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RELATIVISTIC POTENTIAL WITH QCD LARGE Q^2 BEHAVIOUR AND THE DECAY FORM FACTORS OF MESONS

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ABSTRACT

In the framework of single-time formulation of QFT, a quark bound state model, a quasipotential approach and QCD asymptotic freedom are combined to calculate the decay form factors of the pion. For the relativistic potential with QCD large Q^2 behaviour the exact solution of the quasipotential equation for a bound state wave function is used to find explicit expressions for the decay constants of the processes $\pi^0 \to 2\gamma$ and $\pi^0 \to \mu\bar{\nu}$. The connection found between the two constants agrees nicely with experimental data when $m_g \cong 230$ MeV. Further, a relation between the lepton decay constant and asymptotics of the elastic form factor is obtained that coincides in the lowest order with the well-known QCD perturbation theory result.

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At the present stage of the development of the elementary particle theory, when at least all the hadrons turn out to be composite objects and manifest a complicated structure, the majority of problems cannot be solved without a knowledge of hadron wave functions. For example, elastic form factors, deep inelastic structure functions, decay constants, having been studied by many authors during recent years, are mainly determined by properties of the wave functions.

In the simplest case of a quark model, the hadrons are considered as bound states of a few quarks. However, the bound state problem could not be consistently solved in the framework of quantum field theory. QCD, whilst giving good rules for the calculation of various subprocesses of quark interactions, does not answer the question of composite particle wave function as well.

In part, this is related to the fact that we could not go further than the perturbation theory and obtain exact solutions. It seems to us that for solving the problem of bound states it is more reasonable to use relativistic dynamical equations derived in accordance with the general principles of quantum field theory, and to try to find bound state wave functions as exact solutions of these equations. One such equation is the Bethe-Salpeter one, but it contains several internal difficulties, so we shall use here the so-called quasi-potential equation which was first proposed in Ref.1) in the framework of the single-time formulation of QFT.

The analogy with the Schrödinger equation is an attractive feature of this equation which allows us to choose its kernel (quasipotential) phenomenologically in a form of classical potential. In contrast to the Bethe-Salpeter formalism, all particles in the single-time approach are on their mass shells. Therefore it turned out to be possible to use, instead of the usual three-dimensional Fourier transformation in Minkowsky space, a relativistic three-dimensional expansion over the unitary representations of the Lorentz group - the group of motions in Lobachevsky space[2)]. This new transformation defines new relativistic three-dimensional configuration space and in the non-relativistic limit it goes to the usual three-dimensional Fourier transformation.

In the present paper we shall consider decays of the pion consisting of two equal mass scalar quarks within the framework of the single-time formalism. This problem was first solved in Refs.3,4) and later on relativistic corrections[5)] and higher orders in QCD were obtained[6,7)]. We shall try to use the exact solution of the quasipotential equation obtained in Refs.8,9) for the single-time wave function of a quark-antiquark bound state with a quasipotential corresponding to one-gluon exchange in QCD[10)] to calculate the form factors of decays $\pi \to 2\gamma$ and $\pi \to \mu\bar{\nu}$. Thus we shall automatically take into account all relativistic and non-perturbative effects of quark anti-quark interaction in QCD.

In accordance with the general rules of constructing the amplitudes for processes with the participation of composite particles in single-time formalism[11,12] we write the amplitude of the $\pi \to 2\gamma$ decay in c.m.s. in the following form (Fig. 1):

$$M_{\pi\to 2\gamma}(\vec{q}_1/M)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3\vec{k}_1}{2k_1^0}\,T_{q\bar{q}\to 2\gamma}(\vec{q}_1;\vec{k}_1/M)\,\psi_{BM}(\vec{k}_1), \tag{1}$$

where

$$M_{\pi\to 2\gamma}(\vec{q}_1/M)=f_{\pi\to 2\gamma}\,e_1^m e_2^n\,\epsilon_{mnkl}\,q_1^k q_2^l\,u_\pi, \tag{2}$$

$T_{q\bar{q}\to 2\gamma}$ is an annihilation amplitude of the quark-antiquark pair and $\psi_{BM}(\vec{K}_1)$ is a single-time wave function of the pion, i.e. the quark-antiquark bound state, in momentum space.

The wave function expansion over the unitary representations of the group of motions in the Lobachevsky space has the following form[2]:

$$\psi_{BM}(\vec{k}_1)=\frac{2m}{(2\pi)^3}\int\frac{d^3\vec{k}_1}{2k_1^0}\,\xi(\vec{k}_1,\vec{r})\,\psi_{BM}(\vec{r}), \tag{3}$$

or, for the 1 = 0 case which we treat here, due to the spherical symmetry

$$\psi_{BM}(\vec{k}_1)=\frac{4\pi}{m\sinh x}\int_0^\infty dr\, r\sin mrx\,\varphi_{BM}(r), \tag{4}$$

where $x=\ln((k_1^0+k_1)/m)$ is the rapidity of the quark.

The radial wave function $\psi_{BM}(r)$ satisfies (5) the quasipotential equation in configuration $\vec{r}$-space[2]:

$$\hat{H}_0(M-\hat{H}_0)\varphi_{BM}(r)=2mV(r)\varphi_{BM}(r), \tag{5}$$

where $\hat{H}_0$ is a finite-difference operator:

$$\hat{H}_0=2m\cosh\frac{i}{m}\frac{\partial}{\partial r}+\frac{2i}{r}\sinh\frac{i}{m}\frac{\partial}{\partial r}. \tag{6}$$

and V(r) is a quasipotential of quark-antiquark interaction in this $\vec{r}$-space.

For simplicity we consider here only scalar quarks, so the pion should be scalar in this model and we do not concern ourselves with the chiral symmetry. However, just to draw our calculations technically nearer to the real situation, we shall imitate the pseudoscalar coupling by the special choice of the amplitude of quark-antiquark annihilation:

$$T_{q\bar{q}\to 2\gamma}(\vec{q}_1;\vec{k}_1|M) = \frac{4\pi\alpha S}{m^2-(k_1-q_1)^2} \times$$

$$\times\, e_1^m e_2^n \epsilon_{mnkl}\, \bar{u}_{\bar{q}}(\vec{k}_2)(k_1-k)^k(k_2+k)^l u_q(k_1) + (q_1 \leftrightarrow q_2), \qquad (7)$$

where $k_2 = q_1 + q_2 - k_1$ and $S = \frac{1}{2}[e_q^2 - (e_q - 1)^2]$. One can check that further results differ from the spinor quark case only in some numerical factors.

The substitution of the expression (7) into (1) gives the following formula for the decay form factor:

$$f_{\pi\to 2\gamma} = \frac{4\sqrt{2\pi}\,\alpha S}{M}\int_0^\infty \frac{k_1\,dk_1}{k_1^0}\ln\frac{k_1^0+k_1}{m}\,\psi_{BM}(\vec{k}_1). \qquad (8)$$

Using the transformation (4) we can rewrite (8) in configuration space as follows:

$$f_{\pi\to 2\gamma} = \frac{2(2\pi)^{5/2}\alpha S}{M m^2}\,\varphi_{BM}(0). \qquad (9)$$

Thus, the decay constant as in the nonrelativistic case [4,13] is proportional to the wave function at the origin, but in relativistic configuration space.

Here we did not use either perturbation theory or any special input for quark-antiquark interaction. We have used only the transformation (3) of the relativistic wave function which introduces new configuration space conjugated to the mass hyperboloid of a quark.

If we now introduce a new wave function

$$\chi_{BM}(r) = \left[\frac{(2\pi)^3 M}{8m^2}\right]^{1/2}\varphi_{BM}(r) \qquad (10)$$

which in the non-relativistic limit is normalized in the usual way:

$$\int d^3\vec{r}\, |\chi_{BM}(r)|^2 = 1, \tag{11}$$

we shall get the following expression for the decay constant instead of (8):

$$f_{\pi\to 2\gamma} = \frac{4\sqrt{2}\,\alpha_e S m}{\pi M^{3/2}} \int_0^\infty \frac{k_1 dk_1}{k_1^0} \ln \frac{k_1^0 + k_1}{m}\, \chi_{BM}(k_1). \tag{12}$$

This expression coincides with the relativistic formula obtained in Ref.5), except for the absence of two additional factors $\sqrt{2}$ and $\sqrt{3}$ appearing due to spin and colour. However, the authors get the result (9) only in the static limit because they use the usual Fourier transformation for the relativistic wave function. As we have seen, the use of the relativistic transformation (3) leads us straight from eq.(8) to eq.(9).

For the quasipotential V(r) of quark-antiquark interaction in eq.(5) we shall take the usual Coulomb form in our configuration space:

$$V(r) = -\frac{g^2}{r} \tag{13}$$

which was used in Refs. 9,10) in order to calculate the elastic form factor. It was shown in Ref.10) that in this case the one-gluon exchange amplitude has the same asymptotics at very large Q^2 as that in QCD. However, in this formalism, such a behaviour has nothing to do with the exotic dependence of the coupling constant g^2 on Q^2 (asymptotic freedom), and this is a result of using the proper three-dimensional transformation (3) and relativistic configuration space. The potential (13) was used in Refs.5,6) for the description of meson decays, but in usual non-relativistic $\vec{r}$-space.

As we have mentioned above, the exact solution of the quasipotential equation (5) was obtained in Refs.8,9). The wave function of the ground bound state has the form:

$$\varphi_{BM}(r) = C_0 e^{-mr\chi_0}, \tag{14}$$

where the parameter χ_0 links the meson mass with that of the quark through the following formula

$$M = 2m \cos \chi_0, \tag{15}$$

c_0 has to be defined by normalization of the wave function and the quantization condition gives

$$\sin 2x_0 = g^2. \tag{16}$$

Substituting the wave function (14) into (9) we get:

$$f_{\pi\to 2\gamma} = \frac{8(2\pi)^{5/2}\alpha S c_0 \cos^2 x_0}{M^3}. \tag{17}$$

The normalization of the wave function in the single-time formalism (as well as in the Bethe-Salpeter one) requires special consideration, so we shall try to calculate c_0 in our next paper just to express $f_{\pi\to 2\gamma}$ through a quark mass (or the parameter x_0). In the present paper, we shall calculate in the same manner the decay constant $f_{\pi\to\mu\nu}$ and try to eliminate the unknown parameter c_0, thus avoiding the problem of normalization.

The amplitude of the process $\pi^0 \to \mu\bar{\nu}$ (Fig. 2) can be presented in the form:

$$M_{\pi\to\mu\bar{\nu}}(\vec{q}_1/M) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}_1}{2k_1^0}\, T_{q\bar{q}\to\mu\bar{\nu}}(\vec{q}_1;\vec{k}_1/M)\, \Psi_{BM}(\vec{k}_1), \tag{18}$$

where

$$M_{\pi\to\mu\bar{\nu}}(\vec{q}_1 P) = G f_{\pi\to\mu\bar{\nu}}\, \bar{v}_\mu(q_1)\hat{P}(1+\gamma_5)v_{\bar{\nu}}(q_2)u_\pi(P) \tag{19}$$

and P is the momentum of the pion.

For the scalar quarks we shall choose the amplitude $T_{q\bar{q}\to\mu\bar{\nu}}$ of quark-antiquark annihilation in the following form:

$$T_{q\bar{q}\to\mu\bar{\nu}}(\vec{q}_1;\vec{k}_1/M) = \bar{u}_{\bar{q}}(k_2)\left[G_V(k_1-k_2)^n + G_A(k_1+k_2)^n\right]u_q(k_1)\times$$
$$\times\, \bar{v}_\mu(q_1)\gamma_n(1+\gamma_5)v_{\bar{\nu}}(q_2). \tag{20}$$

Substituting (19) and (20) into eq.(18), it is not difficult to calculate that

$$f_{\pi\to\mu\bar{\nu}} = \frac{2G_A}{\sqrt{2\pi}\,GM}\int_0^\infty k_1^2 dk_1\, \varphi_{BM}(\vec{k}_1) \tag{21}$$

or in configuration space

$$f_{\pi\to\mu\bar{\nu}} = \frac{(2\pi)^{3/2}G_A}{GM}\left[\cosh\frac{2i}{m}\frac{\partial}{\partial r}\varphi_{BM}(r)\right]_{r=0}. \tag{22}$$

This result is exact in our relativistic approach and in contrast to the non-relativistic case[4,13], $f_{\pi\to\mu\nu}$ contains an infinite number of derivatives of the wave functions at the origin. It is easy to see that, roughly speaking, the decay constant is determined by the value of the wave function at the point $r = m^{-1}$ rather than at the origin. A similar result was obtained[6,7] in QCD expansions using the perturbation theory.

In the nonrelativistic limit, expanding the expression (22) over powers of the Compton wavelength m^{-1} we get in the lowest orders ($2m = M$):

$$f_{\pi\to\mu\bar{\nu}} = \frac{(2\pi)^{3/2}G_A}{GM}\left[\varphi_{BM}(0) - \frac{8}{M^2}\varphi''_{BM}(0) + \ldots\right]. \tag{23}$$

So in the first approximation, introducing the wave function (10), we obtain the following expression for the decay rate:

$$\Gamma_{\pi\to\mu\bar{\nu}} = \frac{G_A^2\mu^2(M^2-\mu^2)^2}{2\pi M^4}\,|\chi_{BM}(0)|^2 \tag{24}$$

(μ is a muon mass) which coincides with the result of Ref.4).

For the quasipotential (13) using the wave function (14) we find that

$$f_{\pi\to\mu\bar{\nu}} = \frac{(2\pi)^{3/2}g_A C_0\cos 2\chi_0}{M} \tag{25}$$

where $g_A = G_A/G \cong 1.22$.

Now comparing the formulae (17) and (25) we can eliminate the parameter c_0 from both form factors and obtain the following relationship:

$$f_{\pi\to 2\gamma} = \frac{16\pi\alpha S f_{\pi\to\mu\bar\nu}\cos^2 x_0}{g_A M^2 \cos 2x_0} . \tag{26}$$

It is well known that a connection between the two decay constants was found in the framework of the PCAC hypothesis in the limit $M \to 0$. The relationship (26) has a completely different nature and is based on the knowledge of the exact wave function. A separate paper will be devoted to the analysis of low-energy theorems in the single-time approach. It is not difficult to check that, in a model with uncoloured fractionally-charged quarks ($S = 1/6$), the formula (26) is in good agreement with experimental data when $m \simeq 230$ MeV ($x_0 \simeq 1.27$).

In the paper[10)] for the potential (13) we have found the exact expression for the elastic form factor of the pion which at very large Q^2 has the following asymptotics

$$F_\pi(Q^2) \cong \frac{16(2\pi)^4 g^2 c_0^2 \cos^2 x_0}{M^2 Q^2 \ln(Q^2/M^2)} . \tag{27}$$

Now we shall eliminate the parameter c_0 in the formulae (25) and (27) to get the absolute normalization of the form factor asymptotics to the decay constant $f_{\pi\to\mu\bar\nu}$ as is usually done in QCD:

$$F_\pi(Q^2) \cong \frac{32\pi g^2 f^2_{\pi\to\mu\bar\nu}\cos^2 x_0}{g_A^2 Q^2 \ln(Q^2/M^2)\cos^2 2x_0} . \tag{28}$$

In Ref.10) it was shown that as $Q^2 \to \infty$, the quasipotential (13) in momentum space has the asymptotics:

$$V(Q^2) \cong -\frac{8\pi g^2}{Q^2 \ln(Q^2/m^2)} . \tag{29}$$

In accordance with the general idea, this quasipotential should coincide (with the opposite sign) with the one-gluon exchange amplitude in QCD:

$$T^{(2)}_{QCD}(Q^2) \cong \frac{4\pi\alpha_s(Q^2)}{Q^2} , \tag{30}$$

where $\alpha_s(Q^2)$ is a QCD running coupling constant. Comparing (29) and (30) we can write down formally:

$$\alpha_s(Q^2) = \frac{2g^2}{\ln(Q^2/m^2)} \tag{31}$$

Substituting this running coupling constant into (28) we get

$$F_\pi(Q^2) \cong \frac{16\pi f^2_{\pi\to\mu\bar{\nu}}\,\alpha_s(Q^2)\cos^2 x_0}{g_A^2\, Q^2 \cos^2 2x_0}\,, \tag{32}$$

where at large Q^2 we suggest that $\ln Q^2/M^2 \cong \ln Q^2/m^2$. In the case of small binding energy (in the non-relativistic limit) such as $x_0 \to 0$ (or $g^2 \to 0$), we obtain the absolute normalization of the form factor which coincides with the QCD result[14]. The relation (32) differs from that in QCD because we used the exact solution of the quasipotential equation for the single-time wave function, and so we have obtained a more general result.

Thus in the present paper we have managed to calculate the exact expression (in the strong coupling constant) for the decay form factors of the pion in the framework of the single-time formalism, using the exact solution of the quasi-potential equation for the bound state wave function, with the simple Coulomb potential (13) in relativistic configuration space. We have derived the relationship (26) between $f_{\pi\to 2\gamma}$ and $f_{\pi\to\mu\bar{\nu}}$ at physical values of pion mass M, which, in contrast to the usual low-energy formalism, is linear and agrees with experimental data when the quark mass m = 230 MeV. A similar relationship was recently obtained in ref.[15].

The comparison of our results with those obtained in QCD shows that the chosen quasipotential corresponding to one-gluon exchange reproduces the first orders of QCD at large Q^2. However, we have to notice that, in order to describe more prescisely the real picture, one needs to introduce a more complicated quasi-potential including a confinement part and describing the spectra of the quarkonium. We shall try to do this in our following papers as well as to consider the case of spinor quarks with different masses.

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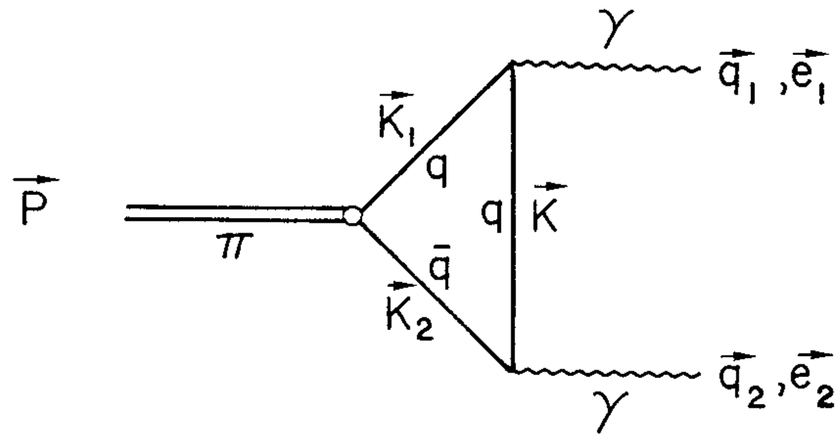
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Fig. 1

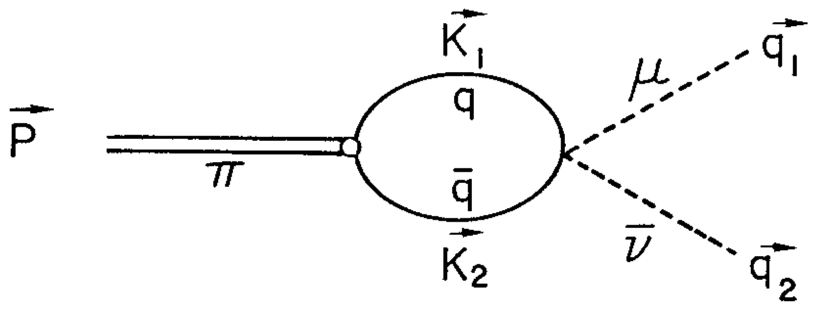
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Fig. 2

12.