THE EUROPEAN
PHYSICAL JOURNAL C

CrossMark

Relativistic Landau levels in the rotating cosmic string spacetime

M. S. Cunha[1], **C. R. Muniz**[2,a], **H. R. Christiansen**[3], **V. B. Bezerra**[4]

[1] Grupo de Física Teórica (GFT), Universidade Estadual do Ceará, Fortaleza, CE 60714-903, Brazil
[2] Universidade Estadual do Ceará, Faculdade de Educação, Ciências e Letras de Iguatu, Rua Deocleciano Lima Verde, Iguatu, CE, Brazil
[3] Instituto Federal de Ciência, Educação e Tecnologia, IFCE Departamento de Física, Sobral 62040-730, Brazil
[4] Departamento de Física, Universidade Federal da Paraíba-UFPB, Caixa Postal 5008, João Pessoa, PB 58051-970, Brazil

Abstract In the spacetime induced by a *rotating* cosmic string we compute the energy levels of a massive spinless particle coupled covariantly to a homogeneous magnetic field parallel to the string. Afterwards, we consider the addition of a scalar potential with a Coulomb-type and a linear confining term and completely solve the Klein–Gordon equations for each configuration. Finally, assuming rigid-wall boundary conditions, we find the Landau levels when the linear defect is itself magnetized. Remarkably, our analysis reveals that the Landau quantization occurs even in the absence of gauge fields provided the string is endowed with spin.

1 Introduction

In the last decade, a renewed interest in cosmic strings has been witnessed after a period of ostracism [1–7]. Cosmic strings are hypothetical massive objects that may have contributed, albeit marginally, to the anisotropy of the cosmic microwave background radiation and, consequently, to the large scale structure of the universe [8]. Actually, their existence is also supported in superstring theories with either compactified or extended extra dimensions. Both static and rotating cosmic strings can be equally responsible for some remarkable effects such as particle self-force [9,10] and gravitational lensing [11], as well as for production of highly energetic particles [12–14].

Rotating cosmic strings, as well as their static counterparts, are one-dimensional stable topological defects probably formed during initial stages of the universe. They are characterized by a wedge parameter α that depends on its linear mass density, μ, and by the linear density of angular momentum J. Initially, they were described as general relativistic solutions of a Kerr spacetime in $(1 + 2)$ dimensions [15], and then naturally extended to the four-dimensional

spacetime [16]. Notably, out of the singularity, cosmic strings (static or rotational) present a flat spacetime geometry with some remarkable global properties. These properties include theoretically predicted effects such as gravitomagnetism and (non-quantum) gravitational Aharonov–Bohm effect [17,18].

Cosmic string may eventually present an internal structure [20] generating a Gödel spacetime featuring an exotic region which allows closed time-like curves (CTC's) around the singularity. The frontier of this region is at a distance proportional to J/α from the string, thus offering a natural boundary condition. Rotating cosmic strings were also studied in the Einstein–Cartan theory [21,22] and in teleparallel gravity [23], in which the region of CTC's was examined. There are also studies of these objects in the extra-dimensional context including their causal structure, which raised criticisms on the real existence of the CTC's region [24].

Regarding Landau levels, in the spacetime of a stationary spinning cosmic string one does not find much literature [25,26] in contrast to what happens with static strings (see [27–30], and references therein). This is probably due to the analogies and possible technological applications [31] found in condensed matter physics (e.g. disclination in crystals). It is precisely this gap what motivates our paper. Thus, to make some progress in this direction, we will present a fully relativistic study of a massive charged particle coupled to a gauge field in the spacetime spanned by a rotating string, with the eventual addition of scalar potentials.

Besides the mathematical challenge on its own, it is phenomenologically meaningful to assess such a calculation for a static magnetic field parallel to the cosmic string and then compare the outcome with the static string results found in the literature [27]. It is also opportune to check the non-relativistic limit in order to improve a previous non-relativistic calculation made with a much simpler approach [26].

[a] e-mail: celio.muniz@uece.br

After such an outset, we will examine the problem when cylindric scalar potentials of coulombian and linear types are also considered. Phenomenologically, the coulombian potential is associated with a self-force acting on a charged particle in the spacetime of a cosmic string [32,33], and the linear term represents a cylindric harmonic oscillator of confining nature. Finally, we will consider the rotating string endowed with an internal magnetic flux and will discuss the raising of the Landau quantization from a pure spacetime rotation.

From the astrophysical point of view, the motivation to the present analysis lies on the possibility of existing scenarios in which charged relativistic particles interact with cosmic strings in the presence of intergalactic magnetic fields, with transitions between the energy levels yielding a spectrum that allows one not only to identify a cosmic string, but also to differentiate a static string from a rotating one. Such scenarios would also allow for getting a reasonable estimate of the angular momentum of the string and, as a consequence, of the size of its CTCs frontier. Indeed, we will do so at the end of the paper.

The paper is organized as follows: in Sect. 2, we obtain the exact energy eigenvalues of the Klein–Gordon equation in the metric of a stationary rotating cosmic string coupled to a static magnetic field. In Sect. 3, we solve the problem along with some additional external potentials. In Sect. 4, we consider a rotating string with an internal magnetic flux. Finally, in Sect. 5 we conclude with some remarks.

2 Spinless charged particle in a rotating cosmic string spacetime surrounded by an external magnetic field

To start, we shall consider a massive, charged, relativistic spinless quantum particle in the spacetime of an idealized stationary rotating cosmic string. It means that the string has no structure and its metric is given by [19]

$$ds^2 = c^2 dt^2 + 2ac\,dt\,d\phi - (\alpha^2\rho^2 - a^2)d\phi^2 - d\rho^2 - dz^2,$$
(1)

where the string is placed along the z axis and the cylindrical coordinates are labeled by (t, ρ, ϕ, z) with the usual ranges. Here, the rotation parameter $a = 4GJ/c^3$ has units of distance and $\alpha = 1 - 4\mu G/c^2$ is the wedge parameter which determines the angular deficit, $\Delta\phi = 2\pi(1 - \alpha)$, produced by the cosmic string. The letters c, G, and μ stand for the light speed, the gravitational Newton constant, and the linear density of the mass of the string.

In order to investigate the relativistic quantum motion in the presence of a gauge potential and in a curved spacetime, let us consider the Klein–Gordon equation whose covariant form is written as

$$\left[\frac{1}{\sqrt{-g}} D_\mu\left(\sqrt{-g}\,g^{\mu\nu} D_\nu\right) + \frac{m^2 c^2}{\hbar^2}\right]\Psi = 0,$$
(2)

where $D_\mu = \partial_\mu - \frac{ie}{\hbar c} A_\mu$, e is the electric charge and m is the mass of the particle; $\hbar$ is as usual the Planck constant, $g^{\mu\nu}$ is the metric tensor, and $g = \det g^{\mu\nu}$. Assuming the existence of a homogeneous magnetic field B parallel to the string, the vector potential can be taken as $\mathbf{A} = (0, A_\phi, 0)$, with $A_\phi = \frac{1}{2}\alpha B\rho^2$.

The cylindrical symmetry of the background space, given by Eq. (1), suggests the factorization of the solution of Eq. (2) as

$$\Psi(\rho, \phi, z; t) = e^{-i\frac{E}{\hbar}t}e^{i(\ell\phi + k_z z)}R(\rho),$$
(3)

where $R(\rho)$ is the solution of the radial equation given by

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \Lambda\frac{R}{\rho^2} - \frac{e^2 B^2}{4\hbar^2 c^2}\rho^2 R + \Delta R = 0,$$
(4)

with

$$\Lambda = \left(\frac{\ell}{\alpha} + \frac{aE}{\alpha\hbar c}\right)^2,$$
(5)

$$\Delta = \frac{E^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} - k_z^2 + \frac{eB}{\hbar c}\left(\frac{\ell}{\alpha} + \frac{aE}{\hbar c\alpha}\right);$$
(6)

k_z and E are z-momentum and energy of the particle, and ℓ the azimuthal angular quantum number. The solutions of Eq. (4) can be found by means of the following transformation:

$$R(\rho) = \exp\left(-\frac{Be\rho^2}{4\hbar c}\right)\rho^{\sqrt{\Lambda}}F(\rho).$$
(7)

Substituting the above expression in Eq. (4) we obtain

$$\rho F''(\rho) + \left(1 + 2\sqrt{\Lambda} - \frac{Be}{\hbar c}\rho^2\right)$$
$$F'(\rho) + \left[\Delta - \frac{Be}{\hbar c}\left(1 + \sqrt{\Lambda}\right)\right]\rho F(\rho) = 0.$$
(8)

Now, let us consider the change of variables $z = (Be/2\hbar c)\rho^2$. Thus, Eq. (8) assumes the familiar form

$$z F''(z) + \left(\sqrt{\Lambda} + 1 - z\right)F'(z)$$
$$- \left[\frac{1}{2}\left(\sqrt{\Lambda} + 1\right) - \frac{\hbar c}{2eB}\Delta\right]F(z) = 0,$$
(9)

which is the well-known confluent hypergeometric equation, whose linearly independent solutions are

$$F^{(1)}(z) = {}_1F_1\left(\frac{1}{2} + \frac{\sqrt{\Lambda}}{2} - \frac{\hbar c}{2eB}\Delta; \sqrt{\Lambda} + 1; z\right),$$
(10)

$$F^{(2)}(z) = z^{-\sqrt{\Lambda}}\,{}_1F_1\left(\frac{1}{2} - \frac{\sqrt{\Lambda}}{2} - \frac{\hbar c}{2eB}\Delta; 1 - \sqrt{\Lambda}; z\right).$$
(11)

Therefore, the radial solutions, $R(\rho)$, can be written as

$$R^{(1)}(\rho) = A_1 \exp\left(-\frac{Be\rho^2}{4\hbar c}\right) \rho^{\sqrt{\Lambda}} \, {}_1F_1$$
$$\times \left(\frac{1}{2} + \frac{\sqrt{\Lambda}}{2} - \frac{\hbar c}{2eB}\Delta; 1 + \sqrt{\Lambda}; \frac{Be\rho^2}{2\hbar c}\right) \quad (12)$$

$$R^{(2)}(\rho) = A_2 \exp\left(-\frac{Be\rho^2}{4\hbar c}\right) \rho^{-\sqrt{\Lambda}} \, {}_1F_1$$
$$\times \left(\frac{1}{2} - \frac{\sqrt{\Lambda}}{2} - \frac{\hbar c}{2eB}\Delta; 1 - \sqrt{\Lambda}; \frac{Be\rho^2}{2\hbar c}\right) \quad (13)$$

where A_1 e A_2 are normalization constants. The second solution is not physically acceptable at the origin and we discard it. Because confluent hypergeometric functions diverge exponentially when $\rho \rightarrow \infty$, in order to have asymptotically acceptable physical solutions we have to impose the condition

$$\frac{1 + \sqrt{\Lambda}}{2} - \frac{\hbar c}{2eB}\Delta = -n, \quad (14)$$

where n is a positive integer. Substituting Λ and Δ given by Eqs. (5) and (6), respectively, into Eq. (14), we obtain the following result:

$$\frac{E^2}{Be\hbar c} + \left(\frac{aE}{\hbar c\alpha} + \frac{\ell}{\alpha}\right) - \left|\frac{aE}{\hbar c\alpha} + \frac{\ell}{\alpha}\right|$$
$$-\frac{c}{Be\hbar}(\hbar^2 k^2 + m^2 c^2) - \frac{1}{4}\frac{Bea^2}{\hbar c^3} = 2n + 1, \quad (15)$$

from which we can read the energy eigenvalues as

$$E_{n,\ell} = \frac{Bea}{2\alpha}\frac{|\ell| - \ell}{\ell}$$
$$\pm \sqrt{m^2 c^4 + k^2\hbar^2 c^2 + \left(\frac{Bea}{2\alpha}\frac{|\ell| - \ell}{\ell}\right)^2 + B\hbar ce\left(2n + 1 + \frac{|\ell|}{\alpha} - \frac{\ell}{\alpha}\right)}. \quad (16)$$

This expression shows that the energy eigenvalues are not invariant under the interchange of positive and negative eigenvalues of the azimuthal quantum number ℓ. This is a consequence of the spacetime topological twist around the spinning string, which now depends not only on α but also on a (see Eq. (1)). It is worth noticing that by turning off the string rotation, i.e. making $a = 0$, we obtain an already known expression [30] valid for the static string. Notice also that for positive ℓ, the energy spectra of both static and rotating strings are identical.

Non-relativistic limit

The non-relativistic expression can be attained by considering $E^2/c^2 - m^2 c^2 \approx 2mE$ in the previous equation. In this case, Eq. (16) turns into

$$E_{n,\ell} \approx \frac{1}{1 + \frac{eBa}{2mc^2\alpha}(1 - |\ell|/\ell)}$$
$$\times \left[\frac{\hbar^2 k^2}{2m} + \frac{Be\hbar}{2mc}\left(2n + 1 + \frac{|\ell|}{\alpha} - \frac{\ell}{\alpha}\right)\right]. \quad (17)$$

As a result, we can see that for $\ell > 0$ (i.e. particle orbiting parallel to the string rotation) the energy levels are the same for both static [27] and spinning strings. Otherwise, for antiparallel orbits ($\ell < 0$), the allowed spectrum depends on the angular momentum density of the string (recall that $a = 4GJ/c^3$).

In this case, if we consider the slow rotation approximation, where the terms $\mathcal{O}(a^2)$ are neglected, we have

$$\Delta E_{n,\ell}/E_{n,\ell}^{(0)} \approx -eBa/\alpha mc^2 \quad (18)$$

where $\Delta E_{n,\ell}$ is the relative difference of our result compared to $E_{n,\ell}^{(0)}$, for the static string levels [27]. This result improves the one found in [26] where further approximations were made.

3 Cylindrically symmetric scalar potential in a rotating cosmic string spacetime surrounded by an external magnetic field

In this section we shall perform a generalization of the analysis above done, through the addition of the following cylindrically symmetric scalar potential [30,35]:

$$S(\rho) = \frac{\kappa}{\rho} + \nu\rho, \quad (19)$$

where κ and ν are constants.

In order to consider the influence of this potential on the quantum dynamics of the particle, we have to modify Eq. (2) by adding Eq. (19) to the mass term in such a way that $\frac{mc}{\hbar}$ is replaced by $\frac{mc}{\hbar} + S(\rho)$. Thus, introducing this modification into Eq. (2) and considering the ansatz given by Eq. (3), we obtain the following radial equation:

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \mathfrak{L}\frac{R}{\rho^2} - 2M\kappa\frac{R}{\rho} - 2M\nu\rho R$$
$$-\Omega^2\rho^2 R + \mathfrak{D}\,R = 0, \quad (20)$$

where

$$M = \frac{mc}{\hbar} \quad (21)$$

$$\Omega^2 = M^2\omega^2 + \nu^2 \quad (22)$$

$$\mathfrak{L} = \left(\frac{\ell}{\alpha} + \frac{a}{\alpha}\mathcal{E}\right)^2 + \kappa^2 \quad (23)$$

$$\mathfrak{D} = \mathcal{E}^2 + 2M\omega\left(\frac{\ell}{\alpha} + \frac{a}{\alpha}\mathcal{E}\right) - M^2 - 2\kappa\nu - k_z^2, \quad (24)$$

$2M\omega = eB/\hbar c\alpha$ and $\mathcal{E} = E/\hbar c$. For convenience, let us define a new funtion $H(\rho)$ such that

$$R(\rho) = \exp\left(-\frac{1}{2}\Omega\rho^2 - \frac{M\nu}{\Omega}\rho\right)\rho^{\sqrt{\mathfrak{L}}}H(\rho). \qquad (25)$$

Thus, using the redefinition $\sqrt{\Omega}\rho \to \rho$, Eq. (20) reads

$$\frac{d^2H}{d\rho^2} + \left(\frac{1+2\sqrt{\mathfrak{L}}}{\rho} - \frac{2M\nu}{\Omega^{3/2}} - 2\rho\right)\frac{dH}{d\rho}$$
$$+ \left[\frac{M^2\nu^2}{\Omega^3} + \frac{\mathfrak{D}}{\Omega} - 2\sqrt{\mathfrak{L}} - 2\right.$$
$$\left. - \frac{1}{2}\left(\frac{4M\kappa}{\sqrt{\Omega}} + (1+2\sqrt{\mathfrak{L}})\frac{2M\nu}{\Omega^{3/2}}\right)\frac{1}{\rho}\right]H = 0. \qquad (26)$$

which corresponds to the biconfluent Heun equation [36,37]. Written in the standard form

$$H_b''(z) + \left(\frac{1+\alpha}{z} - \beta - 2z\right)H_b'(z)$$
$$+ \left[\gamma - \alpha - 2 - \frac{1}{2}[\delta + (1+\alpha)\beta]\frac{1}{z}\right]H_b(z) = 0, \qquad (27)$$

its solutions are the so-called biconfluent Heun functions

$$H_b(z) = C_1 H_b(\alpha, \beta, \gamma, \delta; z) + C_2 z^{-\alpha}H_b(-\alpha, \beta, \gamma, \delta; z), \qquad (28)$$

with C_1 and C_2 being normalization constants. If α is not a negative integer, the biconfluent Heun functions can be written as [38,39]

$$H_b(\alpha, \beta, \gamma, \delta; z) = \sum_{j=0}^{\infty}\frac{A_j}{(1+\alpha)_j}\frac{z^j}{j!} \qquad (29)$$

where the coefficients A_j obey the three-term recurrence relation ($j \geq 0$)

$$A_{j+2} = \left[(j+1)\beta + \frac{1}{2}[\delta + (1+\alpha)\beta]\right]A_{j+1}$$
$$- (j+1)(j+1+\alpha)(\gamma - \alpha - 2 - 2j)A_j \qquad (30)$$

Comparing directly Eqs. (26) and (27), we obtain the following analytical solutions for $H(\rho)$:

$$H^{(1)}(\rho) = c_1 H_b\left(2\sqrt{\mathfrak{L}}, \frac{2M\nu}{\Omega^{3/2}}, \frac{M^2\nu^2}{\Omega^3} + \frac{\mathfrak{D}}{\Omega}, \frac{4M\kappa}{\sqrt{\Omega}}; \sqrt{\Omega}\rho\right) \qquad (31)$$

$$H^{(2)}(z) = c_2 \rho^{-2\sqrt{\mathfrak{L}}}H_b\left(-2\sqrt{\mathfrak{L}}, \frac{2M\nu}{\Omega^{3/2}}, \frac{M^2\nu^2}{\Omega^3} + \frac{\mathfrak{D}}{\Omega}, \frac{4M\kappa}{\sqrt{\Omega}}; \sqrt{\Omega}\rho\right) \qquad (32)$$

where we have substituted back $\rho \to \sqrt{\Omega}\rho$ in the above expressions. In view of Eq. (25) and the fact that the solution given by Eq. (32) is divergent at the origin, we will cast it off. Moreover, the biconfluent Heun functions are highly

divergent at infinity and so we need to focus on their polynomial forms. Indeed, the biconfluent Heun function becomes a polynomial of degree n if the following conditions are both satisfied (see [39] and the references therein),

$$\gamma - \alpha - 2 = 2n, \quad n = 0, 1, 2, \dots \qquad (33)$$
$$A_{n+1} = 0, \qquad (34)$$

where A_{n+1} has $n + 1$ real roots when $1 + \alpha > 0$ and $\beta \in \mathbb{R}$. It is represented as a three-diagonal $(n + 1)$-dimensional determinant, namely,

$$\begin{vmatrix} \delta' & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 2(1+\alpha)n & \delta'-\beta & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 4(2+\alpha)(n-1) & \delta'-2\beta & 1 & 0 & \cdots & 0 \\ 0 & 0 & \gamma_2 & \delta'-3\beta & 1 & \cdots & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \gamma_{j-1} & \delta'_{s-1} & 1 \\ 0 & 0 & 0 & 0 & 0 & \gamma_s & \delta'_s \end{vmatrix} = 0, \qquad (35)$$

where

$$\delta' = -\frac{1}{2}[\delta + (1+\alpha)\beta] \qquad (36)$$
$$\delta'_s = \delta' - (s+1)\beta \qquad (37)$$
$$\gamma_s = 2(s+1)(s+1+\alpha)(n-s). \qquad (38)$$

As an important consequence of Eq. (33), we have

$$\frac{M^2\nu^2}{\Omega^3} + \frac{\mathfrak{D}}{\Omega} - 2\sqrt{\mathfrak{L}} - 2 = 2n, \qquad (39)$$

which means that the energy eigenvalues obey a quantization condition. Differently from Eqs. (14) and (15), now we have a fourth order expression for the energy, which is given by

$$D_4\mathcal{E}^4 + D_3\mathcal{E}^3 + D_2\mathcal{E}^2 + D_1\mathcal{E} + D_0 = 0, \qquad (40)$$

where

$$D_4 = \frac{1}{\Omega^2}$$
$$D_3 = \frac{4M\omega}{\Omega^2}\frac{a}{\alpha}$$
$$D_2 = \frac{2M^2\nu^2}{\Omega^4} - \frac{4(n+1)}{\Omega} + \frac{2}{\Omega^2}\left(L + 2M^2\omega^2\frac{a^2}{\alpha^2}\right)$$
$$D_1 = \left[\frac{2M^2\nu^2}{\Omega^4} - \frac{4(n+1)}{\Omega} + \frac{2L}{\Omega^2}\right]2M\omega\frac{a}{\alpha} - \frac{8a}{\hbar c\alpha}\frac{\ell}{\alpha} \qquad (41)$$
$$D_0 = \frac{M^2\nu^2}{\Omega^3}\left[\frac{M^2\nu^2}{\Omega^3} - 4(n+1)\right]$$
$$+ \left[\frac{2M^2\nu^2}{\Omega^3} - 4(n+1) + \frac{L}{\Omega}\right]\frac{L}{\Omega}$$
$$+ 4(n+1)^2 - \frac{4\ell^2}{\alpha^2} - 4\kappa^2,$$

with $L = 2M\omega\frac{\ell}{\alpha} - M^2 - 2\kappa\nu - k_z^2$. Unfortunately, the analytical solutions for the energy eigenvalues are given by huge (algebraic) expressions. However, we can manage them in some particular cases which will be presented in the following.

3.1 The rotation vanishes (**a = 0**)

In this case, we obtain the following result for the energy eigenvalues:

$$
E/\hbar c = \pm \left[k_z^2 + \frac{M^4\omega^2}{\nu^2 + M^2\omega^2} + 2\kappa\nu - 2M\omega\frac{\ell}{\alpha} + 2\Omega \right.
$$
$$
\left. \times \left(n + 1 + \sqrt{\frac{\ell^2}{\alpha^2} + \kappa^2} \right) \right]^{\frac{1}{2}}, \tag{42}
$$

which coincides with the one already obtained in the literature [30].

3.2 The rotation vanishes and there is no scalar potential (**a = 0, κ = 0, ν = 0**)

In the present situation, we have $\Omega = M\omega$ and then the energy eigenvalues are given by

$$
E/\hbar c = \pm \left[k_z^2 + M^2 + 2M\omega \left(n + 1 + \frac{|\ell|}{\alpha} - \frac{\ell}{\alpha} \right) \right]^{\frac{1}{2}}. \tag{43}
$$

However, in this case the biconfluent Heun solution does not have the odd terms as we can see expanding Eq. (31) or from Eqs. (35)–(38). Therefore, the above expression only make sense when we consider the even terms, or equivalently when $n \to 2n$ [27]. Another way to see this is verifying that

$$
H_b \left(2\sqrt{\Lambda}, 0, \frac{\Delta}{M\omega}, 0, \sqrt{M\omega}\rho \right)
$$
$$
= {}_2F_1 \left(\frac{1 + \sqrt{\Lambda}}{2} - \frac{\Delta}{4M\omega}, 1 + \sqrt{\Lambda}, \sqrt{M\omega}\rho^2 \right) \tag{44}
$$

and, thus, showing the correspondence between conditions (14) and (39) in this particular case.

3.3 Linear confinement ($\kappa = 0$)

In this case, the Coulomb-type potential term is absent, and as a consequence the scalar potential is reduced to the linear term in ρ. Thus, the solutions are now given by

$$
H^{(1)}(\rho) = c_1 H_b \left(2\sqrt{\Lambda}, \frac{2M\nu}{\Omega^{3/2}}, \frac{M^2\nu^2}{\Omega^3} + \frac{\Delta}{\Omega}, 0; \sqrt{\Omega}\rho \right) \tag{45}
$$

$$
H^{(2)}(z) = c_2 \rho^{-2\sqrt{\Lambda}} H_b \left(-2\sqrt{\Lambda}, \frac{2M\nu}{\Omega^{3/2}}, \frac{M^2\nu^2}{\Omega^3} + \frac{\Delta}{\Omega}, 0; \sqrt{\Omega}\rho \right) \tag{46}
$$

Again we discard the second solution because it diverges at $\rho = 0$. The condition to get polynomial solutions is now

$$
\frac{M^2\nu^2}{\Omega^3} + \frac{\Delta}{\Omega} - 2\sqrt{\Lambda} - 2 = 2n \tag{47}
$$

As before, the above condition implies in the quantization of the energy eigenvalues which is equivalent to Eq. (40), with the coefficients given by (41), with $\kappa = 0$.

4 Spinless particle in the rotating cosmic string spacetime with an internal magnetic flux

We will now examine the relativistic Landau levels of a charged spinless particle in the spacetime of a magnetized rotating string (namely, endowed with some intrinsic magnetic flux Φ) with no external electromagnetic field [40,41]. The corresponding gauge coupling is obtained by making $B \to B = \Phi/\alpha\pi\rho^2$ in Eq. (4). In this case, the radial equation reads

$$
\rho^2 \frac{d^2 R}{d\rho^2} + \rho \frac{dR}{d\rho} + (\delta\rho^2 - \Sigma)R = 0. \tag{48}
$$

where Σ and δ are given by

$$
\Sigma = \left(\frac{\ell}{\alpha} + \frac{a}{\alpha}\mathcal{E} - \frac{\epsilon\Phi}{\alpha} \right)^2 \tag{49}
$$

$$
\delta = \mathcal{E}^2 - M^2 - k_z^2 \tag{50}
$$

with $\epsilon = e/2\pi\hbar c$.

The solutions of Eq. (48) are written in terms of Bessel's functions of the first kind, $J_\lambda(z)$, and of the second kind, $Y_\lambda(z)$, as

$$
R(\rho) = C_1 J_{\sqrt{\Sigma}}\left(\sqrt{\delta}\,\rho \right) + C_2 Y_{\sqrt{\Sigma}}\left(\sqrt{\delta}\rho \right), \tag{51}
$$

with C_1 and C_2 being constants. The function $J_\lambda(z)$ is different from zero at the origin when $\lambda = 0$. Otherwise, $Y_{\sqrt{\Sigma}}$ is always divergent at the origin. Thus, we will discard it and consider $\lambda \neq 0$. It is worth pointing out that when $\Phi = 0$, we reobtain the wave function found in [42]. To find the energy eigenvalues, we will impose the so called hard-wall condition. With this boundary condition, the wave function of the particle vanishes at some $\rho = r_w$ which is an arbitrary radius far away from the origin. Thus, we can use the asymptotic expansion for large arguments of $J_\lambda(z)$, given by

$$
J_\lambda(z) \approx \sqrt{\frac{2}{\pi z}} \cos \left(z - \frac{\lambda\pi}{2} - \frac{\pi}{4} \right), \tag{52}
$$

from which we obtain

$$\sqrt{\delta} r_w - \frac{\sqrt{\Sigma} \pi}{2} - \frac{\pi}{4} = \frac{\pi}{2} + n\pi, \tag{53}$$

for $n \in \mathbb{Z}$. Substituting Eqs. (49) and (50) into (53), we get

$$r_\omega \sqrt{\mathcal{E}^2 - M^2 - k_z^2} \mp \frac{\pi}{2} \left(\frac{\ell}{\alpha} + \frac{a}{\alpha} \mathcal{E} - \frac{\epsilon \Phi}{\alpha} \right)$$
$$= \left(n + \frac{3}{4} \right) \pi, \tag{54}$$

where the upper and lower signals correspond to $\ell/\alpha + a\mathcal{E}/\alpha - \epsilon\Phi/\alpha \leq 0$ or $\ell/\alpha + a\mathcal{E}/\alpha - \epsilon\Phi/\alpha > 0$, respectively. Equation (54), can be rewritten as the following second order equation:

$$A_1 \mathcal{E}^2 + A_2 \mathcal{E} + A_3 = 0,$$

with

$$A_1 = r_\omega^2 - \frac{a^2 \pi^2}{4\alpha^2}$$
$$A_2 = -\frac{a\pi^2}{2\alpha} \left[\left(\frac{\ell}{\alpha} - \frac{\epsilon\Phi}{\alpha} \right) \pm \left(2n + \frac{3}{2} \right) \right]$$
$$A_3 = -r_\omega^2 \left(M^2 + k_z^2 \right) - \left(\frac{\ell}{\alpha} - \frac{\epsilon\Phi}{\alpha} \right) \frac{\pi^2}{4} - \left(n + \frac{3}{4} \right)^2 \pi^2$$
$$\mp \left(n + \frac{3}{4} \right) \left(\frac{\ell}{\alpha} - \frac{\epsilon\Phi}{\alpha} \right) \pi^2. \tag{55}$$

Since r_w is very large $\mathcal{E}$ reduces to

$$\mathcal{E}_+ \approx +\sqrt{M^2 + k_z^2} + \frac{a\pi^2}{4\alpha r_\omega^2} \left[\frac{\ell}{\alpha} - \frac{\epsilon\Phi}{\alpha} \pm \left(2n + \frac{3}{2} \right) \right]$$
$$\mathcal{E}_- \approx -\sqrt{M^2 + k_z^2} + \frac{a\pi^2}{4\alpha r_\omega^2} \left[\frac{\ell}{\alpha} - \frac{\epsilon\Phi}{\alpha} \pm \left(2n + \frac{3}{2} \right) \right]. \tag{56}$$
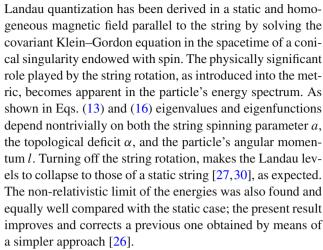
Let us now address $\mathcal{E}_+ (= E_+/\hbar c)$ and assume that $k_z \ll M$. Then, provided that $\ell/\alpha \geq \epsilon\Phi/\alpha$ (see Eq. (54)), we have

$$E_+ \approx mc^2 + \frac{a\pi^2 \hbar c}{4\alpha r_\omega^2} \left[\frac{\ell}{\alpha} - \frac{\epsilon\Phi}{\alpha} + 2n + \frac{3}{2} \right], \tag{57}$$

which shows that in the absence of rotation, the energy eigenvalues reduce to the rest energy of the particle irrespective of α. In other words, the eigenenergies are the same with or without the presence of a (static) magnetized cosmic string in space but split if the string rotates.

5 Conclusions and remarks

We have analyzed the Landau levels of a spinless massive particle in the spacetime of a rotating cosmic string by means of a fully relativistic approach. Specifically, in Sect. 2 the

Landau quantization has been derived in a static and homogeneous magnetic field parallel to the string by solving the covariant Klein–Gordon equation in the spacetime of a conical singularity endowed with spin. The physically significant role played by the string rotation, as introduced into the metric, becomes apparent in the particle's energy spectrum. As shown in Eqs. (13) and (16) eigenvalues and eigenfunctions depend nontrivially on both the string spinning parameter a, the topological deficit α, and the particle's angular momentum l. Turning off the string rotation, makes the Landau levels to collapse to those of a static string [27,30], as expected. The non-relativistic limit of the energies was also found and equally well compared with the static case; the present result improves and corrects a previous one obtained by means of a simpler approach [26].

In Sect. 3 we obtained the spectrum of the particle when a gauge potential together with a scalar one are present in the space around the rotating string. We shown that the eigensates are given by biconfluent Heun functions, which in their polynomial representation allowed finding a quantization condition on the energy levels. The general expression can be analytically obtained but looks rather huge, so we decided to exhibit just some special relevant cases which indeed confirm the results already obtained in [27,30].

We have also tackled the problem of a rotating cosmic string endowed with an internal magnetic flux with a hardwall boundary far away from the source (see Eqs. (52)–(56) in Sect. 4). The resulting eigenfunctions converge to those found in the literature when the magnetic flux vanishes [42], as expected. It is noteworthy that the Landau levels of the spinning string remain the same even when such internal magnetic flux fades away; namely, when there is no gauge field inside nor around. This can be interpreted as an induction of the Landau quantization from the sole rotational condition of the defect. It is interesting to compare this result with that of a rotating spherical source in Kerr spacetime obtained in [43].

Finally, as a phenomenological byproduct of our results, it is possible to provide a reasonable estimate of the angular momentum of the rotating cosmic string, J. Consider a proton orbiting with angular velocity Ω around the string very close to the CTC's frontier. Now, for $a \approx c/\Omega$ and $\Omega = \omega_c = eB/2\alpha mc$ with $B \sim 10^{-6}$ G (which is the value of currently observable intergalactic magnetic fields [34]), we conclude that the CTC's frontier is at about 10^{11} m from the string, which corresponds to $J \sim 10^{47}$ kg m/s. This value is compatible with the one presented in [19] when the upper limit of the photon mass, 10^{-16} eV, is taken into account [44,45].

As a future perspective, we intend to study the problem by considering a spinorial particle.

References

1. A. Vilenkin, in *Inflating Horizons of Particle Astrophysics and Cosmology*, ed. by H. Suzuki, J. Yokoyama, Y. Suto, K. Sato (Universal Academy Press, Tokyo, 2006)
2. T. Kibble, Cosmic strings reborn? COSLAB 2004, Ambleside, Cumbria, United Kingdom (2004). arXiv:astro-ph/0410073v2
3. E. Simanek, Phys. Rev. D **78**, 045014 (2008)
4. H.F. Mota, E.R. Bezerra de Mello, C.H.G. Bessa, V.B. Bezerra, Phys. Rev. D **94**, 024039 (2016)
5. K. Jusufi, Eur. Phys. J. C **76**, 332 (2016)
6. T. Charnock, A. Avgoustidis, E.J. Copeland, A. Moss, Phys. Rev. D **93**, 123503 (2016)
7. M. Salazar-Ramrez, D. Ojeda-Guilln, R.D. Mota, Ann. Phys. **372**, 283 (2016)
8. A. Linde, Chaotic inflation in supergravity and cosmic string production. Phys. Rev. D **88**, 123503 (2013)
9. E.R. Bezerra de Mello, V.B. Bezerra, Yu.V. Grats, Class. Quant. Grav. **15**, 1915 (1998)
10. C.R. Muniz, V.B. Bezerra, Ann. Phys. **340**, 87 (2014)
11. M.V. Sazhin et al., Mon. Not. R. Astron. Soc. **376**, 1731 (2007)
12. V.B. Bezerra, V.M. Mostepanenko, R.M. Teixeira Filho, Int. J. Mod. Phys. D **11**, 437 (2002)
13. V.A. de Lorenci et al., Class. Quant. Grav. **16**, 3047–3055 (1999)
14. J. Audretsch, A. Economou, Phys. Rev. D **44**, 980 (1991)
15. S. Deser, R. Jackiw, G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984)
16. J.R. Gott, M. Alpert, Gen. Relativ. Gravit. **16**, 243 (1984)
17. A. Barros, V.B. Bezerra, C. Romero, Mod. Phys. Lett. A **18**(37), 2673–2679 (2003)
18. V.B. Bezerra, Ann. Phys. **203**, 392 (1990)
19. P.O. Mazur, Phys. Rev. Lett. **57**, 8 (1986)
20. B. Jensen, H.H. Soleng, Phys. Rev. D **45**, 3528 (1992)
21. H. Soleng, Phys. Rev. D **49**, 1124 (1994)
22. N. Ozdemir, Int. J. Mod. Phys. A **20**, 2821 (2005)
23. L.C. Garcia de Andrade, Cosmic strings and closed time-Like curves in teleparallel gravity. arXiv:gr-qc/0102094v1 (2001)
24. R.J. Slagter, Phys. Rev. D **54**, 4873 (1996)
25. A. Mostafazadeh, J. Phys. A Math. Theor. **31**, 7829 (1998)
26. C.R. Muniz, V.B. Bezerra, M.S. Cunha, Ann. Phys. **350**, 105 (2014)
27. G. de A. Marques, C. Furtado, V.B. Bezerra, F. Moraes, J. Phys. A Math. Theor. **34**, 5945 (2001)
28. K. Bakke, L.R. Ribeiro, C. Furtado, J.R. Nascimento, Phys. Rev. D **79**, 024008 (2009)
29. K. Bakke, Braz. J. Phys. **42**, 437 (2012)
30. E.R.F. Medeiros, E.R.B. de Mello, Eur. Phys. J. C **72**, 2051 (2012)
31. M.J. Bueno, C. Furtado, A.M. de M. Carvalho, Eur. Phys. J. B **85**, 53 (2012)
32. B. Linet, Phys. Rev. D **33**, 1833 (1986)
33. T. Souradeep, V. Sahni, Phys. Rev. D **46**, 1616 (1992)
34. R.M. Kulsrud, E.G. Zweibel, Rep. Prog. Phys. **71**, 046901 (2008)
35. J. Audretsch, G. Schaefer, Gen. Relativ. Gravit. **9**, 243 (1978)
36. E.R. Arriola, A. Zarzo, J.S. Dehesa, J. Comput. Appl. Math. **37**, 161–169 (1991)
37. A. Ronveaux (ed.), *Heun Differential Equations* (Oxford University Press, London, 1995)
38. A. Decarreau, P. Maroni, A. Robert, Ann. Soc. Sci. Brux. **92**, 151 (1978)
39. H.S. Vieira, V.B. Bezerra, J. Math. Phys. **56**, 092501 (2015)
40. C. Furtado, F. Moraes, J. Phys. A Math. Gen. **33**, 5513 (2000)
41. J.S. Carvalho, E. Passos, C. Furtado, F. Moraes, Eur. Phys. J. C **57**, 817 (2008)
42. K.D. Krori, P. Borgohain, D. Das, J. Math. Phys. **35**, 1032 (1994)
43. K. Konno, R. Takahashi, Phys. Rev. D **85**, 061502(R) (2012)
44. A.S. Goldhaber, M.M. Nieto, Rev. Mod. Phys. **82**, 939–979 (2010)
45. G. Alencar, C.R. Muniz, R.R. Landim, I.C. Jardim, R.N. Costa Filho, Phys. Lett. B **759**, 138 (2016)