Cent. Eur. J. Phys. • 6(1) • 2008 • 141-152
DOI: 10.2478/s11534-008-0018-0

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Central European Journal of **Physics**

Relativistic solution in *D*-dimensions to a spin-zero particle for equal scalar and vector ring-shaped Kratzer potential

Research Article

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Received 28 June 2007; accepted 20 August 2007

Abstract: The Klein–Gordon equation in D-dimensions for a recently proposed ring-shaped Kratzer potential is solved analytically by means of the conventional Nikiforov-Uvarov method. The exact energy bound states and the corresponding wave functions of the Klein–Gordon are obtained in the presence of the non-central equal scalar and vector potentials. The results obtained in this work are more general and can be reduced to the standard forms in three dimensions given by other works.

PACS (2008): 03.65.-w; 03.65.Fd; 03.65.Ge

Keywords: energy eigenvalues and eigenfunctions • Klein–Gordon equation • magnonics • Kratzer potential • ring-shaped potential • non-central potentials • Nikiforov and Uvarov method

1. Introduction

In various physical applications including those in nuclear physics and high energy physics [1, 2], one of the interesting problems is to obtain exact solutions of relativistic equations like the Klein–Gordon and Dirac equations for mixed vector and scalar potential. The Klein–Gordon and Dirac wave equations are frequently used to describe particle dynamics in relativistic quantum mechanics. The Klein–Gordon equation has also been used to understand the motion of a spin-0 particle in a large class of potentials. In recent years, much effort has been paid to solving these relativistic wave equations for various potentials by using different methods. These relativistic equations contain two objects: the four-vector linear momentum operator and the scalar rest mass. They allow us to introduce two types of potential coupling, which are the four-vector potential (V) and the space-time scalar potential (S).

Recently, many authors have worked on solving these equations with physical potentials including the Morse potential [3, 4], Hulthén potential [5–9], Woods-Saxon potential [10, 11], Pösch-Teller potential [12, 13], reflectionless-type potential [14], pseudoharmonic oscillator [15–17], ring-shaped harmonic oscillator [18], $V_0 \tanh^2(r/r_0)$ potential [19], five-parameter exponential potential [20, 21], Rosen-Morse potential [22], generalized symmetrical

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double-well potential [23], etc. It is remarkable that in most works in this area, the scalar and vector potentials are taken to be almost equal (i.e., $S = V$) [2, 24]. However, in a few other cases, the case is considered where the scalar potential is greater than the vector potential (in order to guarantee the existence of Klein–Gordon bound states) — i.e., $S > V$ [25–30]. Nonetheless, such physical potentials are very few. The bound-state solutions for the last case are obtained for the exponential potential for the s-wave Klein–Gordon equation when the scalar potential is greater than the vector potential [25].

The study of exact solutions of the nonrelativistic equation for a class of non-central potentials with a vector potential and a non-central scalar potential is of considerable interest in quantum chemistry [31–37]. In recent years, numerous studies [38–40] have been made in analyzing the bound states of an electron in a Coulomb field with simultaneous presence of Aharonov-Bohm (AB) [41] field, and/or a magnetic Dirac monopole [42], and Aharonov-Bohm plus oscillator (ABO) systems. In most of this work, the eigenvalues and eigenfunctions are obtained by means of separation of variables in spherical or other orthogonal curvilinear coordinate systems. The path integral for particles moving in non-central potentials is evaluated to derive the energy spectrum of this system analytically [43]. In addition, the idea of SUSY and shape invariance is also used to obtain exact solutions of such non-central but separable potentials [44, 45]. Very recently, the conventional Nikiforov-Uvarov (NU) method [46] has been used to give a clear recipe for how to obtain an explicit exact bound-states solutions for the energy eigenvalues and their corresponding wave functions in terms of orthogonal polynomials for a class of non-central potentials [47].

Another type of non-central potential is the ring-shaped Kratzer potential, which is a combination of a Coulomb potential plus an inverse square potential plus a non-central angular part [48, 49]. The Kratzer potential has been used to describe the vibrational-rotational motion of isolated diatomic molecules [50] and has a mixed-energy spectrum containing both bound and scattering states, with bound-states have been widely used in molecular spectroscopy [51]. The ring-shaped Kratzer potential consists of radial and angle-dependent potentials and is useful in studying ring-shaped molecules [37]. In taking the relativistic effects into account for a spin-0 particle in the presence of a class of non-central potential, Yasuk *et al.* [52] applied the NU method to solve the Klein–Gordon equation for the non-central Coulombic ring-shaped potential [36] for the case $V = S$. Further, Berkdemir [53] used the same method to solve the Klein–Gordon equation for the Kratzer-type potential.

Recently, Chen and Dong [54] proposed a new ring-shaped potential and obtained the exact solution of the Schrödinger equation for the Coulomb potential plus this new ring-shaped potential, which has possible applications to ring-shaped organic molecules like cyclic polyenes and benzene. This type of potential used by Chen and Dong [54] appears to be very similar to the potential used by Yasuk *et al.* [52]. Moreover, Cheng and Dai [55], proposed a new potential consisting of the modified Kratzer's potential [50] plus the new proposed ring-shaped potential in [54]. They have presented the energy eigenvalues for this proposed exactly-solvable non-central potential in the three dimensional (i.e., $D = 3$) Schrödinger equation by means of the NU method. The two quantum systems solved by Refs. [54, 55] are closely relevant to each other as they deal with a Coulombic field interaction, except that a slight change in the angular momentum barrier acts as a repulsive core which for any arbitrary angular momentum ℓ prevents collapse of the system in any dimensional space due to the slight perturbation of the original angular momentum barrier. Very recently, we have also applied the NU method to solve the Schrödinger equation in any arbitrary D dimensions to this new modified Kratzer-type potential [56–58].

The aim of the present paper is to consider the relativistic effects for the spin-0 particle in our recent work [56–58]. We want to present a systematic recipe for solving the D-dimensional Klein–Gordon equation for the new ring-shaped Kratzer potential recently proposed in [55] using the simple NU method. This method is based on solving the Klein–Gordon equation by reducing it to a generalized hypergeometric equation.

This work is organized as follows: in section 2, we shall present the Klein–Gordon equation in spherical coordinates for a spin-0 particle in the presence of equal scalar and vector non-central ring-shaped Kratzer potential and we also separate it into radial and angular parts. Section 3 is devoted to a brief description of the NU method. In section 4 , we present the exact solutions to the radial and angular parts of the Klein–Gordon equation in D dimensions. Finally, the relevant conclusions are given in section 5.

2. The Klein–Gordon equation with equal scalar and vector potentials

In relativistic quantum mechanics, we usually use the Klein–Gordon equation for describing a scalar particle, i.e., spin-0 particle dynamics. Discussion of the relativistic behavior of spin-zero particles requires understanding of the single particle spectrum and exact solutions to the Klein–Gordon equation which are constructed by us-

ing the four-vector potential $\mathbf{A}_\lambda$ ($\lambda = 0,1,2,3$) and the scalar potential (S). In order to simplify solution of the Klein–Gordon equation, the four-vector potential can be written as $\mathbf{A}_\lambda = (A_0, 0, 0, 0)$. The first component of the four-vector potential is represented by a vector potential (V), i.e., $A_0 = V$. In this case, the motion of a relativistic spin-0 particle in a potential is described by the Klein–Gordon equation with the potentials V and S [1]. For the case $S \geq V$, there exist bound-state (real) solutions for a relativistic spin-zero particle [25–30]. On the other hand, for $S = V$, the Klein–Gordon equation reduces to a Schrödinger-like equation and thereby the bound-state solutions are easily obtained by using well-known methods developed in nonrelativistic quantum mechanics [2].

The Klein–Gordon equation describing a scalar particle (spin-0 particle) with scalar $S(r,\theta,\varphi)$ and vector $V(r,\theta,\varphi)$ potentials is given by [2, 24]

$$\left\{\mathbf{P}^2 - \left[E_R - \frac{V(r,\theta,\varphi)}{2}\right]^2 + \right.$$
$$\left. + \left[\mu + \frac{S(r,\theta,\varphi)}{2}\right]^2\right\}\psi(r,\theta,\varphi) = 0\,, \qquad (1)$$

where $E_R, \mathbf{P}$ and μ are the relativistic energy, momentum operator and rest mass of the particle, respectively. The potential terms are scaled in (1) by Alhaidari *et al.* [24] so that in the nonrelativistic limit, the interaction potential becomes V.

In this work, we consider the equal scalar and vector potentials case, that is, $S(r,\theta,\varphi) = V(r,\theta,\varphi)$ with the recently proposed general non-central potential taken in the form of the Kratzer plus ring-shaped potential [55, 56, 58]:

$$V(r,\theta,\varphi) = V_1(r) + \frac{V_2(\theta)}{r^2} + \frac{V_3(\varphi)}{r^2\sin^2\theta}, \qquad (2)$$

$$V_1(r) = -\frac{A}{r} + \frac{B}{r^2}, \quad V_2(\theta) = C\cot^2\theta, \quad V_3(\varphi) = 0, \quad (3)$$

where $A = 2a_0r_0$, $B = a_0r_0^2$ and C is a positive real constant, with a_0 equal to the dissociation energy and r_0 to the equilibrium internuclear distance [50]. The potentials in (3) introduced by Cheng-Dai [55] reduce to the Kratzer potential in the limiting case of $C = 0$ [50]. In fact, the energy spectrum for this potential can be obtained directly by considering it as a special case of the general non-central separable potentials [47].

Our aim is to derive analytically the exact energy spectrum for a moving particle in the presence of a non-central potential given by (2) in a very simple way. We begin by considering the Schrödinger equation in D arbitrary dimensions for our proposed potential [59–72]

$$-\frac{\hbar^2}{2\mu}\nabla_D^2\psi^{(\ell_{D-1}=\ell)}_{\ell_1\cdots\ell_{D-2}}(\mathbf{x}) = [E - V(r)]\,\psi^{(\ell_{D-1}=\ell)}_{\ell_1\cdots\ell_{D-2}}(\mathbf{x}), \qquad (4)$$

where μ and E denote the reduced mass and energy, respectively, of two interacting particles. $\mathbf{x}$ is a D-dimensional position vector with hyperspherical Cartesian components $x_1, x_2, \cdots, x_D$ given as follows [1] [71–79]:

$$x_1 = r\cos\theta_1\sin\theta_2\cdots\sin\theta_{D-1},$$
$$x_2 = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-1},$$
$$x_3 = r\cos\theta_2\sin\theta_3\cdots\sin\theta_{D-1},$$
$$\vdots$$
$$x_j = r\cos\theta_{j-1}\sin\theta_j\cdots\sin\theta_{D-1}, 3 \leq j \leq D-1,$$
$$\vdots$$
$$x_{D-1} = r\cos\theta_{D-2}\sin\theta_{D-1},$$
$$x_D = r\cos\theta_{D-1}, \qquad \sum_{j=1}^{D} x_j^2 = r^2, \qquad (5)$$

for $D = 2, 3, \cdots$. We have $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$ for $D = 2$ and $x_1 = r\cos\varphi\sin\theta$, $x_2 = r\sin\varphi\sin\theta$, $x_3 = r\cos\theta$ for $D = 3$. The Laplace operator ∇_D^2 is defined by

$$\nabla_D^2 = \sum_{j=1}^{D}\frac{\partial^2}{\partial x_j^2}. \qquad (6)$$

The volume element of the configuration space is given by

$$\prod_{j=1}^{D} dx_j = r^{D-1}dr\,d\Omega, \quad d\Omega = \prod_{j=1}^{D-1}(\sin\theta_j)^{j-1}d\theta_j, \quad (7)$$

where $r \in [0,\infty)$, $\theta_1 \in [0, 2\pi]$ and $\theta_j \in [0,\pi]$, $j \in [2, D-1]$. The wave function $\psi^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\mathbf{x})$ with a given angular momentum ℓ can be decomposed as a product of a radial wave function $R_\ell(r)$ and the generalized spherical harmonics $Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}})$ as [71, 72]

$$\psi^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\mathbf{x}) = R_\ell(r)\,Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}}),$$

[1] *It is worth noting that such a definition was introduced by Erdélyi early in the 1950s (cf. [79], pp. 232-5, Chapter 11) even though the notation used by him is quite different from that by Louck and Chatterjee.*

$$Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\widehat{\mathbf{x}}) = Y(\ell_1,\ell_2,\cdots,\ell_{D-2},\ell), \ell = |m| \text{ for } D = 2,$$

$$R_\ell(r) = r^{-(D-1)/2} g(r),$$

$$Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\widehat{\mathbf{x}} = \theta_1,\theta_2,\cdots,\theta_{D-1}) = \Phi(\theta_1 = \varphi) H(\theta_2,\cdots,\theta_{D-1}), \tag{8}$$

which is the simultaneous eigenfunction of L_j^2 :

$$L_1^2 Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\widehat{\mathbf{x}}) = m^2 Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\widehat{\mathbf{x}}),$$

$$L_j^2 Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\widehat{\mathbf{x}}) = \ell_j(\ell_j + j - 1) Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\widehat{\mathbf{x}}),$$

$$\ell = 0,1,\cdots,\ell_k = 0,1,\cdots,\ell_{k+1}, j \in [1, D-1], k \in [2, D-2],$$

$$\ell_1 = -\ell_2, -\ell_2 + 1, \cdots, \ell_2 - 1, \ell_2,$$

$$L_{D-1}^2 Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\widehat{\mathbf{x}}) = \ell(\ell + D - 2) Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\widehat{\mathbf{x}}), \tag{9}$$

where the unit vector along $\mathbf{x}$ is usually denoted by $\widehat{\mathbf{x}} = \mathbf{x}/r$.

Hence for a nonrelativistic treatment with the potential given in (2), the Schrödinger equation in spherical coordinates is

$$\left\{\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) - \frac{L_{D-1}^2}{r^2} + \frac{2\mu}{\hbar^2}\left(E_{NR} - V_1(r) - \frac{V_2(\theta)}{r^2} - \frac{V_3(\varphi)}{r^2\sin^2\theta}\right)\right\}\psi^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\mathbf{x}) = 0\ , \tag{10}$$

where μ and E_{NR} are the reduced mass and the nonrelativistic energy, respectively. The angular momentum operators L_j^2 are defined as [71–78]:

$$L_1^2 = -\frac{\partial^2}{\partial\theta_1^2},$$

$$L_k^2 = \sum_{a<b=2}^{k+1} L_{ab}^2 = -\frac{1}{\sin^{k-1}\theta_k}\frac{\partial}{\partial\theta_k}\left(\sin^{k-1}\theta_k\frac{\partial}{\partial\theta_k}\right) + \frac{L_{k-1}^2}{\sin^2\theta_k},$$

$$2 \le k \le D-1,$$

$$L_{ab} = -i\left[x_a\frac{\partial}{\partial x_b} - x_b\frac{\partial}{\partial x_a}\right]. \tag{11}$$

Application of Eqs. (9) and (11) leads to the separation of (10) into the following set of second-order differential equations:

$$\frac{d^2\Phi(\theta_1 = \varphi)}{d\theta_1^2} + m^2\Phi(\theta_1 = \varphi) = 0, \tag{12}$$

$$\left[\frac{1}{\sin^{j-1}\theta_j}\frac{d}{d\theta_j}\left(\sin^{j-1}\theta_j\frac{d}{d\theta_j}\right) + \ell_j(\ell_j + j - 1) - \frac{\ell_{j-1}(\ell_{j-1} + j - 2)}{\sin^2\theta_j}\right]H(\theta_j) = 0, \quad j \in [2, D-2], \tag{13}$$

$$\left[\frac{1}{\sin^{D-2}\theta_{D-1}}\frac{d}{d\theta_{D-1}}\left(\sin^{D-2}\theta_{D-1}\frac{d}{d\theta_{D-1}}\right) + \lambda_\ell - \frac{1}{\sin^2\theta_{D-1}}\left(L_{D-2}^2 + \frac{2\mu C}{\hbar^2}\cos^2\theta_{D-1}\right)\right]H(\theta_{D-1}) = 0, \tag{14}$$

$$\left[\frac{1}{r^{D-1}}\frac{d}{dr}\left(r^{D-1}\frac{d}{dr}\right) - \frac{\lambda_\ell}{r^2}\right]R_\ell(r) + \frac{2\mu}{\hbar^2}\left[E_{NR} + \frac{A}{r} - \frac{B}{r^2}\right]R(r) = 0, \tag{15}$$

where m^2 and $\lambda_\ell = \ell(\ell + D - 2)$ are the separation constants.

On the other hand, in the relativistic atomic units ($\hbar = c = 1$), the D-dimensional Klein–Gordon equation in (1) becomes [53]:

$$\left\{\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) - \frac{\widetilde{L}_{D-1}^2}{r^2} - (E_R + \mu)\left(V_1(r) + \frac{V_2(\theta)}{r^2}\right) + \left(E_R^2 - \mu^2\right)\right\}\psi_{n\tilde{\ell}m}(r,\theta,\varphi) = 0, \tag{16}$$

with $\psi_{n\tilde{\ell}m}(r,\theta,\varphi)$ being the spherical total wave function separated as follows:

$$\psi_{n\tilde{\ell}m}(r,\theta,\varphi) = R(r)Y_{\tilde{\ell}}^m(\theta,\varphi), \tag{17}$$
$$R(r) = r^{-(D-1)/2}g(r), \quad Y_{\tilde{\ell}}^m(\theta,\varphi) = H(\theta)\Phi(\varphi).$$

The spherical total wave function appearing in (17) has the same representation as in (8), but with the transformation $\widetilde{\ell} \to \ell$. Inserting Eq. (17) into Eq. (16) and using the method of separation of variables leads to the following differential equations:

$$\frac{d^2\Phi(\varphi)}{d\varphi^2} + m^2\Phi(\varphi) = 0, \tag{18}$$

$$\left[\frac{1}{\sin^{j-1}\theta_j}\frac{d}{d\theta_j}\left(\sin^{j-1}\theta_j\frac{d}{d\theta_j}\right) + \widetilde{\ell}_j(\widetilde{\ell}_j + j - 1)\right.$$

$$\left. - \frac{\widetilde{\ell}_{j-1}(\widetilde{\ell}_{j-1} + j - 2)}{\sin^2\theta_j}\right] H(\theta_j) = 0, j \in [2, D-2], \tag{19}$$

$$\left[\frac{1}{\sin^{D-2}\theta_{D-1}}\frac{d}{d\theta_{D-1}}\left(\sin^{D-2}\theta_{D-1}\frac{d}{d\theta_{D-1}}\right) + \lambda_{\widetilde{\ell}}\right.$$

$$\left. - \frac{\widetilde{L}^2_{D-2} + C\alpha_2^2\cos^2\theta_{D-1}}{\sin^2\theta_{D-1}}\right] H(\theta_{D-1}) = 0, \tag{20}$$

$$\frac{1}{r^{D-1}}\frac{d}{dr}\left(r^{D-1}\frac{dR(r)}{dr}\right)$$
$$-\left[\frac{\lambda_{\widetilde{\ell}}}{r^2} + \alpha_2^2\left(\alpha_1^2 - \frac{A}{r} + \frac{B}{r^2}\right)\right] R(r) = 0, \tag{21}$$

where $\alpha_1^2 = \mu - E_R, \alpha_2^2 = \mu + E_R, m$ and $\widetilde{\ell}$ are constants with m^2 and $\lambda_{\widetilde{\ell}} = \widetilde{\ell}(\widetilde{\ell}+D-2)$ as the separation constants. Equations (18)-(21) have the same functional form as Eqs (12)-(15). Therefore, solution of the Klein–Gordon equation can be reduced to solution of the Schrödinger equation with the appropriate choice of parameters: $\widetilde{\ell} \to \ell, \alpha_1^2 \to -E_{NR}$ and $\alpha_2^2 \to 2\mu/\hbar^2$.

The solution of Eq. (18) is a well-known periodic and must satisfy the period boundary condition $\Phi(\varphi + 2\pi) = \Phi(\varphi)$ which is the azimuthal angle solution:

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}}\exp(\pm im\varphi), m = 0, 1, 2, \ldots. \tag{22}$$

Additionally, Eqs. (19)-(21), the polar angle and radial equations, are to be solved by using the Nikiforov-Uvarov (NU) method [46] which is given briefly in the following section.

3. Nikiforov-Uvarov method

The NU method is based on reducing the second-order differential equation to a generalized equation of hypergeometric type [46, 47, 80–84]. In this sense, the Schrödinger equation, after employing an appropriate coordinate transformation $s = s(r)$, transforms to the following form:

$$\psi_n''(s) + \frac{\widetilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{23}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are polynomials, at most of second-degree, and $\widetilde{\tau}(s)$ is a first-degree polynomial. Using a wave function, $\psi_n(s)$, of the simple ansatz:

$$\psi_n(s) = \phi_n(s)y_n(s), \tag{24}$$

(23) reduces into an equation of a hypergeometric type:

$$\sigma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0, \tag{25}$$

where

$$\sigma(s) = \pi(s)\frac{\phi(s)}{\phi'(s)}, \tag{26}$$

$$\tau(s) = \widetilde{\tau}(s) + 2\pi(s), \tau'(s) < 0, \tag{27}$$

and λ is a parameter defined as

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), n = 0, 1, 2, \ldots. \tag{28}$$

The polynomial $\tau(s)$ with the parameter s and prime factors shows the differentials at first degree to be negative. It is worthwhile to note that λ or λ_n are obtained from a particular solution of the form $y(s) = y_n(s)$ which is a polynomial of degree n. Further, the other part $y_n(s)$ of the wave function (24) is the hypergeometric-type function whose polynomial solutions are given by the Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right], \tag{29}$$

where B_n is the normalization constant and the weight function $\rho(s)$ must satisfy the condition [46]

$$\frac{d}{ds}w(s) = \frac{\tau(s)}{\sigma(s)}w(s), w(s) = \sigma(s)\rho(s). \tag{30}$$

The function π and the parameter λ are defined as

$$\pi(s) = \frac{\sigma'(s) - \widetilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \widetilde{\tau}(s)}{2}\right)^2 - \widetilde{\sigma}(s) + k\sigma(s)}, \tag{31}$$

$$\lambda = k + \pi'(s). \tag{32}$$

In principle, since $\pi(s)$ has to be a polynomial of degree at most one, the expression under the square root sign in (31) can be arranged to be the square of a polynomial of first degree [46]. This is possible only if its discriminant is zero. In this case, an equation for k is obtained. After solving this equation, the obtained values of k are substituted into (31). In addition, by comparing Eqs. (28) and (32), we obtain the energy eigenvalues.

4. Exact solutions of the radial and angle-dependent equations

4.1. The solutions of the D-dimensional angular equations

At the beginning, we rewrite Eqs. (19) and (20) representing the angular wave equations in the following simple forms:

$$\frac{d^2H(\theta_j)}{d\theta_j^2} + (j-1)ctg\theta_j \frac{dH(\theta_j)}{d\theta_j} + \left(\Lambda_j - \frac{\Lambda_{j-1}}{\sin^2\theta_j}\right) H(\theta_j) = 0, \tag{33}$$

with $j \in [2, D-2], D > 3$ and

$$\frac{d^2H(\theta_{D-1})}{d\theta_{D-1}^2} + (D-2)\cot\theta_{D-1}\frac{dH(\theta_{D-1})}{d\theta_{D-1}}$$

$$+ \left[\widetilde{\ell}(\widetilde{\ell}+D-2) - \frac{\Lambda_{D-2} + C\alpha_2^2\cos^2\theta_{D-1}}{\sin^2\theta_{D-1}}\right] H(\theta_{D-1}) = 0, \tag{34}$$

where $\Lambda_p = \widetilde{\ell}_p(\widetilde{\ell}_p + p - 1), p = j-1, j$ which is well-known in three-dimensional space[2]. Eqs. (33) and (34) will be solved in the following subsection. In order to apply the NU method, we introduce a new variable $s = \cos\theta_j$. Hence, Eq. (33) is rearranged in the form of the universal associated-Legendre differential equation

$$\frac{d^2H(s)}{ds^2} - \frac{js}{1-s^2}\frac{dH(s)}{ds} + \frac{\Lambda_j - \Lambda_{j-1} - \Lambda_j s^2}{(1-s^2)^2} H(s) = 0, \tag{35}$$

where $j \in [2, D-2], D > 3$. By comparing Eqs. (35) and (23), the corresponding polynomials are obtained:

$$\widetilde{\tau}(s) = -js, \quad \sigma(s) = 1 - s^2, \quad \widetilde{\sigma}(s) = -\Lambda_j s^2 + \Lambda_j - \Lambda_{j-1}. \tag{36}$$

[2] $\Lambda_{D-2} = m^2$ *for* $D = 3$.

Inserting the above expressions into Eq. (31) and taking $\sigma'(s) = -2s$, one obtains the following function:

$$\pi(s) = \frac{(j-2)}{2}s \pm \sqrt{\left[\left(\frac{j-2}{2}\right)^2 + \Lambda_j - k\right]s^2 + k - \Lambda_j + \Lambda_{j-1}}. \tag{37}$$

Following the method, the polynomial $\pi(s)$ is found to have four possible values:

$$\pi(s) = \begin{cases} \left(\frac{j-2}{2} + \widetilde{\Lambda}_{j-1}\right)s & \text{for } k_1 = \Lambda_j - \Lambda_{j-1}, \\ \left(\frac{j-2}{2} - \widetilde{\Lambda}_{j-1}\right)s & \text{for } k_1 = \Lambda_j - \Lambda_{j-1}, \\ \frac{(j-2)}{2}s + \widetilde{\Lambda}_{j-1} & \text{for } k_2 = \Lambda_j + \left(\frac{j-2}{2}\right)^2, \\ \frac{(j-2)}{2}s - \widetilde{\Lambda}_{j-1} & \text{for } k_2 = \Lambda_j + \left(\frac{j-2}{2}\right)^2, \end{cases} \tag{38}$$

where $\widetilde{\Lambda}_p = \widetilde{\ell}_p + (p-1)/2$, with $p = j-1, j$ and $j \in [2, D-2], D > 3$. Imposing the condition $\tau'(s) < 0$ for Eq. (27), one selects the following physically valid solutions with $\tau' = \tau'(\widetilde{\ell}_{j-1})$; that is, a function of the angular momentum:

$$k_1 = \Lambda_j - \Lambda_{j-1} \text{ and } \pi(s) = \left(\frac{j-2}{2} - \widetilde{\Lambda}_{j-1}\right)s. \tag{39}$$

This condition leads to writing

$$\tau(s) = -2(1 + \widetilde{\Lambda}_{j-1})s. \tag{40}$$

Making use of Eqs. (28) and (32), the following expressions for λ are obtained, respectively,

$$\lambda = \lambda_n = 2n_j(1 + \widetilde{\Lambda}_{j-1}) + n_j(n_j - 1), \quad j \in [2, D-2],\ D > 3 \tag{41}$$

$$\lambda = \Lambda_j - \Lambda_{j-1} - \widetilde{\Lambda}_{j-1} + \frac{j-2}{2}. \tag{42}$$

Upon comparing Eqs. (41) and (42), we obtain

$$n_j = \widetilde{\ell}_j - \widetilde{\ell}_{j-1}. \tag{43}$$

Additionally, using Eqs. (24)-(26) and (29)-(30), we obtain the following useful parts of the wavefunctions:

$$\phi(s) = \left(1-s^2\right)^{\left(\frac{\widetilde{\Lambda}_{j-1}}{2} - \frac{(j-2)}{4}\right)}, \quad \rho(s) = \left(1-s^2\right)^{\widetilde{\Lambda}_{j-1}}, \tag{44}$$

where $j \in [2, D-2]$, $D > 3$. We also substitute the weight function $\rho(s)$ given in (44) into the Rodrigues relation (29) to obtain one of the wavefunctions in the form

$$y_{n_j}(s) = A_{n_j}\left(1-s^2\right)^{-\widetilde{\Lambda}_{j-1}} \frac{d^n}{ds^n}\left(1-s^2\right)^{n_j+\widetilde{\Lambda}_{j-1}}, \quad (45)$$

where A_{n_j} is the normalization factor. Finally the angular wavefunction is

$$H_{n_j}(\theta_j) = N_{n_j}\left(\sin\theta_j\right)^{\left(\widetilde{\Lambda}_{j-1}-\frac{(j-2)}{2}\right)} P_{n_j}^{(\widetilde{\Lambda}_{j-1},\widetilde{\Lambda}_{j-1})}(\cos\theta_j), \quad (46)$$
$$j \in [2,\ D-2],\ D > 3$$

with the normalization factor

$$\begin{aligned} N_{n_j} &= \sqrt{\frac{(2\ell'+1)\,\Gamma(\ell'-m')}{2\Gamma(\ell'+m')}} \\ &= \sqrt{\frac{\left(2n_j+2\widetilde{\ell}_{j-1}+j-1\right)n_j!}{2\Gamma\left(n_j+2\widetilde{\ell}_{j-1}+j-2\right)}}, \\ & j \in [2, D-2], \qquad D > 3. \end{aligned} \quad (47)$$

Likewise, in solving Eq. (34), we introduce a new variable $s = \cos\theta_{D-1}$. Thus, we can also rearrange it as the universal associated-Legendre differential equation

$$\frac{d^2H(s)}{ds^2} - \frac{(D-1)s}{1-s^2}\frac{dH(s)}{ds} + \frac{\nu'(1-s^2)-\Lambda'_{D-2}}{(1-s^2)^2}H(s) = 0, \quad (48)$$

where

$$\begin{aligned} \nu' &= \ell'(\ell'+D-2) = \widetilde{\ell}(\widetilde{\ell}+D-2) + C\alpha_2^2 \quad (49) \\ \text{and} \quad & \Lambda'_{D-2} = \Lambda_{D-2} + C\alpha_2^2. \end{aligned}$$

Equation (48) has been recently solved in 2D and 3D by the NU method in [55–58]. However, the aim in this subsection is to solve it in D-dimensions. Upon letting $D = 3$, we can readily restore the $3D$ solution given in [55]. By comparing Eqs. (48) and (23), the corresponding polynomials are obtained:

$$\widetilde{\tau}(s) = -(D-1)s,\ \sigma(s) = 1-s^2,\ \widetilde{\sigma}(s) = -\nu's^2 + \nu' - \Lambda'_{D-2}. \quad (50)$$

Inserting the above expressions into Eq. (31) and taking $\sigma'(s) = -2s$, one obtains the following function:

$$\begin{aligned} \pi(s) &= \frac{(D-3)}{2}s \quad (51) \\ &\pm \sqrt{\left[\left(\frac{D-3}{2}\right)^2 + \nu' - k\right]s^2 + k - \nu' + \Lambda'_{D-2}} \end{aligned}$$

Following the method, the polynomial $\pi(s)$ is found to have four possible values:

$$\pi(s) = \begin{cases} \left(\frac{D-3}{2} + \widetilde{\Lambda}_{D-2}\right)s & \text{for } k_1 = \nu' - \Lambda'_{D-2}, \\ \left(\frac{D-3}{2} - \widetilde{\Lambda}_{D-2}\right)s & \text{for } k_1 = \nu' - \Lambda'_{D-2}, \\ \frac{(D-3)}{2}s + \widetilde{\Lambda}_{D-2} & \text{for } k_2 = \nu' + \left(\frac{D-3}{2}\right)^2, \\ \frac{(D-3)}{2}s - \widetilde{\Lambda}_{D-2} & \text{for } k_2 = \nu' + \left(\frac{D-3}{2}\right)^2, \end{cases} \quad (52)$$

where $\widetilde{\Lambda}_{D-2} = \sqrt{\left(\widetilde{\ell}_{D-2} + \frac{D-3}{2}\right)^2 + C\alpha_2^2}$. Imposing the condition $\tau'(s) < 0$ for Eq. (27), one selects the following physically valid solutions with $\tau' = \tau'(\widetilde{\ell}_{D-2})$; that is, a function of the angular momentum,[3]

$$k_1 = \nu' - \Lambda'_{D-2} \text{ and } \pi(s) = \left(\frac{D-3}{2} - \widetilde{\Lambda}_{D-2}\right)s, \quad (53)$$

which yields from Eq. (27) that

$$\tau(s) = -2(1+\widetilde{\Lambda}_{D-2})s. \quad (54)$$

Making use of Eqs. (28) and (32), the following expressions for λ are obtained, respectively:

$$\lambda = \lambda_n = 2n_{D-1}(1+\widetilde{\Lambda}_{D-2}) + n_{D-1}(n_{D-1}-1), \quad (55)$$

$$\lambda = \nu' - \Lambda'_{D-2} - \widetilde{\Lambda}_{D-2} + \frac{D-3}{2}. \quad (56)$$

We compare Eqs. (55) and (56) and from the definition $\nu' = \ell'(\ell'+D-2)$, the new angular momentum $\ell', \widetilde{\ell}_{D-2}$ and $\widetilde{\ell}$ values are obtained as

$$\widetilde{\ell} = -\frac{(D-2)}{2} + \sqrt{\left(n_j + m' + \frac{1}{2}\right)^2 - \alpha_2^2 C},$$

$$\ell' = -\frac{1}{2} + \sqrt{\left(\widetilde{\ell} + \frac{D-2}{2}\right)^2 + C\alpha_2^2} = n_j + m',\ j = D-1$$

$$m' = \sqrt{\left(\widetilde{\ell}_{D-2} + \frac{D-3}{2}\right)^2 + \alpha_2^2 C}, \quad (57)$$

[3] *The physical significance of this choice of parameters is that theeigenvalue and the eigenfunction equations can be directly reduced to the ones obtained in Ref. [55] once we set D=3.*

where $\widetilde{\ell} = \widetilde{\ell}_{D-1}$, which can be easily reduced to the well-known definition

$$\ell' = n + \sqrt{m^2 + \alpha_2^2 C}, \tag{58}$$

where $n_{D-1} = n$, $\widetilde{\ell}_{D-2} = m$, $\widetilde{\Lambda}_{D-2} = m'$in $3D$ space [55]. Using Eqs. (24)-(26) and (29)-(30), we obtain the following useful parts of the wavefunctions:

$$\phi(s) = \left(1 - s^2\right)^{(2\widetilde{\Lambda}_{D-2}+3-D)/4}, \rho(s) = \left(1 - s^2\right)^{\widetilde{\Lambda}_{D-2}}. \tag{59}$$

We also substitute the weight function $\rho(s)$ given in (59) into the Rodrigues relation (29) and obtain one of the wavefunctions in the form

$$y_{n_j}(s) = B_{n_j}\left(1 - s^2\right)^{-\widetilde{\Lambda}_{D-2}} \frac{d^{n_j}}{ds^{n_j}}\left(1 - s^2\right)^{n_j+\widetilde{\Lambda}_{D-2}}, j = D-1 \tag{60}$$

where $B_{n_{D-1}}$ is the normalization factor. Finally the angular wavefunction is

$$H_{n_j}(\theta_j) = N_{n_j}\left(\sin\theta_j\right)^{\widetilde{\Lambda}_{D-2}-\frac{(D-3)}{2}} P_{n_j}^{(\widetilde{\Lambda}_{D-2},\widetilde{\Lambda}_{D-2})}(\cos\theta_j), \tag{61}$$

where the normalization factor

$$N_{n_j} = \sqrt{\frac{\left(2n_j + 2m' + 1\right) n_j!}{2\Gamma\left(n_j + 2m'\right)}}, j = D-1 \tag{62}$$

where m' is given in Eq. (57).

4.2. Eigensolutions of the radial equation

We seek to solve the radial part of the Klein–Gordon equation given by Eq. (21) by simply writting it in the following simple form [56–70]:

$$\frac{d^2g(r)}{dr^2} - \left[\frac{(M-1)(M-3)}{4r^2} - \right. \\ \left. - \alpha_2^2\left(\frac{A}{r} - \frac{B}{r^2}\right) + \alpha_1^2\alpha_2^2\right] g(r) = 0, \tag{63}$$

where

$$M = D + 2\widetilde{\ell}. \tag{64}$$

In what follows, we present the exact bound-states (real) solution of Eq. (63). Letting

$$\varepsilon^2 = \alpha_1^2\alpha_2^2, 4\gamma^2 = (M-1)(M-3) + 4B\alpha_2^2, \beta^2 = A\alpha_2^2, \tag{65}$$

and substituting these expressions into Eq. (63), one gets

$$\frac{d^2g(r)}{dr^2} + \left(\frac{-\varepsilon^2r^2 + \beta^2r - \gamma^2}{r^2}\right) g(r) = 0. \tag{66}$$

To apply the conventional NU-method, Eq. (66) is compared with Eq. (23), resulting in the following expressions:

$$\widetilde{\tau}(r) = 0, \sigma(r) = r, \widetilde{\sigma}(r) = -\varepsilon^2r^2 + \beta^2r - \gamma^2. \tag{67}$$

Substituting the above expressions into Eq. (31) gives

$$\pi(r) = \frac{1}{2} \pm \frac{1}{2}\sqrt{4\varepsilon^2r^2 + 4(k - \beta^2)r + 4\gamma^2 + 1}. \tag{68}$$

Therefore, we can determine the constant k by using the condition that the discriminant of the square root is zero—that is

$$k = \beta^2 \pm \varepsilon\sqrt{4\gamma^2 + 1}, 4\gamma^2 + 1 = (D + 2\widetilde{\ell} - 2)^2 + 4B\alpha_2^2. \tag{69}$$

In view of that, we arrive at the following four possible functions of $\pi(r)$:

$$\pi(r) = \begin{cases} \frac{1}{2} + \left[\varepsilon r + \frac{1}{2}\sqrt{4\gamma^2 + 1}\right], & k_1 = \beta^2 + \varepsilon\sqrt{4\gamma^2 + 1}, \\ \frac{1}{2} - \left[\varepsilon r + \frac{1}{2}\sqrt{4\gamma^2 + 1}\right], & k_1 = \beta^2 + \varepsilon\sqrt{4\gamma^2 + 1}, \\ \frac{1}{2} + \left[\varepsilon r - \frac{1}{2}\sqrt{4\gamma^2 + 1}\right], & k_2 = \beta^2 - \varepsilon\sqrt{4\gamma^2 + 1}, \\ \frac{1}{2} - \left[\varepsilon r - \frac{1}{2}\sqrt{4\gamma^2 + 1}\right], & k_2 = \beta^2 - \varepsilon\sqrt{4\gamma^2 + 1}. \end{cases} \tag{70}$$

The correct value of $\pi(r)$ is chosen such that the function $\tau(r)$ given by Eq. (27) will have negative derivative [46]. So we can select the physical values to be

$$k = \beta^2 - \varepsilon\sqrt{4\gamma^2 + 1} \text{ and } \pi(r) = \frac{1}{2} - \left[\varepsilon r - \frac{1}{2}\sqrt{4\gamma^2 + 1}\right], \tag{71}$$

which yield

$$\tau(r) = -2\varepsilon r + (1 + \sqrt{4\gamma^2 + 1}), \tau'(r) = -2\varepsilon < 0. \tag{72}$$

Using Eqs. (28) and (32), the following expressions for λ are obtained, respectively,

$$\lambda = \lambda_n = 2n\varepsilon, \qquad n = 0, 1, 2, \ldots, \tag{73}$$

$$\lambda = \beta^2 - \varepsilon(1 + \sqrt{4\gamma^2 + 1}). \tag{74}$$

So we can obtain the Klein–Gordon energy eigenvalues from the following relation:

$$\left[1+2n+\sqrt{\left(2\widetilde{\ell}+D-2\right)^2+4(\mu+E_R)B}\right] \cdot\sqrt{\mu-E_R}=A\sqrt{\mu+E_R}, \quad (75)$$

and hence for the Kratzer plus the new ring-shaped potential, it becomes

$$\left[1+2n+\sqrt{\left(2\widetilde{\ell}+D-2\right)^2+4a_0r_0^2(\mu+E_R)}\right] \cdot\sqrt{\mu-E_R}=2a_0r_0\sqrt{\mu+E_R}, \quad (76)$$

where $\widetilde{\ell}$ is defined by Eq. (57). Although Eq. (76) is exactly solvable for E_R, it looks to be a little complicated. Furthermore, it is interesting to investigate the solution for the Coulomb potential. Therefore, applying the following transformations: $A=Ze^2$, $B=0$, and $\widetilde{\ell}=\ell$, the central part of the potential in (2) turns into the Coulomb potential with Klein–Gordon solution for the energy spectra given by

$$E_R=\mu\left(1-\frac{2q^2e^2}{q^2e^2+(2n+2\ell+D-1)^2}\right), \quad n,\ell=0,1,2,..., \quad (77)$$

where $q=Ze$ is the charge of the nucleus. Furthermore, Eq. (77) can be expanded as a series in the nucleus charge as

$$E_R = \mu-\frac{2\mu q^2e^2}{(2n+2\ell+D-1)^2} + \frac{2\mu q^4e^4}{(2n+2\ell+D-1)^4}-\mathcal{O}(qe)^6, \quad (78)$$

The physical meaning of each term in the last equation was given in detail by Ref. [53]. The difference from the conventional nonrelativistic form is due to the choice of the vector $V(r,\theta,\varphi)$ and scalar $S(r,\theta,\varphi)$ parts of the potential in Eq. (1).

Furthermore, if the value of $\widetilde{\ell}$ obtained from Eq. (57) is inserted into the eigenvalues of the radial part of the Klein–Gordon solution in Eq. (76), we finally arrive at the energy eigenvalues for a bound electron in the presence of a non-central potential given by Eq. (2) as

$$\left[1+2n+\sqrt{(2\ell'+D-2)^2+4(a_0r_0^2-C)(\mu+E_R)}\right] \cdot\sqrt{\mu-E_R}=2a_0r_0\sqrt{\mu+E_R}, \quad (79)$$

where

$$2\ell'+D-2=1+2n_{D-1}+\sqrt{\left(2\widetilde{\ell}_{D-2}+D-3\right)^2+4C(\mu+E_R)}. \quad (80)$$

On the other hand, the solution of the Schrödinger equation, Eq. (10), for this potential has already been obtained by using the same method in Ref. [56, 57] and it is in Coulombic-like form:

$$E_{NR} = -\frac{8\mu a_0^2r_0^2}{\left[2n+1+\sqrt{(2\ell'+D-2)^2+8\mu(a_0r_0^2-C)}\right]^2}, \quad n = 0,1,2,... \quad (81)$$

and

$$2\ell'+D-2 = \sqrt{(D-2)^2+(2\widetilde{n}+2m'+1)^2-1}, \quad \widetilde{n} = 0,1,2,... \quad (82)$$

where $m'=\sqrt{m^2+2\mu C}$ in 3D space. Also, applying a suitable transformation ($\mu+E_R\to 2\mu$, $\mu-E_R\to -E_{NR}$, $\widetilde{\ell}\to\ell$) to Eq. (79) provides exactly the nonrelativistic limit given by Eqs. (81)-(82).

In what follows, let us now turn attention to finding the radial wavefunctions for this potential. Substituting the values of $\sigma(r)$, $\pi(r)$ and $\tau(r)$ in Eqs. (67), (71) and (72) into Eqs. (26) and (30), we find

$$\phi(r)=r^{(\zeta+1)/2}e^{-\varepsilon r}, \quad (83)$$

$$\rho(r)=r^{\zeta}e^{-2\varepsilon r}, \quad (84)$$

where $\zeta=\sqrt{4\gamma^2+1}$. Then, from Eq. (20), we obtain

$$y_{nj}(r)=B_{nj}r^{-\zeta}e^{2\varepsilon r}\frac{d^n}{dr^n}\left(r^{n+\zeta}e^{-2\varepsilon r}\right) \quad (85)$$

and the wave function $g(r)$ can be written in the form of the generalized Laguerre polynomials as

$$g(\rho)=C_{nj}\left(\frac{\rho}{2\varepsilon}\right)^{(1+\zeta)/2}e^{-\rho/2}L_n^{\zeta}(\rho), \quad (86)$$

where for Kratzer's potential we have

$$\zeta=2\sqrt{\left(\widetilde{\ell}+\frac{D-2}{2}\right)^2+a_0r_0^2(\mu+E_R)}, \rho=2\varepsilon r. \quad (87)$$

Finally, the radial wave functions of the Klein–Gordon equation are obtained:

$$R(\rho)=C_{nj}\left(\frac{\rho}{2\varepsilon}\right)^{(\zeta+2-D)/2}e^{-\rho/2}L_n^{\zeta}(\rho), \quad (88)$$

where C_{nj} is the normalization constant to be determined below. Using the normalization condition, $\int_0^\infty R^2(r)r^{D-1}dr = 1$, and the orthogonality relation of the generalized Laguerre polynomials,

$$\int_0^\infty z^{\eta+1}e^{-z}\left[L_n^\eta(z)\right]^2 dz = \frac{(2n+\eta+1)(n+\eta)!}{n!},$$

we have [85, 86]

$$C_{nj} = \left(2\sqrt{\mu^2 - E_R^2}\right)^{1+\frac{\zeta}{2}} \sqrt{\frac{n!}{(2n+\zeta+1)\,\Gamma(n+\zeta+1)}}. \tag{89}$$

Finally, we may express the normalized total wave functions as

$$\psi^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\mathbf{x}) = \frac{\left(2\sqrt{\mu^2 - E_R^2}\right)^{1+\frac{\zeta}{2}}}{2^{m'}\Gamma(n+m'+1)} \sqrt{\frac{n!}{2\pi\,(2n+\zeta+1)\,\Gamma(n+\zeta+1)}}\, r^{\frac{(\zeta+2-D)}{2}} \exp(-\sqrt{\mu^2 - E_R^2}r) L_n^\zeta(2\sqrt{\mu^2 - E_R^2}r)$$
$$\cdot \exp(\pm im\varphi) \prod_{j=2}^{D-2} \sqrt{\frac{\left(2n_j + 2\widetilde{\ell}_{j-1} + j - 1\right) n_j!}{2\Gamma\left(n_j + 2\widetilde{\ell}_{j-1} + j - 2\right)}} \left(\sin\theta_j\right)^{\left(\widetilde{\Lambda}_{j-1} - \frac{(j-2)}{2}\right)} P_{n_j}^{(\widetilde{\Lambda}_{j-1}, \widetilde{\Lambda}_{j-1})}(\cos\theta_j)$$
$$\cdot \sqrt{\frac{(2n_{D-1} + 2m' + 1)\, n_{D-1}!}{2\Gamma\left(n_{D-1} + 2m'\right)}} \sin(\theta_{D-1})^{\widetilde{\Lambda}_{D-2} - (D-3)/2} P_{n_{D-1}}^{(\widetilde{\Lambda}_{D-2}, \widetilde{\Lambda}_{D-2})}(\cos\theta_{D-1}), \tag{90}$$

where m' and ζ are given in Eqs. (57) and (87), respectively.

5. Conclusions

The relativistic exact bound state solutions of the D-dimensional Klein–Gordon equation for a spin-0 particle have been obtained easily for equal scalar and vector ring-shaped Kratzer potential by means of the conventional NU method. The analytic expressions for the bound state energy levels and eigenfunctions of this nontrivial dynamical system can be reduced to the well-known three-dimensional space expressions upon setting $D = 3$. Furthermore, such non-central potentials as have recently been investigated in [47] are to be introduced as perturbations to the Kratzer potential while adjusting the strength of the coupling constant C in terms of the parameter a_0, the coupling constant of the Kratzer potential. Additionally, the analytical solution of the radial and polar angle wave functions of Klein–Gordon equation are found in terms of Laguerre and Jacobi polynomials, respectively. The analytical method presented in this paper is general and worth extending to the solution of other interaction problems. This method is very simple and useful in solving other complicated systems analytically without restriction conditions on the solution of quantum systems as is the case in other models. We have seen that for the nonrelativistic model, the exact energy spectra can be obtained either by solving the Schrödinger equation in (10) (cf. Ref. [56, 57] or Eq. (55)) or by applying an appropriate transformation to the relativistic solution given by Eq. (76). Finally, we point out that these exact analytical solutions obtained for the new proposed form of the non-central potential (2) constitute an interesting investigation of a nontrivial dynamical system. Furthermore, there are interesting applications in studying various quantum mechanical systems in atomic and molecular physics.

Acknowledgments

This research was partially supported by the Scientific and Technical Research Council of Turkey. We thank the referees for their invaluable comments.

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