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Relativistic Rotators and Bilocal Theory

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The notion of relativistic, or "particle" rotator, which is the system of four "beingrössen" centered on a moving point in Minkowski space, has recently been introduced to describe kinematically the average motion of extended particles in space time. In this paper, we further study this system as such and show its similarity with the bilocal theory introduced by Yukawa. The special example of the hyper-spherical rotator is treated in detail by replacing the original beingrössen variables with complex triad variables and relativistic Euler angles.

Introduction

By definition let us call relativistic or "particle" rotator the system kinematically defined,

a. by the coordinates $x_\mu(\tau)$ of its origin (x), and

b. by the set of four orthogonal and unitary four vectors centered on (x), or, in other words, by a moving tetrad in Minkowski space.[1)]

The corresponding set of parameters can be used to describe several types of physical problems which, very characteristically, cannot be described on the basis of the point particle model. We shall mention only two.

In a series of recent papers[2)] it has been shown that the hydrodynamical description of the spinor wave equations of quantum mechanics could not be carried out on the basis of point-like elements. Indeed, in order to describe the existence of an angular momentum density in such waves one has to introduce, alongside the coordinates $x_\mu(\tau)$ of the lines of flow and the invariant conserved density ρ supplementary "beingrössen" kinematical variables $b_\mu{}^\xi$ describing the local "spin".

On the other hand, on the basis of a general model of extended particles proposed by Møller[3)] and developed by two of us (D. B. and J. P. V.)[4)] one can show that the motion of average variables can be developed on the basis of the rotator kinematical variables. The motion of the origin of the tetrad just corresponds to the behaviour of a central geometrical point (the so-called "center of matter density") while the space and time like part of the tetrad's instantaneous

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rotation $\omega_{\mu\nu}$ describes the rotation and acceleration of matter in its neighbourhood.

Furthermore, de Broglie and two of us (P. H. and J. P. V.) have proposed the idea that the time has come to substitute relativistic rotators for point particles as classical foundation of quantum mechanics.

In the present paper, we shall make no specific physical assumption about the possible physical signification of the rotator variables and study directly relativistic rotators with the help of the usual Lagrangian and Hamiltonian method.

In section **1** we shall discuss the behaviour of isolated rotators and show that the conservation equations resulting from general Lagrangian imply the existence of a second remarkable point center of a particular inertial frame, showing the similarity of our point of view with Yukawa's bilocal theory[5] in Minkowski space.

In section **2** we shall discuss a special type of Lagrangian and show in particular that this allows a simple classical interpretation of de Broglie's relation $E=h\nu$.

Finally, in section **3** we shall introduce relativistic Euler angles as internal variables and show that these new variables greatly simplify subsequent quantization of the theory.

§ 1

According to our program let us first recall certain general results on the Lagrangian and Hamiltonian method. Let us make two basic assumptions:

A. That our rotator can be described by:

a) The coordinates $x_\mu(\tau)$ of the origin of the tetrad, where τ represents the proper time of the world line l followed by this point (x).

b) The components $b_\mu{}^\xi$ of the four vectors constituting the tetrad. Let us adopt the usual symbolic conventions. The index μ represents tensor components, varying from one to four (Latin indices representing space components vary only from one to three), and its repetition implies the usual summation. Since we calculate in Minkowski space all fourth components are purely imaginary for real vectors. In $b_\mu{}^\xi$ the index ξ (which also varies from one to four) is not a usual tensor component but rather differentiates the vectors themselves. We thus have three space-like vectors $b_\mu{}^\xi$ and one time-like vector which we write $ib_\mu{}^4$. Their orthogonal and unitary character is represented by the relations:

$$b_\mu{}^\xi b_\nu{}^\xi=\delta_{\mu\nu}, \qquad b_\mu{}^\xi b_\mu{}^\eta=\delta^{\xi\eta}. \tag{1a}$$

From $b_\mu{}^\xi$'s which are functions of the proper time τ of their origin, the instantaneous rotational velocity of the tetrad is defined by the skew tensor:

$$\omega_{\mu\nu}=b_\mu{}^\xi \dot{b}_\nu{}^\xi.$$

As we have to assume[1] that $ib_\mu{}^4$ points along the four velocity of the origin such that

$$\dot{x}_\mu = i c b_\mu{}^4, \tag{1b}$$

we see that the acceleration of x_μ and the instantaneous rotation of the tetrad in any rest frame Σ (satisfying the relations $b_k^4 = b_4^r = 0$) are determined by the vectors:

$$\alpha_\mu = \omega_{\mu\nu} b_\nu{}^4$$

$$\omega_\mu = \tilde{\omega}_{\mu\nu} b_\nu{}^4 = \frac{i}{2} \varepsilon_{\mu\nu\alpha\beta} \mathring{b}_\alpha{}^\xi b_\beta{}^\xi b_\nu{}^4$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is the completely antisymmetrical Ricci-Levi-Civita tensor and the symbol $\mathring{A}$ represents the derivative $dA/d\tau$.

B. The second assumption is that its laws of motion can be deduced from a variation principle with a scalar Lagrangian $L(\dot{x}_\alpha, \ddot{x}_\alpha, b_\alpha{}^\xi, \mathring{b}_\alpha{}^\xi)$ where all variables depend on τ only. This implies that the physical motion between two points M_1 and M_2 will correspond to the minimum of the world-line integral $\int_{M_1}^{M_2} L\, d\tau$, that is,

$$\delta \int_{M_1}^{M_2} L(\dot{x}_\alpha, \ddot{x}_\alpha, b_\alpha{}^\xi, \mathring{b}_\alpha{}^\xi)\, d\tau = 0 \tag{2}$$

for arbitrary variations δx_μ, $\delta b_\alpha{}^\xi$ which vanish at M_1 and M_2.

Following Noether[6] we then see that the corresponding Euler equations:

$$\frac{d}{d\tau} G_\mu = 0 \qquad \text{with} \qquad G_\mu = \frac{\partial L}{\partial \dot{x}_\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial \ddot{x}_\mu} \tag{3a}$$

and

$$\frac{d}{d\tau} \frac{\partial L}{\partial \mathring{b}_\mu{}^\xi} = \frac{\partial L}{\partial b_\mu{}^\xi}, \tag{3b}$$

which determine the equations of motion, imply two conservation equations. Indeed (2) must remain unchanged under any variation resulting from δx_μ's and $\delta b_\mu{}^\xi$'s determined by an arbitrary infinitesimal Lorentz transform $\delta\varepsilon_{\alpha\beta} = -\delta\varepsilon_{\beta\alpha}$. For such a transformation we have

$$\delta x_\mu = \delta\varepsilon_{\mu\nu} x_\nu, \qquad \delta b_\mu{}^\xi = \delta\varepsilon_{\mu\nu} b_\nu{}^\xi. \tag{4}$$

From the invariance of L with respect to this infinitesimal Lorentz transformation we obtain the relation

$$\mathring{M}_{\alpha\beta} = G_\alpha \dot{x}_\beta - G_\beta \dot{x}_\alpha, \tag{5}$$

where

$$M_{\alpha\beta} = \left(b_\alpha{}^\xi \frac{\partial L}{\partial \mathring{b}_\beta{}^\xi} + \dot{x}_\alpha \frac{\partial L}{\partial \ddot{x}_\beta} \right) - \left(b_\beta{}^\xi \frac{\partial L}{\partial \mathring{b}_\alpha{}^\xi} + \dot{x}_\beta \frac{\partial L}{\partial \ddot{x}_\alpha} \right).$$

As is known[7] relations (3) and (5) represent the two essential relations of conservation of momentum and angular momentum of particles with extended structures.

Starting from A. and B., the next step is to pass to Hamiltonian formalism. If L depends in general on a certain set of kinematical variables q_μ^ξ (where α denotes the usual components and ξ differentiates the variables) so that $L=L(q_\alpha^\xi, \dot{q}_\alpha^\xi)$ we can introduce the set of canonical variables Π_α^ξ by the relations

$$\Pi_\alpha^\xi=\partial L/\partial\dot{q}_\alpha^\xi$$

and define a relativistic Hamiltonian H by the expression

$$H(\Pi_\alpha^\xi, q_\alpha^\xi)=\Pi_\alpha^\xi\dot{q}_\alpha^\xi-L(q_\alpha^\xi, \Pi_\alpha^\xi) \tag{6}$$

where we have expressed in $L(q_\alpha^\xi, \dot{q}_\alpha^\xi)$ $\dot{q}_\alpha^\xi$ in terms of Π_α^ξ. The equations of motion can then be written in the well-known form:

$$\dot{\Pi}_\alpha^\xi=-\partial H/\partial q_\alpha^\xi \quad \text{and} \quad \dot{q}_\alpha^\xi=\partial H/\partial\Pi_\alpha^\xi. \tag{7}$$

H is evidently a constant of the motion since

$$\dot{H}=\frac{\partial H}{\partial\Pi_\alpha^\xi}\dot{\Pi}_\alpha^\xi+\frac{\partial H}{\partial q_\alpha^\xi}\dot{q}_\alpha^\xi=0$$

in view of (7). We shall see that H is just proportional to the rest mass term as in the point particle case. From relations (7) we deduce moreover that the proper time derivation of any function f along the world line followed by x_μ is given by

$$\dot{f}=df/d\tau=[f, H] \tag{8}$$

where [ ] denotes the usual Poisson bracket with respect to the variables q_μ^ξ and their canonical momenta.

As an example, let us describe in this way the usual point particle. We write $L=\frac{1}{2}m\dot{x}_\mu\dot{x}_\mu$ and obtain immediately

$$G_\mu=m\dot{x}_\mu,$$

$$H=\frac{1}{2m}G_\mu G_\mu. \tag{9}$$

The canonical equations become $\dot{G}_\mu=0$, so that

$$H=-\frac{1}{2}mc^2 \quad \text{since} \quad \dot{x}_\mu\dot{x}_\mu=-c^2.$$

The Hamiltonian formalism for the case where

$$L=L(x_\mu, \dot{x}_\mu, \ddot{x}_\mu, q_\mu, \dot{q}_\mu)$$

is performed by the Ostrogradski's method.[8] Introducing the new variables

$$\dot{x}_\mu=v_\mu, \qquad n_\mu=\frac{\partial L}{\partial\ddot{x}_\mu}$$

and putting

$$G_\mu=\frac{\partial L}{\partial\dot{x}_\mu}-\dot{n}_\mu,$$

and

$$H=G_\mu v_\mu+n_\mu \ddot{x}_\mu-L(x_\mu,\ v_\mu,\ \ddot{x}_\mu),$$

we get

$$\dot{x}_\mu=\partial H/\partial G_\mu, \qquad \dot{G}_\mu=-\partial H/\partial x_\mu,$$

$$\dot{v}_\mu=\partial H/\partial n_\mu, \qquad \dot{n}_\mu=-\partial H/\partial v_\mu,$$

which determine the evolution of the canonically conjugated variables (x_μ, G_μ) and (v_μ, n_μ). This form will be employed later.

If we now depart from the point particle model and introduce new parameters such as b_μ^ξ into the Lagrangian, we can show immediately that such a step implies to pass from local to bilocal theory.

Indeed, let us call r_μ the four vector defined by

$$M^2 r_\mu=M_{\mu\nu}G_\nu, \tag{10}$$

where $-M^2=G_\mu G_\mu$ is evidently a constant of the motion, and we assume $M>0$.

If we introduce the world point y with coordinates defined by

$$y_\mu(\tau)=x_\mu-r_\mu=x_\mu(\tau)-M_{\mu\nu}(\tau)G_\nu(\tau)/M^2, \tag{11}$$

we can show that this point (which corresponds in the extended droplet theory to Møller's so-called "center of gravity") moves, in the absence of exterior forces, along a straight world line l_0 with a four velocity $u_\mu=G_\mu/M$ where M is the preceding constant of the motion.

To show this, let us differentiate (11) with respect to τ. We get

$$dy_\mu/d\tau=\dot{x}_\mu-(G_\mu\dot{x}_\nu-G_\nu\dot{x}_\mu)G_\nu/M^2=\dot{x}_\mu+G_\mu\frac{m}{M^2}-\dot{x}_\mu=\frac{m}{M^2}G_\mu, \tag{12}$$

where we have written $m=-G_\mu\dot{x}_\mu$. This proves that y_μ moves with a four velocity parallel to G_μ, the relation between the proper time $d\tau'$ of l_0 and $d\tau$ being $d\tau'=(m/M)d\tau$.

Thus the introduction of new "line" variables alongside the coordinate x_μ determines a second point y_μ moving in a straight line l_0 around which l spirals in a more or less complex way according to the exact form of L. Moreover, if we introduce an inertial frame II (in which $G_i=0$) we see that r_μ is purely space-like in that frame since $r_\mu G_\mu=0$.

Both points are in a sense canonically associated, x_μ being connected with kinematical and y_μ with dynamical variables. We notice immediately that our model is quite similar to Yukawa's classical model of bilocal theory[5] and thus the introduction of new "line" variables $b_\mu^\xi(\tau)$ necessarily implies a passage from "local" to "bilocal" theory. This is not very astonishing since Yukawa's model was precisely proposed as the simplest possible extension of the point particle idea. Our model, however, implies more degrees of freedom than the simple bilocal model. As was said before, the b_μ^ξ variables associated with the relativistic rotator can be understood as describing "internal" average motions of matter in the immediate neighbourhood of x_μ in the case of extended relativistic particles. In this light Yukawa's model appears as a very simplified schematization of extended particle models.

On the other hand the new variables may be thought as a possible classical example of the "hidden" variables introduced *a priori* in the causal interpretation of quantum mechanics. Naturally, they are only "hidden" in the sense that we usually neglect them when we suppress particle extension in space time, reduce world tubes to world lines and leave aside internal motions (rotations, etc.) of matter around typical average points such as the center of matter density.

§ 2

We shall now study as a typical example of the preceding formalism the relativistic rotator described by the Lagrangian:

$$L=-\frac{1}{4}I\omega_{\alpha\beta}\omega_{\alpha\beta}+\lambda_{\mu\nu}\left(b_\mu{}^r b_\nu{}^r-\frac{1}{c^2}\dot{x}_\mu\dot{x}_\nu-\delta_{\mu\nu}\right)-\frac{\lambda}{4}(\omega_{\alpha\beta}\omega_{\alpha\beta}-K^2) \tag{13}$$

with $r\sim 1, 2, 3$ and

$$\omega_{\alpha\beta}=\dot{b}_\alpha{}^r b_\beta{}^r-\frac{1}{c^2}\ddot{x}_\alpha\dot{x}_\beta\,.$$

We shall assume moreover that the four vectors $\dot{x}_\alpha$ and $b_\alpha{}^r$ are physically initially well determined but that $b_\alpha{}^1$ and $b_\alpha{}^2$ can be arbitrarily rotated around the hyperplane containing $b_\alpha{}^3$ and $\dot{x}_\alpha=icb_\alpha{}^4=v_\alpha$ without changing the motion. This corresponds, as we shall see, to gauge invariance.

In (13) the first term

$$T=-\frac{1}{4}I\omega_{\alpha\beta}\omega_{\alpha\beta}=-\frac{1}{4}I\dot{b}_\mu{}^\xi b_\mu{}^\xi \tag{14}$$

$$(\dot{x}_\alpha=icb_\alpha{}^4),\quad \xi,\ \eta\sim 1, 2, 3, 4$$

is a formal relativistic generalization of the rotation energy of the three-dimensional spherical rigid body, the term $I+\lambda$ playing the part of the non-relativistic moment of inertia. This results from the fact that the corresponding angular momentum tensor is

$$M_{\alpha\beta}=\dot{x}_\alpha\frac{\partial L}{\partial\ddot{x}_\beta}-\dot{x}_\beta\frac{\partial L}{\partial\ddot{x}_\alpha}+b_\alpha{}^r\frac{\partial L}{\partial\dot{b}_\beta{}^r}-b_\beta{}^r\frac{\partial L}{\partial\dot{b}_\alpha{}^r}=(I+\lambda)\,\omega_{\alpha\beta}\,, \tag{15}$$

if we take into account the relations

$$b_\mu{}^r b_\nu{}^r-\frac{1}{c^2}\dot{x}_\mu\dot{x}_\nu=\delta_{\mu\nu}\,.$$

This results immediately from the second term $\lambda_{\mu\nu}(b_\mu{}^r b_\nu{}^r-1/c^2\cdot\dot{x}_\mu\dot{x}_\nu-\delta_{\mu\nu})$ in (13), with the symmetrical Lagrange multipliers $\lambda_{\mu\nu}$. The third term is also a Lagrange condition which means the constancy of $\omega_{\alpha\beta}\omega_{\alpha\beta}$.

We here note that our rotator is apparently similar to the one considered by Nakano[9]. However, our forth axis plays a distinguished rôle through the identification (1b), while Nakano does not make such identification because his beingrössen parameters are associated to an Euclidian space in contrast to our $b_\mu^{\,\xi}$ which defines an internal Lorentz space. On the other hand he imposes the dynamical subsidiary condition which is not assumed in our case, i. e., $M_{\mu\nu}G_\nu=0$. This condition means that the point y coincides with x and thus the bilocal feature degenerates.

Since (13) is symmetrical in the internal Lorentz space (i. e., for $\xi=1,2,3,4$) we shall call the present model "hyper-spherical" rotator, which is also a quite special example of possible relativistic rotators.

Now the equations of motion result immediately by applying the formulas given in § 1. We get with $n_\alpha=\partial L/\partial\ddot{x}_\alpha$

$$\dot{G}_\alpha=\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{x}_\alpha}-\dot{n}_\alpha\right)=\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{x}_\alpha}-\frac{d}{d\tau}\frac{\partial L}{\partial\ddot{x}_\alpha}\right)=0 \tag{16a}$$

$$\dot{M}_{\alpha\beta}=G_\alpha\dot{x}_\beta-G_\beta\dot{x}_\alpha \tag{16b}$$

and

$$M_{\alpha\beta}\dot{x}_\beta=(I+\lambda)\ddot{x}_\alpha \tag{16c}$$

which constitute a simple generalization of Weyssenhoff's usual relations.[7]

To solve the equations of motion we first change variables and write:

$$\dot{x}_\alpha=icb_\alpha^{\,4}=v_\alpha\,. \tag{17}$$

This is legitimate since L does not depend directly on x_α. Starting then from (16c), we get

$$\begin{aligned}M_{\alpha\beta}G_\alpha v_\beta&=\frac{1}{2}M_{\alpha\beta}(G_\alpha v_\beta-G_\beta v_\alpha)=\frac{1}{2}M_{\alpha\beta}\dot{M}_{\alpha\beta}\\&=(I+\lambda)G_\alpha\ddot{x}_\alpha=-(I+\lambda)\dot{m}c^2.\end{aligned} \tag{18}$$

This is an important relation which shows that $M_{\alpha\beta}M_{\alpha\beta}$ and m are simultaneous constants of the motion, since we shall show that λ is a constant of the motion. To prove this we first note that if we insert the expression (15) into (18) we get

$$\frac{1}{2}M_{\alpha\beta}\dot{M}_{\alpha\beta}=\frac{1}{2}(I+\lambda)\dot{\lambda}K^2,$$

so that the relation (18) is written as

$$\dot{m}c^2=-\frac{1}{2}\dot{\lambda}K^2. \tag{19}$$

Next we apply the Hamiltonian formalism stated in the preceding section to the form (13) of the Lagrangian.

We get

$$\begin{cases} \Pi_\alpha^{\ r} = \dfrac{\partial L}{\partial \dot{b}_\alpha^{\ r}} = -\dfrac{1}{2}(I+\lambda)\dot{b}_\alpha^{\ r}, \\ n_\alpha = \dfrac{\partial L}{\partial \ddot{x}_\alpha} = \dfrac{1}{2c^2}(I+\lambda)\ddot{x}_\alpha, \end{cases} \tag{20}$$

so that

$$\Pi_\alpha^{\ r}\dot{b}_\alpha^{\ r} + n_\alpha \ddot{x}_\alpha = \frac{1}{2}(I+\lambda)\,\omega_{\alpha\beta}\,\omega_{\alpha\beta},$$

and

$$\begin{aligned} H &= G_\alpha v_\alpha + \Pi_\alpha^{\ r}\dot{b}_\alpha^{\ r} + n_\alpha \ddot{x}_\alpha - L \\ &= -mc^2 - \frac{1}{4}(I+\lambda)\,\omega_{\alpha\beta}\,\omega_{\alpha\beta} - \lambda_{\alpha\beta}\left(b_\alpha^{\ r} b_\beta^{\ r} - \frac{1}{c^2}\dot{x}_\alpha \dot{x}_\beta - \delta_{\alpha\beta}\right) + \frac{\lambda}{4}(\omega_{\alpha\beta}\,\omega_{\alpha\beta} - K^2) \end{aligned} \tag{21}$$

As we know that H and $\omega_{\alpha\beta}\,\omega_{\alpha\beta}$ are constants of the motion, the derivation of (21) yields that

$$\dot{m}c^2 = -\frac{1}{4}\dot{\lambda}\omega_{\alpha\beta}\,\omega_{\alpha\beta} = -\frac{1}{4}\dot{\lambda}K^2. \tag{22}$$

The relation (22) compared with (19) proves the fact we anticipated $\dot{\lambda}=0$. We thus conclude therefore that m and $M_{\alpha\beta}M_{\alpha\beta}$ are also constants of the motion.

On the basis of this result,* the complete integration of the motion can be carried out explicitly as has been developed by Halbwachs.[10] Starting from the basic relations (16) and the constancy of m and $M_{\alpha\beta}M_{\alpha\beta}$, he has obtained the following results:

a. The " spin " $S_\mu = \widetilde{M}_{\nu\mu}\dot{x}_\nu$ is a constant of the motion ($\dot{S}_\mu = 0$) with a constant length, S: $S^2 = S_\mu S_\mu$

where
$$\widetilde{M}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}M_{\alpha\beta}.$$

b. The acceleration $\ddot{x}_\mu$ has also a constant length and is always parallel to

$$r_\mu = \frac{1}{M^2 c^2}M_{\mu\nu}G_\nu.$$

c. The " radius " r_μ has also a constant length ($r_\mu r_\mu$=constant) and remains perpendicular in the frame Π (for which $G_i=0$) to the spin. As a consequence, in that frame, the motion of x_μ around y_μ reduces to a circular motion with constant angular velocity Ω with

$$\Omega^2 \simeq M^2 c^4 / S_\mu S_\mu.$$

* More general analysis of the motion (without necessarily requiring the constancy of $M_{\alpha\beta}{}^2$) was performed by one of the authors (T.T.). This reveals the relations between six conserved quantities of internal motion which are identified as the known intrinsic properties of elementary particles (to be published).

This, as Møller[3)] remarked, is just the classical analogue of Schrödinger's Zitterbewegung.

d. The rotation of the frame defined by b_μ^r is necessarily constant. This can be shown in the following way: we first notice that the quantities

$$\omega^i = \frac{1}{2}\varepsilon^{ijk}\dot{b}_\mu{}^j b_\mu{}^k$$

are constants of the motion ($\dot{\omega}^i=0$) and represent the projections of the instantaneous rotation velocity on the moving frame (b_μ^r). According to our preceding results the "spin" has as projections on the axes of the moving tetrad constant components $I\omega^1$, $I\omega^2$, $I\omega^3$ and 0. If it points initially along $b_\mu{}^3$ (that is, if we start with $\omega^1=\omega^2=0$), it will remain so and $b_\mu{}^1$ and $b_\mu{}^2$ will rotate around $b_\mu{}^3$ with a constant angular velocity $\omega=\dot{b}_\mu{}^1 b_\mu{}^2$, so that $\dot{b}_\mu{}^1=\omega b_\mu{}^2$. Naturally in that case $b_\mu{}^1$ and $b_\mu{}^2$ are not physically completely determined since we can always rotate them by a constant angle around $b_\mu{}^3$.

The preceding motion can also be understood as the classical analogue of De Broglie's relation $E=h\nu$. Indeed if we start with the initial relation,

$$mc^2=-G_\alpha\dot{x}_\alpha=S_\mu\omega_\mu=(I\omega_\mu)\,\omega_\mu, \tag{23}$$

this relation will be conserved by the motion so that if we suppose that $I\omega=S=\hbar$ initially we get

$$E=h\nu=mc^2$$

in the rest frame. The total motion then appears as a spiral motion of the origin x combined with a space rotation of the tetrad on itself, a behaviour already strangely similar to quantum theoretical motions.

§ 3

We shall now introduce a new set of kinematical parameters: the relativistic Euler angles. This step is justified by the fact that it allows simple comparison with the non-relativistic theory of rotating spherical rigid bodies and also, as we shall see in a subsequent paper, facilitates the subsequent quantization of the theory.

The introduction of the new variables rests on the remark that the $b_\mu{}^\xi$ parameters define only six independent quantities taking relations (1) into account. This means that the orientation of the $b_\mu{}^\xi$ frame Σ_0 with respect to any fixed laboratory frame Π_0 (defined by the tetrad $a_\mu{}^\xi$ also satisfying (1)) is determined (except for an arbitrary constant rotation) by the six parameters[11)] of the homogeneous Lorentz transformation which transforms Π_0 into Σ_0.

As one knows[12)] these relativistic Euler angles determines the Lorentz transformation from $a_\mu{}^\xi$ to $b_\mu{}^\xi$ by the matrix relation:

$$\begin{pmatrix} b_\mu{}^1 \\ b_\mu{}^2 \\ b_\mu{}^3 \\ b_\mu{}^4 \end{pmatrix} = \begin{pmatrix} \cos(\varphi^+/2) & \sin(\varphi^+/2) & 0 & 0 \\ -\sin(\varphi^+/2) & \cos(\varphi^+/2) & 0 & 0 \\ 0 & 0 & \cos(\varphi^+/2) & \sin(\varphi^+/2) \\ 0 & 0 & -\sin(\varphi^+/2) & \cos(\varphi^+/2) \end{pmatrix}$$

$$\times \begin{pmatrix} \cos(\varphi^-/2) & \sin(\varphi^-/2) & 0 & 0 \\ -\sin(\varphi^-/2) & \cos(\varphi^-/2) & 0 & 0 \\ 0 & 0 & \cos(\varphi^-/2) & -\sin(\varphi^-/2) \\ 0 & 0 & \sin(\varphi^-/2) & \cos(\varphi^-/2) \end{pmatrix} \begin{pmatrix} \cos(\theta^+/2) & 0 & -\sin(\theta^+/2) & 0 \\ 0 & \cos(\theta^+/2) & 0 & -\sin(\theta^+/2) \\ \sin(\theta^+/2) & 0 & \cos(\theta^+/2) & 0 \\ 0 & \sin(\theta^+/2) & 0 & \cos(\theta^+/2) \end{pmatrix}$$

$$\times \begin{pmatrix} \cos(\theta^-/2) & 0 & -\sin(\theta^-/2) & 0 \\ 0 & \cos(\theta^-/2) & 0 & \sin(\theta^-/2) \\ \sin(\theta^-/2) & 0 & \cos(\theta^-/2) & 0 \\ 0 & -\sin(\theta^-/2) & 0 & \cos(\theta^-/2) \end{pmatrix} \begin{pmatrix} \cos(\psi^+/2) & \sin(\psi^+/2) & 0 & 0 \\ -\sin(\psi^+/2) & \cos(\psi^+/2) & 0 & 0 \\ 0 & 0 & \cos(\psi^+/2) & \sin(\psi^+/2) \\ 0 & 0 & -\sin(\psi^+/2) & \cos(\psi^+/2) \end{pmatrix}$$

$$\times \begin{pmatrix} \cos(\psi^-/2) & \sin(\psi^-/2) & 0 & 0 \\ -\sin(\psi^-/2) & \cos(\psi^-/2) & 0 & 0 \\ 0 & 0 & \cos(\psi^-/2) & -\sin(\psi^-/2) \\ 0 & 0 & \sin(\psi^-/2) & \cos(\psi^-/2) \end{pmatrix} \begin{pmatrix} a_\mu{}^1 \\ a_\mu{}^2 \\ a_\mu{}^3 \\ a_\mu{}^4 \end{pmatrix}. \tag{24}$$

The complex angles $\omega^+ = \{\theta^+, \varphi^+, \psi^+\}$ and $\omega^- = \{\theta^-, \varphi^-, \psi^-\}$ correspond to the relativistic generalization of the three-dimensional Euler angles. They are defined by the relations:

$$\varphi^\pm = \varphi_1 \pm i\varphi_2, \quad \theta^\pm = \theta_1 \pm i\theta_2, \quad \psi^\pm = \psi_1 \pm i\psi_2 \tag{25}$$

where the real quantities φ_1, θ_1, ψ_1 correspond to the usual space Euler angles, while φ_2, θ_2, ψ_2 represent hyperbolic angles (varying from $-\infty$ to $+\infty$) expressing pure Lorentz transforms. The two sets ω^+ and ω^- are thus complex conjugates of each other.

Before we express the Hamiltonian as a function of these new variables we shall briefly discuss their geometrical meaning.

As already implied in the work of Einstein and Mayer,[13] we construct from $b_\mu{}^\xi$ a set of complex self-dual antisymmetrical tensors:

$$\Gamma_{\mu\nu}{}^{r\pm} = \varepsilon^{rst} b_\mu{}^s b_\nu{}^t \pm (b_\mu{}^r b_\nu{}^4 - b_\nu{}^r b_\mu{}^4),$$

of which independent components are

$$B_k^{r\pm} = b_4^r b_k^4 - b_k^r b_4^4 \pm \varepsilon_{ijk} b_i^r b_j^4. \tag{26}$$

Then it can be shown that each set of $\{B_k^{r+}\}$ and $\{B_k^{r-}\}$ behaves as a set of three complex vectors spanning a complex three-dimensional orthonormal frame of axes:

$$B_i^{r\pm} B_j^{r\pm} = \delta_{ij}, \quad B_k^{r\pm} B_k^{s\pm} = \delta^{rs}. \tag{27}$$

In exactly the same way one can construct from $a_\mu{}^\xi$ the corresponding quantities by

$$A_{\mu\nu}{}^{r\pm}=\varepsilon^{rst}a_\mu{}^s a_\nu{}^t\pm(a_\mu{}^r a_\nu{}^4-a_\nu{}^r a_\mu{}^4),$$

$$A_k^{r\pm}=a_4{}^r a_k{}^4-a_k{}^r a_4{}^4\pm\varepsilon_{ijk}a_i{}^r a_j{}^4, \tag{26'}$$

with

$$A_i^{r\pm}A_j^{r\pm}=\delta_{ij},\quad A_k^{r\pm}A_k^{s\pm}=\delta^{rs}. \tag{27'}$$

Now, utilizing definition (26) one can demonstrate after a short calculation that

$$\begin{pmatrix} B_k^{1\pm} \\ B_k^{2\pm} \\ B_k^{3\pm} \end{pmatrix}
=\begin{pmatrix} \cos\varphi^\pm\cos\theta^\pm\cos\psi^\pm-\sin\varphi^\pm\sin\psi^\pm, & \cos\varphi^\pm\cos\theta^\pm\sin\psi^\pm+\sin\varphi^\pm\cos\psi^\pm, & -\cos\varphi^\pm\sin\theta^\pm \\ -\sin\varphi^\pm\cos\theta^\pm\cos\psi^\pm-\cos\varphi^\pm\sin\psi^\pm, & -\sin\varphi^\pm\cos\theta^\pm\sin\psi^\pm+\cos\varphi^\pm\cos\psi^\pm, & -\sin\varphi^\pm\sin\theta^\pm \\ \sin\theta^\pm\cos\psi^\pm, & \sin\theta^\pm\sin\psi^\pm, & \cos\theta^\pm \end{pmatrix}
\times\begin{pmatrix} A_k^{1\pm} \\ A_k^{2\pm} \\ A_k^{3\pm} \end{pmatrix}. \tag{28}$$

This shows that defining four complex three-dimensional frames A^+ with A_k^{r+}, A^- with A_k^{r-}, B^+ with B_k^{r+}, and B^- with B_k^{r-}, we can pass from A^+ to B^+ and from A^- to B^- by two complex three-dimensional rotations defined by the complex Euler angles ω^+ and ω^-, since relations (28) are idential (except for the complex character of all quantities) with the classical expression for real three-dimensional ordinary rotations (in terms of real Euler angles) in ordinary space.

As the set B_k^{r+}, A_k^{r+} (or B_k^{r-}, A_k^{r-}) are equivalent to $b_\mu{}^\xi$ and $a_\mu{}^\xi$ this redemonstrates in a very simple way the well-known isomorphism established by Einstein[14] and Cartan[15] between Lorentz transforms and three-dimensional complex rotation. These complex rotations can be represented in two complex conjugated three-dimensional spaces, E^+ and E^-. If we choose $a_\mu{}^\xi$ and $b_\mu{}^\xi$ as rest frames, we see that $A_k^{r\pm}=a_k^r$, and $B_k^{r\pm}=b_k^r$ so that the complex rotations ω^+ and ω^- can be understood as taking place in the three-dimensional ordinary space.

If we now return to Minkowski space we must remember that any rotation (defined by a skew symmetrical tensor) takes place around a bivector which glides on itself under the corresponding matrix transformation. The meaning of relations (24) and (28) then becomes clear. Relations (24) define a set of six successive rotations around six bivectors in a certain determined order (right to left) which bring Π_0 into Σ_0. Relation (28) defines the corresponding transformation which brings the system of bivectors $A_k^{r\pm}$ on $B_k^{r\pm}$. One notes also that any self-dual skew tensor of the $(+)$ type (such as $m_{\mu\nu}+i\tilde{m}_{\mu\nu}$) is orthogonal to the $(-)$ type $(n_{\mu\nu}-i\tilde{n}_{\mu\nu})$, so that we have identically $(m_{\mu\nu}+i\tilde{m}_{\mu\nu})(n_{\mu\nu}-i\tilde{n}_{\mu\nu})=0$.

For more details on relativistic Euler angles the reader may consult specialized mathematical papers.[12),16)]

Let us now return to the hyper-spherical rotator. Let us call T (kinematical energy) the first term in the Lagrangian (13).

In order to prepare the transition to the use of relativistic Euler angles we can first express T as a function of $B_k^{r\pm}$. A rather long but simple calculation gives

$$T=\frac{I}{2}\varepsilon_{ijk}\varepsilon_{ilm}(\dot{B}_j^{r+}B_k^{r+}\dot{B}_l^{s+}B_m^{s+}+\dot{B}_j^{r-}B_k^{r-}\dot{B}_l^{s-}B_m^{s-})=\frac{I}{2}(\dot{B}_k^{r+}\dot{B}_k^{r+}+\dot{B}_k^{r-}\dot{B}_k^{r-}). \tag{29}$$

We see here appearing for the first time an essential property of the hyper-spherical rotator: its Hamiltonian can be split into two complex conjugated parts the first depending on ω^+, the second on ω^-.

This is very natural if one remarks that the skew tensor can be split into a sum of two self-dual tensors of the types $(+)$ and $(-)$. We have

$$\omega_{\mu\nu}=\frac{1}{2}(\omega_{\mu\nu}+i\tilde{\omega}_{\mu\nu})+\frac{1}{2}(\omega_{\mu\nu}-i\tilde{\omega}_{\mu\nu})=\frac{1}{2}\omega_{\nu\mu}^{+}+\frac{1}{2}\omega_{\mu\nu}^{-}, \tag{30}$$

so that

$$\omega_{\mu\nu}\omega_{\mu\nu}=\frac{1}{4}\omega_{\mu\nu}^{+}\omega_{\mu\nu}^{+}+\frac{1}{4}\omega_{\mu\nu}^{-}\omega_{\mu\nu}^{-}=\frac{1}{2}\omega_{ik}^{+}\omega_{ik}^{+}+\frac{1}{2}\omega_{ik}^{-}\omega_{ik}^{-}, \tag{31}$$

taking into account the self-dual character of $\omega_{\mu\nu}^{+}$ and $\omega_{\mu\nu}^{-}$. Relation (31) multiplied by $I/2$ is equivalent to (29).

Introducing the expressions

$$\omega_k^{\pm}=\varepsilon_{ijk}\dot{B}_i^{r\pm}B_j^{r\pm}=\frac{1}{2}\varepsilon_{ijk}\omega_{ij}^{\pm},$$

we see that the hyper-spherical Hamiltonian can be written in the new form.

$$H=mc^2+\frac{1}{2}\vartheta\,\omega_k^{+}\omega_k^{+}+\frac{1}{2}\vartheta\,\omega_k^{-}\omega_k^{-}$$
$$+\sum_{\pm}\lambda_{ij}^{\pm}(B_i^{r\pm}B_j^{r\pm}-\delta_{ij})+\lambda/2(\omega_k^{+}\omega_k^{+}+\omega_k^{-}\omega_k^{-}-K^2) \tag{32}$$

where $$\vartheta=I+\lambda.$$

That is exactly[12)] the sum of two complex conjugated three-dimensional spherical rigid rotators (the proper time τ playing the part of ordinary time). This simplifies everything.

The angular momenta associated to $\omega_k^{\pm}$ are just

$$S_k^{\pm}=\vartheta\,\omega_k^{\pm} \tag{33}$$

and correspond to the splitting of the angular momentum $M_{\alpha\beta}$ into two self-dual parts:

$$M_{\alpha\beta}=\frac{1}{2}(M_{\alpha\beta}+i\widetilde{M}_{\alpha\beta})+\frac{1}{2}(M_{\alpha\beta}-i\widetilde{M}_{\alpha\beta})=\frac{1}{2}M_{\alpha\beta}{}^{+}+\frac{1}{2}M_{\alpha\beta}{}^{-}, \tag{34}$$

and we wrote the space components as

$$S_k{}^{\pm}=\frac{1}{2}\varepsilon_{ijk}M_{ij}^{\pm}, \tag{35}$$

while the space-time parts are just proportional to them (since $M_{\alpha\beta}{}^{\pm}=\pm i\widetilde{M}_{\alpha\beta}{}^{\pm}$), so that we can write:

$$T=\frac{1}{4\vartheta}S_k^{+}S_k^{+}+\frac{1}{4\vartheta}S_k^{-}S_k^{-}. \tag{36}$$

The physical meaning of expressions like $S_k^{\pm}$ is clear. These quantities represent the projections of the internal angular momentum $M_{\alpha\beta}$ on the fixed self-dual bivectors (or rather their space-like parts) out of which we have built the fixed frames A^{+} and A^{-}. For example, we have

$$S_3{}^{+}=M_{\alpha\beta}A_{\alpha\beta}{}^{3+}=M_{\alpha\beta}{}^{+}A_{\alpha\beta}{}^{3+} \tag{37}$$

and similar expressions for all such quantities.

Our last step is to make a new change of variables and express the Lagrangian and Hamiltonian in terms of relativistic Euler angles. This, with the help of expressions (28), can be carried out exactly like in the three-dimensional case, so we shall just recall the results.

We first calculate the projections of the angular velocities $\vec{\omega}^{\pm}$ (with components $\omega_k{}^{\pm}$) on the axes of the moving frames B^{+} and B^{-}.

We get (denoting by primed quantities such projections in order to differentiate them from unprimed quantities which represent projections on the fixed frames $A^{\pm}$):

$$\begin{cases} \omega'^{1\pm}=\cos\varphi^{\pm}\sin\theta^{\pm}\dot{\psi}^{\pm}+\sin\varphi^{\pm}\dot{\theta}^{\pm} \\ \omega'^{2\pm}=-\sin\varphi^{\pm}\sin\theta^{\pm}\dot{\psi}^{\pm}+\cos\varphi^{\pm}\dot{\theta}^{\pm} \\ \omega'^{3\pm}=\dot{\varphi}^{\pm}+\cos\theta^{\pm}\dot{\psi}^{\pm} \end{cases} \tag{38}$$

and the Lagrangian becomes with the help of (38):

$$\begin{aligned} L=&\vartheta/2(\dot{\varphi}^{+2}+\dot{\psi}^{+2}+\dot{\theta}^{+2}+2\dot{\varphi}^{+}\dot{\psi}^{+}\cos\theta^{+}) \\ &+\vartheta/2(\dot{\varphi}^{-2}+\dot{\psi}^{-2}+\dot{\theta}^{-2}+2\dot{\varphi}^{-}\dot{\psi}^{-}\cos\theta^{-})+\lambda K^2=L^{+}+L^{-}+\lambda K^2 \end{aligned} \tag{39}$$

where as we know λ and K^2 are constants.

This is exactly the sum $L^{+}+L^{-}$ of the Lagrangians of two complex conjugate three-dimensional spherical rigid bodies. Naturally the Lagrange condition terms with the multipliers $\lambda_{ij}^{\pm}$ have disappeared since the orthogonality conditions are automatically satisfied by our six Euler angles.

We can now calculate the canonical momenta associated with the new variables. Naturally, we could vary separately in the Lagrangian L^+ (and in L^-) φ_1, φ_2, θ_1, θ_2, ψ_1, and ψ_2, (that is the real and imaginary part of ω^+ and ω^-), and consider L^+ as a function of these variables; the equations of motion being defined by:

$$\delta\int L^+(\varphi_1, \varphi_2, \theta_1, \theta_2, \psi_1, \psi_2, \dot{\varphi}_1, \dot{\varphi}_2, \dot{\theta}_1, \dot{\theta}_2, \dot{\psi}_1, \dot{\psi}_2)\, d\tau=0 \tag{40}$$

and the same variational equation with L^-. However, we know that one obtains exactly the same equations of motion if one considers L^+ as a function of $\omega^+=(\theta^+, \varphi^+, \psi^+)$ and L^- as a function of $\omega^-=(\theta^-, \varphi^-, \psi^-)$ and vary these quantities independently. This can be shown directly. We get for the corresponding momenta:

$$\begin{cases} p_{\varphi\pm}=\partial L^\pm/\partial\dot{\varphi}^\pm=\mathcal{J}(\dot{\varphi}^\pm+\dot{\psi}^\pm\cos\theta^\pm) \\ p_{\psi\pm}=\partial L^\pm/\partial\dot{\psi}^\pm=\mathcal{J}(\dot{\psi}^\pm+\dot{\varphi}^\pm\cos\theta^\pm) \\ p_{\theta\pm}=\partial L^\pm/\partial\dot{\theta}^\pm=\mathcal{J}\dot{\theta}^\pm \end{cases} \tag{41}$$

so that the Hamiltonian $H=p_i\dot{q}_i-L$ becomes

$$H=\frac{1}{2}(p_{\varphi+}\dot{\varphi}^++p_{\psi+}\dot{\psi}^++p_{\theta+}\dot{\theta}^+)+\frac{1}{2}(p_{\varphi-}\dot{\varphi}^-+p_{\psi-}\dot{\psi}^-+p_{\theta-}\dot{\theta}^-), \tag{42}$$

if we neglect constant terms. Namely,

$$\begin{aligned} H&=\frac{1}{2\mathcal{J}}\left[p_{\theta+}^2+\frac{1}{\sin^2\theta^+}(p_{\varphi+}^2+p_{\psi+}^2-2p_{\varphi+}p_{\psi+}\cos\theta^+)\right] \\ &+\frac{1}{2\mathcal{J}}\left[p_{\theta-}^2+\frac{1}{\sin^2\theta^-}(p_{\varphi-}^2+p_{\psi-}^2-2p_{\varphi-}p_{\psi-}\cos\theta^-)\right] \\ &=\frac{1}{2}(s_i'^+\omega_i'^++s_i'^-\omega_i'^-) \\ &=H^++H^-, \end{aligned} \tag{43}$$

or

$$H=\frac{1}{2\mathcal{J}}(S_k^+S_k^++S_k^-S_k^-) \tag{44a}$$

$$=\frac{1}{2\mathcal{J}}(S_k'^+S_k'^++S_k'^-S_k'^-). \tag{44b}$$

One then checks immediately with the help of the usual Poisson brackets that $[S_k^+, S_j^-]=0$, $[S_k'^+, S_j'^-]=0$.

Comparing (44a) and (44b), we obtain the connection between our angular momenta and the projections of the momentum $S_k'^\pm$ in the form

$$\begin{cases} p_{\varphi\pm}=S_3'^{\pm} \\ p_{\theta\pm}=S_1'^{\pm}\sin\varphi^{\pm}+S_2'^{\pm}\cos\varphi^{\pm} \\ p_{\psi\pm}=-S_1'^{\pm}\sin\theta^{\pm}\cos\varphi^{\pm}+S_2'^{\pm}\sin\theta^{\pm}\sin\varphi^{\pm}+S_3'^{\pm}\cos\theta^{\pm} \end{cases} \tag{45}$$

so that the projections of the momentum on the frames $B^{\pm}$ and $A^{\pm}$ take the familiar form:

$$\begin{cases} S_1'^{\pm}=p_{\theta\pm}\sin\varphi^{\pm}+p_{\varphi\pm}\cot\theta^{\pm}\cos\varphi^{\pm}-p_{\psi\pm}\dfrac{\cos\varphi^{\pm}}{\sin\theta^{\pm}} \\ S_2'^{\pm}=p_{\theta\pm}\cos\varphi^{\pm}-p_{\varphi\pm}\cot\theta^{\pm}\sin\varphi^{\pm}+p_{\psi\pm}\dfrac{\sin\varphi^{\pm}}{\sin\theta^{\pm}} \\ S_3'^{\pm}=p_{\varphi\pm} \end{cases} \tag{46}$$

on $B^{\pm}$, and

$$\begin{cases} S_1^{\pm}=-p_{\theta\pm}\sin\psi^{\pm}-p_{\psi\pm}\cot\theta^{\pm}\cos\psi^{\pm}+p_{\varphi\pm}\dfrac{\cos\psi^{\pm}}{\sin\theta^{\pm}} \\ S_2^{\pm}=p_{\theta\pm}\cos\psi^{\pm}-p_{\psi\pm}\cot\theta^{\pm}\sin\psi^{\pm}+p_{\varphi\pm}\dfrac{\sin\psi^{\pm}}{\sin\theta^{\pm}} \\ S_3^{\pm}=p_{\psi\pm} \end{cases} \tag{47}$$

on $A^{\pm}$.

H^{+} and H^{-} are then evidently constants of the motion and we get, writing $(S^{\pm})^2=S_k'^{\pm}S_k'^{\pm}=S_k^{\pm}S_k^{\pm}$, the relations

$$[H,\ (S^{\pm})^2]=0,\quad [H,\ p_{\psi\pm}]=0,\quad [H,\ p_{\varphi\pm}]=0, \tag{48}$$

which correspond to classical three-dimensional properties of the non-relativistic rotators.

Concluding remarks

In conclusion we want to add a few remarks.

First we note that if we want to connect such a rotator with real physical movements inside relativistic fluid masses, it must be understood that its behaviour constitutes only an average and very crude abstraction of real internal motions since we then necessarily neglect an infinite number of possible degrees of freedom.

Secondly the introduction of this model as possible starting point for quantum theory (or elements of a new sub-quantum-mechanical level) raises many difficult but interesting problems. It is closely connected, for example, with very recent researches and ideas in quantum field theories, such as indefinite metric and non-linear waves, and we intend to discuss them in subsequent papers. Indeed, as three of us have shown (D. B., P. H. and J. P. V.), the quantization of the above rotator leads to the introduction of quantum numbers and energy levels which can be classified according to the well-known Nishijima-Gell-Mann scheme of elementary particles.

Evidently the present model is a quite special one of relativistic rotators. One of us (T. T.) has classified possible relativistic rotators from wider point of view which makes the physical meanings of various quantities and conditions clearer. Indeed, we only need the three-dimensional symmetry in the internal Lorentz space (i.e., the rotator be *spherical*) so as to obtain the conserved iso-spin components, and it is then shown that only its third component remains conserved when electromagnetic interaction is introduced. General theory of such spherical rotators includes, as its special examples, various models: the hyper-spherical rotator treated in this paper, Matthison-Weyssenhoff's particle,[7] Nakano's rigid body, Hönl-Papapetrou's particle, and others. These will also be given in separate papers.

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References

1) T. Takabayasi, Nuovo Cimento **13** (1959), 532.
F. Halbwachs, P. Hillion, and. J. P. Vigier, Nuovo Cimento (to be published).
2) D. Bohm, R. Schiller, and J. Tiomno, Nuovo Cimento Suppl. **1** (1955), 48.
T. Takabayasi and J. P. Vigier, Prog. Theor. Phys. **18** (1957), 573.
T. Takabayasi, Nuovo Cimento **7** (1958) 118; C. R. Ac. Sci. (Paris), **246** (1958), 64, 1010.
P. Hillion, Thesis (Paris, 1958).
3) C. Møller, An. Inst. H. Poincaré **11-12** (1949), 251.
4) D. Bohm and J. P. Vigier, Phys. Rev. **109** (1958), 1882.
5) H. Yukawa, Phys. Rev. **77** (1950), 219.
6) E. Noether, Götting. Nachr. (1918), 235.
7) J. Weyssenhoff and Raabe, Act. Phys. Pol. **9** (1947), 7.
8) F. Bopp, Z. Naturforsch. **3a** (1948), 564.
J. Weyssenhoff, Act. Phys. Pol. **11** (1951), 4.
9) T. Nakano, Prog. Theor. Phys. **15** (1956), 333.
10) F. Halbwachs, *Théorie relativiste des fluides à spin* (Gauthier-Villars, 1960); C. R. Ac. Sci. (Paris), **249** (1959), 2500.
11) J. L. Synge, *Relativity—The Special Theory* (North Holland Pub., Amsterdam, 1956).
12) F. Halbwachs, P. Hillion, and J. P. Vigier, An. Inst. H. Poincaré (to be published).
13) A. Einstein and W. Mayer, Sitz. Akad. Berlin (1932), 522.
14) G. Racah, lecture note (Varenna Summer school, 1958).
15) E. Cartan, *Leçons sur la théorie des spineurs*, (Herman, Paris, 1938).
16) T. Takabayasi, autocopie (1958).
C. Van Winter, Thesis (Gröningen, 1957).