3.7) eliminated, it is evident from (3.16) that we are free to choose $a_+ \in (0, \infty)$. Taking a_+ equal to $ka_-/2$, $k \in \mathbb{N}^*$, from (3.16) we obtain

$$K(x, p) = 2\mathrm{e}^{\mathrm{i}\pi k/2}[c_-(x + p) - (-)^k c_-(x - p)]\exp(\mathrm{i}\pi xp/a_+a_-) \qquad (a_+ = ka_-/2). \tag{3.22}$$

Hence we have

$$K(x, p) = c_k s_-(x)s_-(p)\exp(\mathrm{i}\pi xp/a_+a_-) \qquad (k \;\; \text{even}) \tag{3.23}$$

$$K(x, p) = c_k c_-(x)c_-(p)\exp(\mathrm{i}\pi xp/a_+a_-) \qquad (k \;\; \text{odd}). \tag{3.24}$$

Defining the weight functions

$$\hat{w}_{\mathrm{hyp},0}(u) \equiv 1/s_-(u)^2 \tag{3.25}$$

$$\hat{w}_{\mathrm{hyp},1/2}(u) \equiv 1/c_-(u)^2 \tag{3.26}$$

it is evident that a suitable multiple of $K(x, p)$ yields the kernel of a unitary operator from $L^2(\mathbb{R}, \hat{w}_{\mathrm{hyp},s}(p)\,\mathrm{d}p)$ onto $L^2(\mathbb{R}, \hat{w}_{\mathrm{hyp},s}(x)\,\mathrm{d}x)$, with $s = 0$ for k even and $s = 1/2$ for k odd.

Next, we introduce

$$\hat{w}_{\mathrm{hyp}}(u) \equiv 1/s_-(u + \mathrm{i}a_+)s_-(u - \mathrm{i}a_+). \tag{3.27}$$

Then it can be shown that for $a_+ \in \mathcal{A}_0$ (3.5) a suitable multiple of $K(x, p)$ (3.16) yields the kernel of a unitary operator from the odd subspace of $L^2(\mathbb{R}, \hat{w}_{\mathrm{hyp}}(p)\,\mathrm{d}p)$ onto the odd subspace of $L^2(\mathbb{R}, \hat{w}_{\mathrm{hyp}}(x)\,\mathrm{d}x)$, whereas isometry is violated on the even subspace; for $a_+ \in \mathcal{A}_M$ with $M > 0$ isometry is violated on both subspaces.

It should be noted that these results tie in with the elliptic orthogonality and non-orthogonality results obtained in subsection 2.2. The proofs of the assertions in the previous paragraph are, however, quite different, and involve some new machinery. This also applies to the Hilbert space results paralleling those in subsection 2.4, to which we now turn. (We will address these functional-analytic aspects elsewhere.)

Taking $x \to x + ia_-/2$ in the joint B_δ-eigenfunctions $K(x, p)$ (3.16), we obtain joint eigenfunctions of B_+ (3.4) and the 'crossed channel' AΔO

$$\tilde{B}_- = \frac{c_-(x + ia_+)}{c_-(x)} T_{ia_+} + (i \to -i) \tag{3.28}$$

with eigenvalues $2c_+(p)$ and $2c_-(p)$, respectively. Omitting a multiplicative constant, these can be written

$$\tilde{K}(x, p) = 2[qs_-(x + p) - \overline{q}s_-(x - p)] \exp(i\pi xp/a_+a_-). \tag{3.29}$$

Introducing

$$\tilde{w}_{\mathrm{hyp}}(x) \equiv 1/c_-(x + ia_+)c_-(x - ia_+) \tag{3.30}$$

we now detail the state of affairs concerning isometry properties. First, we choose $a_+ \in (0, a_-/2)$. Then a suitable multiple of $\tilde{K}(x, p)$ yields the kernel of a unitary operator from the even subspace of $L^2(\mathbb{R}, \hat{w}_{\mathrm{hyp}}(p)\,\mathrm{d}p)$ onto the odd subspace of $L^2(\mathbb{R}, \tilde{w}_{\mathrm{hyp}}(x)\,\mathrm{d}x)$. The odd subspace of $L^2(\mathbb{R}, \hat{w}_{\mathrm{hyp}}(p\,\mathrm{d}p)$ is mapped isometrically onto the orthocomplement in the even subspace of $L^2(\mathbb{R}, \tilde{w}_{\mathrm{hyp}}(x)\,\mathrm{d}x)$ of the constant functions. (Note that $\tilde{B}_-$ has eigenvalue $2\cos(\pi a_+/a_-)$ on the latter, while B_+ has eigenvalue -2.)

For $a_+ > a_-/2$ and $a_+ \neq ka_-/2$, $k \in \mathbb{N}$, these isometry properties break down. Again, this is analogous to our elliptic results, cf subsection 2.4. Observe also that the even bound state

$$\tilde{K}(x) \equiv 1 \tag{3.31}$$

can be viewed as the limit of the band functions $\mathcal{H}^b(x, y)$ for $r \downarrow 0$. More precisely, from the product representation (1.6) one easily deduces

$$\lim_{r\downarrow 0} c_r\, s(r, a; \pi/2r + x) = 1 \qquad c_r \equiv \exp(-\pi^2/4ar)\pi/2a \qquad \text{(uniformly on compacts)}. \tag{3.32}$$

Thus, when the functions $\mathcal{H}^b(x, y)$ (2.20) are multiplied by the renormalizing constant c_r, they all converge to 1 as $r \downarrow 0$, uniformly on x-compacts.

To conclude this section, we consider the relation of the above functions to the infinite-volume delta-function potential eigenfunctions. Of course, the formulae in subsection 2.3 are easily specialized for $r = 0$, but the somewhat involved reasoning in the proof of theorem 2.1 can be bypassed by taking $K(x, p)$ (3.16) as a starting point.

Indeed, when one substitutes

$$a_+ \to a_- - a_-^2 c/\pi \tag{3.33}$$

in the function

$$E(x, k) \equiv \frac{K(x, ka_+a_-/\pi)}{4|s_-(x + ia_+)s_-(ka_+a_-/\pi + ia_+)|} \qquad k \in \mathbb{R} \tag{3.34}$$

then it is quite easy to check directly that one has

$$E(\pm x, k) = i\left(\frac{c \pm ik}{c \mp ik}\right)^{1/2} \exp(\pm ixk) + O(a_-) \qquad x \in (0, \infty) \quad a_- \downarrow 0. \tag{3.35}$$

Furthermore, the crossed channel eigenfunction

$$\tilde{E}(x, k) \equiv \frac{\tilde{K}(x, ka_+a_-/\pi)}{4|c_-(x + ia_+)s_-(ka_+a_-/\pi + ia_+)|} \qquad k \in \mathbb{R} \tag{3.36}$$

has almost the same limiting behaviour:

$$\tilde{E}(\pm x, k) = \pm \mathrm{i} \left(\frac{c \pm \mathrm{i}k}{c \mp \mathrm{i}k} \right)^{1/2} \exp(\pm \mathrm{i}xk) + \mathrm{O}(a_-) \qquad x \in (0, \infty) \quad a_- \downarrow 0. \tag{3.37}$$

There is yet a second, essentially different way to tie in $\tilde{K}(x, p)$ (3.29) with the delta-function eigenfunctions, however. Specifically, let us put

$$a_+ \to a_-/2 - a_-^2 c/2\pi \tag{3.38}$$

in the function

$$F(x, k) \equiv \frac{\tilde{K}(xa_+a_-/\pi, k)}{4|c_-(xa_+a_-/\pi + \mathrm{i}a_+)s_-(k + \mathrm{i}a_+)|} \qquad x \in \mathbb{R}. \tag{3.39}$$

Then it follows from the same calculation as before that one has

$$F(x, \pm k) = \pm \left(\frac{c \pm \mathrm{i}x}{c \mp \mathrm{i}x} \right)^{1/2} \exp(\pm \mathrm{i}xk) + \mathrm{O}(a_-) \qquad k \in (0, \infty) \quad a_- \downarrow 0. \tag{3.40}$$

As is well known, the odd part of the 'distinguishable particle' delta-function transform yields the sine-transform on $L^2((0, \infty))$, whereas the even part yields the unitary operator on $L^2((0, \infty))$ with kernel

$$D(y, p) = \mathrm{i}(2\pi)^{-1/2} \left(\frac{c + ip}{c - ip} \right)^{1/2} \exp(\mathrm{i}yp) + \mathrm{c.c.} \qquad y, p \in (0, \infty). \tag{3.41}$$

In view of the above limits, one needs the *odd* transform associated with K and the *even* transforms associated with $\tilde{K}$ to obtain the kernel D in the pertinent limits. It should be stressed that for none of these scaling limits there is an operator in sight that has the formal limit

$$H_{\mathrm{delta}} \equiv -\mathrm{d}^2/\mathrm{d}y^2 + 2c\delta(y) \tag{3.42}$$

of which $D(y, p)$ is an eigenfunction with eigenvalue p^2. Rather, the existence of the transform limits was suggested by considerations from scattering theory, cf [3, section 4C].

4. The trigonometric case

We proceed by studying the trigonometric specialization of the above elliptic results. Thus $s(x)$ reduces to $r^{-1} \sin rx$, cf equation (1.9). The $a \to \infty$ limit of the AΔO B_+ (1.17) does not exist, so we wind up with the AΔO B_- (1.16). Omitting the prefactor $\exp(-2\beta r)$, we obtain the trigonometric AΔO

$$B = \frac{\sin r(x + \mathrm{i}\beta)}{\sin rx} T_{\mathrm{i}\beta} + (\mathrm{i} \to -\mathrm{i}). \tag{4.1}$$

Instead of (1.20), we now have only one β-interval $(0, \infty)$, and the equations (2.9), (2.10) and (2.13) have no trigonometric counterparts.

Choosing first $z = \pi/2r + \mathrm{i}\gamma$, $\gamma \in \mathbb{R}$, in (the trigonometric specialization of) the constraint (2.7), we obtain

$$\frac{\mathrm{ch}\, r(\gamma - \beta)}{\mathrm{ch}\, r(\gamma + \beta)} = \mathrm{e}^{-4\beta r - 2\beta y}. \tag{4.2}$$

Thus we find a uniquely determined solution $y = f(\gamma) \in (-3r, -r)$, yielding a B-eigenfunction

$$\mathcal{H}^b(x, y) = r^{-1} \cos r(x + \mathrm{i}\gamma(y)) \mathrm{e}^{2\mathrm{i}rx + \mathrm{i}xy} \qquad y \in (-3r, -r). \tag{4.3}$$

Clearly, we have

$$\mathcal{H}_1(x) \equiv \mathcal{H}^b(x, -2r) = r^{-1}\cos rx \tag{4.4}$$

$$\mathcal{H}_2(x) \equiv 2r \lim_{y\uparrow -r} \mathrm{e}^{-r\gamma(y)}\,\mathcal{H}^b(x, y) = 2r \lim_{y\downarrow -3r} \mathrm{e}^{r\gamma(y)}\,\mathcal{H}^b(x, y) = 1. \tag{4.5}$$

Next choosing $z \in \mathrm{i}(\beta, \infty)$, the constraint (2.7) entails

$$y'(z) = \frac{r\sin(2\mathrm{i}\beta r)}{-2\beta\sin(r(z-\mathrm{i}\beta))\sin(r(z+\mathrm{i}\beta))}. \tag{4.6}$$

Consequently, $y'(z)$ is non-zero and $y(z)$ decreases from ∞ to $-r$ as z goes from $\mathrm{i}\beta$ to $\mathrm{i}\infty$. Switching to the new parameter

$$\kappa \equiv -\mathrm{i}z \in (\beta, \infty) \tag{4.7}$$

we obtain the B-eigenfunctions

$$\mathcal{H}(x, y) = r^{-1}\sin r(x + \mathrm{i}\kappa(y))\mathrm{e}^{\mathrm{i}x(y+2r)} \qquad y \in (-r, \infty) \tag{4.8}$$

with $\kappa(y)$ uniquely determined by

$$\frac{\operatorname{sh} r(\kappa-\beta)}{\operatorname{sh} r(\kappa+\beta)} = \mathrm{e}^{-2\beta(y+2r)} \qquad y \in (-r, \infty). \tag{4.9}$$

Comparing equations (4.9) and (3.9), we obtain equality for $\alpha = \kappa r$, $\gamma = \beta(y + 2r)$ and $t = \beta r$. From (3.13) (with $\eta = -\mathrm{i}rx$) it then follows that we may write

$$\begin{aligned}\mathcal{H}(x, y) = \mathrm{i}(2r)^{-1}\big[\operatorname{sh}\beta(y+3r)\operatorname{sh}\beta(y+r)\big]^{-1/2}\big[&\operatorname{ch}(\beta(y+2r) - \mathrm{i}rx)\mathrm{e}^{\beta r} \\ &- \operatorname{ch}(\beta(y+2r) + \mathrm{i}rx)\mathrm{e}^{-\beta r}\big]\mathrm{e}^{\mathrm{i}x(y+2r)}.\end{aligned} \tag{4.10}$$

Similarly, specializing (2.8) we deduce from (3.12) that the associated eigenvalue can be rewritten

$$E = 2\operatorname{ch}\beta(y + 2r). \tag{4.11}$$

Though we have assumed $y \in (-r, \infty)$ in deriving (4.10) and (4.11), it follows from (2.11) and (2.14) that we also obtain a B-eigenfunction (4.10) with eigenvalue (4.11) for $y \in (-\infty, -3r)$. Moreover, it is not hard to check that for $y \in (-3r, -r)$ the function $\mathcal{H}(x, y)$ (4.10) amounts to the eigenfunction $\mathcal{H}^b(x, y)$ (4.3), so that the B-eigenvalue on $\mathcal{H}^b(x, y)$ is once again given by (4.11).

Next, we consider the eigenfunctions

$$\psi_n(x) = \mathcal{H}(x, nr) - \mathcal{H}(-x, nr) \qquad n \in \mathbb{N} \tag{4.12}$$

in relation to the Hilbert space

$$\mathcal{H}_{\hat{w}} = L^2((0, \pi/r), r^2[\sin r(x+\mathrm{i}\beta)\sin r(x-\mathrm{i}\beta)]^{-1}\,\mathrm{d}x). \tag{4.13}$$

Specializing (2.72) and (2.80), one readily obtains

$$(\psi_n, \psi_m) = \begin{cases} 0 & n \neq m \\ 2\pi/r & n = m. \end{cases} \tag{4.14}$$

More is true: the B-eigenfunctions ψ_n are in fact an orthogonal base for $\mathcal{H}_{\hat{w}}$.

To prove this, it suffices to show that the functions

$$P_n(x) \equiv \big[\sin r(x+\mathrm{i}\beta)\sin rx\sin r(x-\mathrm{i}\beta)\big]^{-1}\psi_n(x) \tag{4.15}$$

are total in the Hilbert space

$$L^2((0, \pi/r), \sin r(x+\mathrm{i}\beta)\sin^2 rx\sin r(x-\mathrm{i}\beta)\,\mathrm{d}x). \tag{4.16}$$

We switch to the functions $P_n(x)$, since they are polynomials in $\cos rx$ of degree n. Taking this assertion for granted, it is plain that $P_0, P_1, \ldots$, span the space (4.16).

To prove the assertion, we note that $P_n(x)$ is clearly a rational function of the variable $z = e^{irx}$. The poles of the prefactor in (4.15) at $z = \pm 1, \pm e^{\beta r}$ and $\pm e^{-\beta r}$ are cancelled by zeros of ψ_n, so that $P_n(x)$ equals a Laurent polynomial $Q_n(z, z^{-1})$. Now $P_n(x)$ is even in x, so Q_n is invariant under $z \leftrightarrow z^{-1}$. Hence Q_n may be viewed as a polynomial in $z + z^{-1} = 2\cos rx$. Taking $z \to \infty$ one sees that the pertinent degree is n, so our assertion follows.

As a consequence of the orthogonal base property, it follows that the AΔO B gives rise to a self-adjoint operator in the Hilbert space $\mathcal{H}_{\hat{w}}$. It should be stressed that it is the polynomial character of the functions $P_n(x)$ that renders completeness obvious in the trigonometric case. There is no analog of this feature at the elliptic level, which is why completeness of the functions $\{\psi_n\}_{n=0}^{\infty}$ is left open in that case.

Specializing the non-orthogonality results for the even eigenfunctions $\mathcal{H}(x, y) + \mathcal{H}(-x, y)$ with $y = nr$, $n \in \mathbb{N}$, and for the Floquet/Bloch eigenfunctions (cf the end of subsection 2.2), we also obtain non-orthogonality in the trigonometric regime. On the other hand, the former eigenfunctions are obviously polynomials R_k of degree $k = n + 3$ in $\cos rx$, cf equation (4.10). Therefore, one might be inclined to believe that there exists a weight function $W(x)$ on $(0, \pi/r)$ that differs from $\hat{w}(x)$, such that the orthogonal polynomials associated with $W(x)$ yield B-eigenfunctions coinciding with R_k for $k \geqslant 3$.

This contingency can be ruled out, however. To be sure, for $y = -2r$ the eigenfunction $\mathcal{H}(x, y)$ (4.10) reduces to a multiple of $\cos rx$, and omitting the square-root factor one can put $y = -3r$ or $y = -r$ to obtain a constant eigenfunction, cf also equations (4.4) and (4.5). Thus B does admit polynomial eigenfunctions R_k of degrees $k = 0, 1, 3, 4, \ldots$. But we claim that B has no degree-two polynomial as an eigenfunction.

Indeed, a straightforward calculation yields

$$B\cos^2(rx) = 2\,\mathrm{ch}(\beta r)(\cos^2(rx) - \mathrm{sh}^2(\beta r)). \tag{4.17}$$

Thus, B has a non-trivial Jordan form in the invariant vector space spanned by the two functions $\cos^2 rx$ and 1, and so our claim follows.

We proceed by detailing a connection between the hyperbolic and trigonometric settings, which naturally leads to duality properties of the latter. First, we observe that the hyperbolic eigenfunction $\mathcal{H}(x)$ (3.6) gives rise to the trigonometric eigenfunction $\mathcal{H}(x, y)$ (4.8) via the substitutions

$$a_+ \to \beta \qquad a_- \to \pi/\mathrm{i}r \qquad \pi p/a_- \to \beta(y + 2r) \qquad z \to \mathrm{i}\kappa. \tag{4.18}$$

Comparing equations (3.3) and (4.1), we see that (4.18) entails

$$B_- \to B \tag{4.19}$$

in agreement with (4.11). Similarly, from (3.4) we obtain

$$B_+ \to -Q \tag{4.20}$$

where Q is the quasi-periodicity AΔO

$$Q \equiv T_{\pi/r} + T_{-\pi/r}. \tag{4.21}$$

Next substituting (4.18) in $K(x, p)$ (3.16) and in the AΔOs $\hat{B}_-$ (3.17) and $\hat{B}_+$ (3.18), we obtain

$$L(x, y) = 2[e^{-\beta r}\,\mathrm{ch}(\mathrm{i}rx + \beta(y + 2r)) - e^{\beta r}\,\mathrm{ch}(\mathrm{i}rx - \beta(y + 2r))]e^{\mathrm{i}x(y+2r)} \tag{4.22}$$

$$\tilde{B}_- = \frac{\mathrm{sh}(\beta(y + 3r))}{\mathrm{sh}(\beta(y + 2r))}\tilde{T}_r + \frac{\mathrm{sh}(\beta(y + r))}{\mathrm{sh}(\beta(y + 2r))}\tilde{T}_{-r} \tag{4.23}$$

$$\tilde{B}_+ = -(\tilde{T}_{\mathrm{i}\pi/\beta} + (\mathrm{i} \to -\mathrm{i})) \tag{4.24}$$

where $\tilde{T}_\alpha$ acts on meromorphic functions of y by

$$(\tilde{T}_\alpha G)(y) = G(y - \alpha) \qquad \alpha \in \mathbb{C}. \tag{4.25}$$

The eigenvalues of $\tilde{B}_-$ and $\tilde{B}_+$ on $L(x, y)$ read

$$\tilde{E}_- = 2\cos rx \qquad \tilde{E}_+ = 2\,\mathrm{ch}(\pi x/\beta). \tag{4.26}$$

Since we clearly have (cf equation (4.10))

$$\mathcal{H}(x, y) = (4\mathrm{i}r)^{-1}[\mathrm{sh}(\beta(y+3r))\,\mathrm{sh}(\beta(y+r))]^{-1/2} L(x, y) \tag{4.27}$$

the function $\mathcal{H}(x, y)$ is an eigenfunction of the analytic difference operator

$$\tilde{H}_- = \left(\frac{\mathrm{sh}\,\beta(y+3r)}{\mathrm{sh}\,\beta(y+2r)}\right)^{1/2} \tilde{T}_r \left(\frac{\mathrm{sh}\,\beta(y+r)}{\mathrm{sh}\,\beta(y+2r)}\right)^{1/2} + \left(\frac{\mathrm{sh}\,\beta(y+r)}{\mathrm{sh}\,\beta(y+2r)}\right)^{1/2} \tilde{T}_{-r} \left(\frac{\mathrm{sh}\,\beta(y+3r)}{\mathrm{sh}\,\beta(y+2r)}\right)^{1/2} \tag{4.28}$$

with eigenvalue $\tilde{E}_-$. Hence it follows that $(\psi_0(x), \psi_1(x), \ldots)$ is an improper eigenfunction of the *discrete* difference operator

$$D = \left(\frac{\mathrm{sh}(n+3)\beta r}{\mathrm{sh}(n+2)\beta r}\right)^{1/2} S \left(\frac{\mathrm{sh}(n+1)\beta r}{\mathrm{sh}(n+2)\beta r}\right)^{1/2} + \text{h.c.} \tag{4.29}$$

on the Hilbert space $l^2(\mathbb{N})$. Here, S is the right shift

$$(Sf)_n \equiv \begin{cases} 0 & n = 0 \\ f_{n-1} & n > 0 \end{cases} \tag{4.30}$$

with $f = (f_0, f_1, \ldots) \in l^2(\mathbb{N})$, and h.c. stands for Hermitian conjugate. To be quite precise, $(r/2\pi)^{1/2}\psi_n(x)$ may be viewed as the kernel of a unitary operator from $l^2(\mathbb{N})$ onto $\mathcal{H}_{\hat{w}}$ (4.13), diagonalizing the bounded self-adjoint operator D as multiplication by $2\cos rx$, cf also theorem IV.1 in II.

5. The rational case

The rational specialization of the above can be most easily obtained by letting $r \downarrow 0$ in the trigonometric quantities. To begin with, this yields the rational AΔO

$$B = \frac{x + \mathrm{i}\beta}{x} T_{\mathrm{i}\beta} + (\mathrm{i} \to -\mathrm{i}) \tag{5.1}$$

cf equation (4.1). The (renormalized) band functions (4.3) reduce to constant functions, on which B has eigenvalue 2. From equations (4.8) and (4.9) we get B-eigenfunctions

$$\mathcal{H}(x, y) = (x + \mathrm{i}\kappa(y))\mathrm{e}^{\mathrm{i}xy} \qquad y \in (0, \infty) \tag{5.2}$$

with $\kappa(y)$ given by

$$\frac{\kappa - \beta}{\kappa + \beta} = \mathrm{e}^{-2\beta y} \qquad y \in (0, \infty). \tag{5.3}$$

Moreover, the B-eigenvalue reads

$$E = 2\,\mathrm{ch}(\beta y) \tag{5.4}$$

cf equation (4.11).

Taking $r \downarrow 0$ in (4.10), we can eliminate κ altogether:

$$\mathcal{H}(x, y) = (x + \mathrm{i}\beta\mathrm{cth}\,(\beta y))\mathrm{e}^{\mathrm{i}xy}. \tag{5.5}$$

Alternatively, this formula follows directly from the constraint (5.3), since the latter amounts to

$$\kappa(y) = \beta \operatorname{cth}(\beta y). \tag{5.6}$$

The odd combination

$$\psi(x, y) \equiv \mathcal{H}(x, y) - \mathcal{H}(-x, y) = 2x\cos(xy) - 2\beta \operatorname{cth}(\beta y)\sin(xy) \tag{5.7}$$

gives rise to an isometry from $L^2((0,\infty), \mathrm{d}y)$ onto $L^2((0,\infty), (x^2+\beta^2)^{-1}\,\mathrm{d}x)$ (after multiplication by a suitable constant). The even combination has a non-integrable singularity as $y \downarrow 0$, and so it does not yield an isometry.

Turning to duality properties, we note first that $\mathcal{H}(x, y)$ is an eigenfunction of the AΔO

$$\tilde{H}_+ = \tilde{T}_{\mathrm{i}\pi/\beta} + (\mathrm{i} \to -\mathrm{i}) \tag{5.8}$$

with eigenvalue

$$\tilde{E}_+ = 2\operatorname{ch}(\pi x/\beta). \tag{5.9}$$

It is also easy to verify directly that $\mathcal{H}(x, y)$ is an eigenfunction of the *differential* operator

$$\tilde{H}_-^{(0)} \equiv -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{2\beta^2}{\operatorname{sh}^2(\beta y)} \tag{5.10}$$

with eigenvalue

$$\tilde{E}_-^{(0)} = x^2. \tag{5.11}$$

This can be understood from $\mathcal{H}(x, y)$ (4.8) being an eigenfunction of the AΔO $\tilde{H}_-$ (4.28) with eigenvalue $2\cos rx$: writing $\tilde{T}_r = \exp(-r\mathrm{d}/\mathrm{d}y)$, subtracting 2 and dividing by $-r^2$, we obtain $\tilde{H}_-^{(0)}$ and $\tilde{E}_-^{(0)}$ for $r \downarrow 0$, respectively.

Of course, $\tilde{H}_-^{(0)}$ amounts to the hyperbolic specialization of the non-relativistic Lamé operator $H_{\mathrm{nr}}(2)$ (1.1). This state of affairs can also be understood from a study of the non-relativistic limit, with which we now proceed.

6. The non-relativistic limit

We conclude this paper by studying the non-relativistic limit $\beta \downarrow 0$. Beginning with the elliptic case, we subtract 1 from the left-hand side and right-hand side of (2.7), divide by β, and let $\beta \downarrow 0$ to obtain

$$\mathrm{i}s'(z)/s(z) = y + 2r. \tag{6.1}$$

Thus the function (2.6) has limit

$$\mathcal{H}_0(x) = s(x+z)\exp(-xs'(z)/s(z)). \tag{6.2}$$

Clearly, it is an eigenfunction of the $\beta \downarrow 0$ limit

$$B_+^{(0)} = -\mathrm{e}^{-3ar}(\mathrm{e}^{-2\mathrm{i}rx}T_{\mathrm{i}a} + (\mathrm{i} \to -\mathrm{i})) \tag{6.3}$$

of B_+ (1.17), with eigenvalue

$$E_+^{(0)} = 2\mathrm{e}^{-2ar}\operatorname{ch}(2\mathrm{i}zr + \mathrm{i}as'(z)/s(z)). \tag{6.4}$$

Writing $T_{\mathrm{i}\beta} = \exp(-\mathrm{i}\beta\mathrm{d}/\mathrm{d}x)$ in $\mathrm{e}^{2\beta r}B_-$, subtracting 2 and dividing by β^2, we obtain the $\beta \downarrow 0$ limit

$$B_-^{(0)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{s''(x)}{s(x)} + 2\frac{s'(x)}{s(x)}\frac{\mathrm{d}}{\mathrm{d}x}. \tag{6.5}$$

Using equations (2.8) and (2.7) to expand $\beta^{-2}(e^{2\beta r}E_- - 2)$, it readily follows that $\mathcal{H}_0(x)$ is a $B_-^{(0)}$-eigenfunction with eigenvalue

$$E_-^{(0)} = -\wp(z) + 4\eta r/\pi. \tag{6.6}$$

(To verify this directly is not a trivial matter.)

The weight function $\hat{w}(x)$ (1.14) has limit

$$\hat{w}_0(x) = 1/s(x)^2. \tag{6.7}$$

Setting

$$H_\delta^{(0)} \equiv \hat{w}_0(x)^{1/2} B_\delta^{(0)} \hat{w}_0(x)^{-1/2} \qquad \delta = +, - \tag{6.8}$$

we obtain

$$\begin{aligned} H_-^{(0)} &= -\frac{d^2}{dx^2} - 2\frac{s''(x)}{s(x)} + 2\left(\frac{s'(x)}{s(x)}\right)^2 \\ &= -\frac{d^2}{dx^2} + 2[\wp(x) + 2\eta r/\pi] \\ &= H_{\text{nr}}(2) + 4\eta r/\pi \end{aligned} \tag{6.9}$$

and also

$$H_+^{(0)} = e^{-2ar} T_{ia} + (i \to -i). \tag{6.10}$$

One can easily check that the constraint (6.1) and the eigenvalues (6.4) and (6.6) are invariant under (2.10)–(2.12), and that the transformation properties (2.13)–(2.15) still hold when $\mathcal{H}(x)$ is replaced by $\mathcal{H}_0(x)$. Choosing first $z \in \pi/2r + i\mathbb{R}$, we may use (2.16) to obtain $y = f_0(\gamma) \in \mathbb{R}$, with

$$f_0(\gamma) \equiv -2r + r\,\text{th}(r\gamma) + 4r \sum_{k=1}^{\infty} \frac{\exp(-2kar)\,\text{sh}(2r\gamma)}{1 + \exp(-4kar) + 2\exp(-2kar)\,\text{ch}(2r\gamma)}. \tag{6.11}$$

Hence f_0 is monotonically increasing and maps $\mathbb{R}$ onto $\mathbb{R}$. It is not hard to see that f_0 satisfies (2.19), so we need only choose $\gamma \in [-a/2, a/2]$ and $y \in [-3r, -r]$, as before. The band eigenfunctions are then again given by

$$\mathcal{H}_0^b(x, y) = s(x + \pi/2r + i\gamma(y))e^{2irx + ixy} \qquad y \in [-3r, -r] \tag{6.12}$$

with $\gamma(y)$ the inverse of $f_0(\gamma)$; also, (2.21) and (2.22) still apply when $\mathcal{H}^b$ is replaced by $\mathcal{H}_0^b$.

Letting next z ascend the imaginary axis from 0 to ia, it is clear that y varies over $\mathbb{R}$ from ∞ to $-\infty$, with $y(ia/2) = -r$; furthermore, $y(z)$ is monotonically decreasing in view of (2.32) and (2.34). Denoting the inverse by $z_0(y)$, we get joint $B_\delta^{(0)}$-eigenfunctions

$$\mathcal{H}_0(x, y) = s(x + z_0(y))e^{2irx + ixy} \qquad y \in \mathbb{R}. \tag{6.13}$$

As before, we have

$$\mathcal{H}_3(x) \equiv \mathcal{H}_0(x, -r) = s(x + ia/2)e^{irx} \tag{6.14}$$

cf equation (2.27).

In contrast to the relativistic case, the $H_{\text{nr}}(2)$-eigenfunctions $\hat{w}_0(x)^{1/2}\mathcal{H}_0(x)$ are not square-integrable over $(0, \pi/r)$, since $1/s(x)^2$ has a non-integrable singularity at $x = 0$ and $x = \pi/r$. Consider next the functions

$$\begin{aligned} \Phi_n(x) &\equiv \hat{w}_0(x)^{1/2}[\mathcal{H}_0(x, nr) - \mathcal{H}_0(-x, nr)] \\ &= \frac{1}{s(x)}[s(x + z_n)e^{i(n+2)rx} + s(x - z_n)e^{-i(n+2)rx}] \qquad x \in (0, \pi/r) \end{aligned} \tag{6.15}$$

with $z_n \equiv z_0(nr) \in \mathrm{i}(0, a/2)$ given by

$$\mathrm{i}s'(z_n)/s(z_n) = (n+2)r \qquad n \in \mathbb{N}. \tag{6.16}$$

The function in square brackets vanishes at $x = 0$ and is π/r-periodic (antiperiodic) for n odd (even). Hence $\Phi_n(x)$ does give rise to a vector in the Hilbert space $\mathcal{H}$ (1.24). Note that (the analytic continuation of) $\Phi_n(x)$ is even, as opposed to its relativistic generalization $\hat{w}(x)^{1/2}\psi_n(x)$, which is odd for real x. (This parity change can be readily understood by comparing the functions $x \mapsto x^2$ and $x \mapsto x(x^2+\epsilon^2)^{1/2}$, $\epsilon > 0$.)

It is not hard to see that the differential operator $H_{\mathrm{nr}}(2)$ gives rise to an essentially self-adjoint operator on the subspace $C_0^\infty((0, \pi/r))$ of $\mathcal{H}$, and that the vectors $\Phi_n(\cdot)$ are in the domain of the self-adjoint closure $\overline{H}_{\mathrm{nr}}(2)$. Using the Weyl–Kodaira–Titchmarsh theory we expect one can show they are actually an orthogonal base of eigenvectors for $\overline{H}_{\mathrm{nr}}(2)$, but to our knowledge the details have not been worked out in the literature. (Of course, only completeness is at issue; orthogonality is plain in the differential operator setting.)

Taking completeness for granted, we can exploit the eigenvectors to associate a self-adjoint Hamiltonian $\overline{H}_+^{(0)}$ with the AΔO $H_+^{(0)}$ (6.10), by setting (cf equation (6.4))

$$\overline{H}_+^{(0)}\Phi_n \equiv E_{+,n}^{(0)}\Phi_n \qquad E_{+,n}^{(0)} = 2\mathrm{e}^{-2ar}\,\mathrm{ch}(2\mathrm{i}z_n r + (n+2)ar) \qquad n \in \mathbb{N} \tag{6.17}$$

extending linearly, and taking the closure. It should be observed that the AΔO $H_+^{(0)}$ has constant coefficients, whereas the eigenfunctions $\Phi_n(x)$ are not in any sense 'free'.

To be sure, we have a similar state of affairs at the relativistic level, cf H_- (1.12) and H_+ (1.13). In that case, though, the pertinent eigenfunctions $\hat{w}(x)^{1/2}\psi_n(x)$ of the defining AΔO H_- are singled out in the infinite-dimensional eigenfunction space by requiring that they be eigenfunctions of the 'free' AΔO H_+, too, cf the end of subsection 2.1. By contrast, for the Schrödinger operator $H_{\mathrm{nr}}(2)$ the eigenfunctions span a two-dimensional space, and self-adjointness requirements uniquely determine the relevant eigenfunctions. To our knowledge, the existence of a self-adjoint, commuting operator $\overline{H}_+^{(0)}$ with a very simple action (namely, by the AΔO $H_+^{(0)}$ (6.10)) on a core for $\overline{H}_{\mathrm{nr}}(2)$ has not been observed before, neither for the elliptic potential $2\wp(x)$ nor for its hyperbolic specialization, which we study below.

Before doing so, we add some remarks concerning the band functions (6.12). First, we recall their role in the spectral analysis of the operator

$$\tilde{H}_{\mathrm{nr}}(2) \equiv -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\wp(x + \mathrm{i}a/2) \tag{6.18}$$

viewed as a self-adjoint operator on $L^2(\mathbb{R})$ in the obvious way. The key point is that besides the functions $\mathcal{H}_0(x + \mathrm{i}a/2, y)$, $y \in \mathbb{R}$ (cf equation (6.2)), the functions $\mathcal{H}_0^b(x + \mathrm{i}a/2, y)$, $y \in (-3r, -r]$, are the only eigenfunctions of the differential operator on the right-hand side of (6.18) that have a real eigenvalue and a π/r-multiplier that is a phase. (This well known fact follows from a consideration of the discriminant of the periodic Schrödinger operator (6.18) at hand, but a quite short proof will be given in a moment.) Thus it follows that $\tilde{H}_{\mathrm{nr}}(2)$ has a purely absolutely continuous spectrum $[-e_1, -e_2] \cup [-e_3, \infty)$ with multiplicity two, cf [11, section XIII.16] and references therein.

Comparing this state of affairs to our findings in subsection 2.4, the reader will see why the completeness conjecture made there is plausible. A suitable generalization of the well known lore on periodic Schrödinger operators to AΔOs with periodic coefficients might settle this open problem.

With the above formulae at our disposal, it is actually quite simple to demonstrate the key point just mentioned. First, we note that the eigenvalue $-\wp(z)$ takes all values in $(-\infty, \infty)$ as z varies over the rectangle with corners $0, \pi/2r, \pi/2r + \mathrm{i}a/2$ and $\mathrm{i}a/2$ (with the first corner excluded, of course). Thus we need only consider the functions $\mathcal{H}_0(x)/s(x)$ (given by equation (6.2)) and their complex conjugates for z on the rectangle. The latter have the same eigenvalue $-\wp(z)$ and are linearly independent of the former unless z equals one of the last three corners. (A second eigenfunction $E_j(x)$ independent of $\mathcal{H}_j(x)/s(x)$, $j \in \{1, 2, 3\}$, can be easily constructed via reduction of order, but $E_j(x + \pi/r)$ does not equal $\mu E_j(x)$ for some $\mu \in \mathbb{C}$.) Hence it suffices to prove that the y-value given by (6.1) is not real for z on the horizontal sides of the rectangle.

Now for z between $\mathrm{i}a/2$ and $\mathrm{i}a/2 + \pi/2r$ we may write

$$\frac{s'(z)}{s(z)} = -\mathrm{i}r + \int_{\mathrm{i}a/2}^{z} (-\wp(w) - 2\eta r/\pi)\,\mathrm{d}w \qquad z - \mathrm{i}a/2 \in (0, \pi/2r) \quad (6.19)$$

cf equations (2.32) and (2.36). The integral equals 0 for $z = \mathrm{i}a/2 + \pi/2r$ and the integrand decreases monotonically as w goes from $\mathrm{i}a/2$ to $\mathrm{i}a/2 + \pi/2r$. Since $-e_3 - 2\eta r/\pi$ is positive (cf equation (2.34)), we deduce that the integral yields a positive number for z between $\mathrm{i}a/2$ and $\mathrm{i}a/2 + \pi/2r$. Therefore, the associated y-value has a non-zero imaginary part.

It also follows from the previous paragraph that $-e_2 - 2\eta r/\pi$ is a negative number. *A fortiori*, $-\wp(w) - 2\eta r/\pi$ is negative for $w \in (0, \pi/2r]$. Now for z between 0 and $\pi/2r$ we have

$$\frac{s'(z)}{s(z)} = \int_{z}^{\pi/2r} (\wp(w) + 2\eta r/\pi)\,\mathrm{d}w \qquad z \in (0, \pi/2r) \quad (6.20)$$

so it follows that $s'(z)/s(z)$ is positive. Hence y is not real, and the proof is complete.

Proceeding with the hyperbolic specialization, we let $r \downarrow 0$ in the above formulae, obtaining the operators

$$B_+^{(0)} = -T_{\mathrm{i}a} - T_{-\mathrm{i}a} \qquad H_+^{(0)} = T_{\mathrm{i}a} + T_{-\mathrm{i}a} \quad (6.21)$$

$$B_-^{(0)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\pi^2}{a^2} + \frac{2\pi}{a}\,\mathrm{cth}\left(\frac{\pi x}{a}\right)\frac{\mathrm{d}}{\mathrm{d}x} \quad (6.22)$$

$$H_-^{(0)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\pi^2}{a^2\,\mathrm{sh}^2(\pi x/a)} = \mathrm{sh}^{-1}(\pi x/a) B_-^{(0)}\,\mathrm{sh}(\pi x/a). \quad (6.23)$$

The constraint (6.1) becomes

$$\mathrm{cth}(\pi z/a) = -\mathrm{i}ay/\pi \quad (6.24)$$

which amounts to

$$\exp(2\pi z/a) = \frac{y + \mathrm{i}\pi/a}{y - \mathrm{i}\pi/a}. \quad (6.25)$$

Thus $\mathcal{H}_0(x)$ (6.2) can be rewritten

$$\mathcal{H}_0(x, y) = \frac{a}{\pi}\frac{1}{(a^2y^2 + \pi^2)^{1/2}}\left[ay\,\mathrm{sh}\left(\frac{\pi x}{a}\right) + \mathrm{i}\pi\,\mathrm{ch}\left(\frac{\pi x}{a}\right)\right]\mathrm{e}^{\mathrm{i}xy} \quad (6.26)$$

and the eigenvalues become

$$E_+^{(0)} = 2\,\mathrm{ch}(ay) \qquad E_-^{(0)} = y^2. \quad (6.27)$$

A suitable multiple of the odd combination

$$\begin{aligned}\psi_0(x, y) &\equiv \mathcal{H}_0(x, y) - \mathcal{H}_0(-x, y)\\ &= \frac{2a}{\pi}\frac{1}{(a^2y^2 + \pi^2)^{1/2}}\left[ay\,\mathrm{sh}\left(\frac{\pi x}{a}\right)\cos(xy) - \pi\,\mathrm{ch}\left(\frac{\pi x}{a}\right)\sin(xy)\right] \end{aligned} \quad (6.28)$$

yields an isometry from $L^2((0,\infty),\mathrm{d}y)$ onto $L^2((0,\infty),\mathrm{sh}^{-2}(\pi x/a)\,\mathrm{d}x)$. The even combination does not vanish for $x\to 0$, so it does not give rise to an isometry.

Comparing the above hyperbolic formulae and the rational specialization in section 5, one reads off that they are related via the substitutions

$$x \leftrightarrow y \qquad \beta \to \pi/a. \tag{6.29}$$

From this the pertinent duality property of the joint $B_\delta^{(0)}$-eigenfunction $\mathcal{H}_0(x,y)$ (6.26) is clear: it is also an eigenfunction of the AΔO

$$\tilde{H} \equiv \left(\frac{y-2\mathrm{i}\pi/a}{y}\right)^{1/2} \tilde{T}_{\mathrm{i}\pi/a} \left(\frac{y+2\mathrm{i}\pi/a}{y}\right)^{1/2} + (\mathrm{i}\to -\mathrm{i}) \tag{6.30}$$

with eigenvalue

$$\tilde{E} = 2\,\mathrm{ch}(\pi x/a). \tag{6.31}$$

(The arbitrary-g generalization of this non-relativistic-hyperbolic versus relativistic-rational duality was first pointed out in [3, subsection 3B2].)

Just as in section 3, we can also take $x \to x + \mathrm{i}a/2$ to get operators

$$\tilde{B}_-^{(0)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\pi^2}{a^2} + \frac{2\pi}{a}\,\mathrm{th}\left(\frac{\pi x}{a}\right)\frac{\mathrm{d}}{\mathrm{d}x} \tag{6.32}$$

$$\tilde{H}_-^{(0)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{2\pi^2}{a^2\,\mathrm{ch}^2(\pi x/a)} = \mathrm{ch}^{-1}(\pi x/a)\,\tilde{B}_-^{(0)}\,\mathrm{ch}(\pi x/a). \tag{6.33}$$

The $\tilde{B}_-^{(0)}$-eigenfunction

$$\tilde{\mathcal{H}}_0(x,y) = \frac{a}{\pi}\frac{1}{(a^2y^2+\pi^2)^{1/2}}\left[ay\,\mathrm{ch}\left(\frac{\pi x}{a}\right) + \mathrm{i}\pi\,\mathrm{sh}\left(\frac{\pi x}{a}\right)\right]\mathrm{e}^{\mathrm{i}xy} \tag{6.34}$$

gives rise to an isometry from the even subspace of $L^2(\mathbb{R},\mathrm{d}y)$ onto the odd subspace of $L^2(\mathbb{R},\mathrm{ch}^{-2}(\pi x/a)\,\mathrm{d}x)$, whereas the odd $L^2(\mathbb{R},\mathrm{d}y)$-subspace is mapped isometrically onto the orthocomplement in the even $L^2(\mathbb{R},\mathrm{ch}^{-2}(\pi x/a)\,\mathrm{d}x)$-subspace of the constant functions. (The bound state energy equals $-\pi^2/a^2$, cf equation (6.32).)

We continue by specializing to the non-relativistic trigonometric regime. This can be done in three distinct ways, each of which yields the same results: we can take $a\uparrow\infty$ in the elliptic formulae, perform a suitable analytic continuation in the hyperbolic formulae, or let $\beta\downarrow 0$ in the formulae of section 4. We now detail the latter option.

The AΔO B (4.1) yields the non-relativistic limit

$$B^{(0)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + r^2 + 2r\cot(rx)\frac{\mathrm{d}}{\mathrm{d}x} \tag{6.35}$$

corresponding to the Schrödinger operator

$$H^{(0)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2r^2}{\sin^2(rx)} = \sin^{-1}(rx)B^{(0)}\sin(rx). \tag{6.36}$$

The constraints (4.2) and (4.9) become

$$r\,\mathrm{th}(r\gamma) = y+2r \qquad r\,\mathrm{cth}(r\kappa) = y+2r \tag{6.37}$$

respectively. Eliminating γ and κ yields the $B^{(0)}$-eigenfunction

$$\mathcal{H}_0(x,y) = \frac{1}{r}\frac{1}{(y^2+4ry+3r^2)^{1/2}}\left[(y+2r)\sin(rx) + \mathrm{i}r\cos(rx)\right]\mathrm{e}^{\mathrm{i}x(y+2r)} \tag{6.38}$$

with eigenvalue

$$E^{(0)} = (y+2r)^2. \tag{6.39}$$

The odd eigenfunctions

$$\psi_n^{(0)}(x) = \mathcal{H}_0(x, nr) - \mathcal{H}_0(-x, nr) \qquad n \in \mathbb{N} \tag{6.40}$$

give rise to an orthonormal base for the Hilbert space

$$L^2((0, \pi/r), r^3(2\pi \sin^2(rx))^{-1}\, \mathrm{d}x). \tag{6.41}$$

(Just as in section 4, completeness follows from the functions $\sin^{-3}(rx)\psi_n^{(0)}(x)$ being polynomials in $\cos rx$ of degree n.) The $\beta \downarrow 0$ limit of (4.17) reads

$$B^{(0)} \cos^2(rx) = r^2(\cos^2(rx) - 2) \tag{6.42}$$

and is readily checked directly from (6.35).

Using (4.28) one deduces that $\mathcal{H}_0(x, y)$ is an eigenfunction of the dual AΔO

$$\tilde{H}_-^{(0)} = \left(\frac{y+3r}{y+2r}\right)^{1/2} \tilde{T}_r \left(\frac{y+r}{y+2r}\right)^{1/2} + \left(\frac{y+r}{y+2r}\right)^{1/2} \tilde{T}_{-r} \left(\frac{y+3r}{y+2r}\right)^{1/2} \tag{6.43}$$

with eigenvalue $2\cos rx$. The discrete difference operator

$$D^{(0)} = \left(\frac{n+3}{n+2}\right)^{1/2} S \left(\frac{n+1}{n+2}\right)^{1/2} + \text{h.c.} \tag{6.44}$$

on $l^2(\mathbb{N})$ is then diagonalized as multiplication by $2\cos rx$ on the Hilbert space (6.41) via the unitary with kernel $(\psi_0^{(0)}(x), \psi_1^{(0)}(x), \ldots)$.

Finally, we turn to the rational case. As before, this most degenerate case can be reached via various paths, yielding the same results: one gets the operators

$$B^{(0)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x} \qquad H^{(0)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2}{x^2} = \frac{1}{x} B^{(0)} x \tag{6.45}$$

and $B^{(0)}$-eigenfunctions and -eigenvalues

$$\mathcal{H}_0(x, y) = \left(x + \frac{\mathrm{i}}{y}\right)\mathrm{e}^{\mathrm{i}xy} \qquad E^{(0)} = y^2. \tag{6.46}$$

The $H^{(0)}$-eigenfunction $\mathcal{H}_0(x, y)/x$ is manifestly self-dual (symmetric under $x \leftrightarrow y$). The even combination

$$\Phi^{(0)}(x, y) = 2\cos(xy) - 2\sin(xy)/xy \tag{6.47}$$

yields the kernel of an isometry on $L^2((0, \infty))$, whereas the odd combination does not give rise to a bounded operator on $L^2((0, \infty))$.

Note added. After completion of this paper a preprint by Billey [15] appeared that bears out the scenario sketched in the paragraph below (1.23). More precisely, she shows that $N > 2$ elliptic relativistic eigenfunctions can be found via a suitable (nested) Bethe ansatz, provided the coupling g is an integer. Unfortunately, it is not obvious that her $g = 2$ Bethe ansatz equations and eigenfunctions can be made to converge to those of Lieb and Liniger by appropriate substitutions, but we have little doubt that this is feasible.

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