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Relativistic Spin-One Boson Plasma

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We derive the polarization tensor for a system of charged spin-one bosons and antibosons in the case of no external magnetic field. This requires a thorough exposition of relativistic spin-one quantum mechanics, and thus we initially focus upon the Sakata-Taketani equation and its free-field solutions. We employ these results to evaluate the matrix elements required for the calculation of the polarization tensor, which itself is derived via the self-consistent random-phase approximation (RPA) method. It is from this tensor that we obtain the longitudinal and transverse dielectric response functions for this plasma. We evaluate these response functions at zero temperature, and exhibit the characteristic modes of oscillation. Finally, we discuss possible generalizations of this work, in particular to a finite-temperature plasma, and to one with an external magnetic field.

§ 1. Introduction

In this paper, we present a study of the relativistic spin-one particle-antiparticle plasma, in the presence of no external magnetic field. The course of the investigation begins with a review of single particle theories of spin-one (vector) bosons, which we present in § 2 of this work. In § 3 of the paper, we present a detailed treatment of the six-component formalism for spin-one bosons, first developed by Sakata and Taketani,[1] which we then employ in our linear response theory calculations for the plasma. To our knowledge, this particular formalism is under-represented in the literature pertinent to spin-one bosons, and hence our exposition is an attempt to clarify its general features, and underscore its particular utility in work of the nature we have undertaken.

The latter part of the paper (§§ 4 and 5) contains the linear response calculations proper. We set about deriving the polarization four-tensor, employing a method first proposed by Harris,[2] this being the self-consistent random-phase approximation (RPA) method. We then obtain the characteristic modes of oscillation of the plasma.

At present, we concern ourselves solely with the plasma properties of the polarization tensor, leaving a complete study of the vacuum modes of oscillation and their renormalization to a later paper, in which we also propose to introduce the presence of an external magnetic field. Employing our plane-wave solutions of the Sakata-Taketani equation for the case of no external fields, we proceed to calculate the longitudinal and transverse dielectric response functions, which are obtained via the employment of the relationship between the covariant polarization four-tensor and the dielectric three-tensor. In the case of the longitudinal response function, we present the formal result which is valid for all temperatures, and we then evaluate the longitudinal and transverse response functions at zero temperature, from which we obtain the modes of oscillation.

Our work follows on from that of Kowalenko, Frankel and Hines (KFH),[3] who studied the spin-zero pair plasma by employing a self-consistent field method to find the longitudinal modes of oscillation of this plasma at zero temperature, and with no fields present. Witte, Kowalenko and Hines (WKH)[4] extended this work to the presence of strong magnetic fields, and also to finding the full polarization tensor by generalizing the RPA technique of Harris[2] for the study of non-relativistic quantum mechanical plasmas to that of relativistic pairs. Indeed, our paper may be viewed as the generalization of some of the work of KFH and WKH to the spin-one case. The only previous authors, to our knowledge, who study the relativistic spin-one pair plasma are Williams and Melrose.[5] These authors employ a method of ensemble-averaged propagators in their work, and find that the longitudinal modes of oscillation of the spin-one plasma at zero temperature are precisely those of the spin-zero plasma found by KFH[3] and WKH,[4] but that the transverse modes differ to that found by WKH. Indeed, they find that three branches are present, as opposed to the single transverse mode WKH find for the relativistic pair spin-zero plasma. As we shall show in § 5 of this paper, whilst the longitudinal modes of the spin-zero and spin-one boson plasma are identical at zero temperature, there is only *one* transverse mode of oscillation, which differs markedly from the zero-temperature transverse mode of the spin-zero plasma. This is a new result which disagrees with the only comparable calculation, in Ref. 5), which we discuss at length in this section.

It is to be noted that the system which we study is a homogeneous, isotropic, infinite, three-dimensional plasma, and consists of particles of zero anomalous magnetic and electric moment. This would correspond to a plasma of ρ- or ω-bosons, for instance. However, W-bosons have an anomalous magnetic moment, and would also require the inclusion of weak-force couplings, and hence our treatment is not appropriate for a W-boson plasma for these reasons.[32]

The motivation for the study is to further add to the considerable collection of the linear response theory work in general, and to that of relativistic pair plasmas in particular. We also wish to compare our results to those obtained by previous authors[5] employing techniques manifestly different to our own, but which should contain the same physics. Furthermore, we envision the possible manifestation of a relativistic spin-one pair plasma in the early universe, and hence the work may have astrophysical and cosmological applications. In closing the Introduction, we note the corresponding behaviour of the relativistic electron plasma, studied by Tsytovich, Jancovici, Kalman, Cover and Bakshi, ourselves and others is reviewed in depth in Refs. 3) and 4), along with its astrophysical significance.

§ 2. Relativistic field theories for spin-one particles

A. *Requirements of the RPA method*

A familiar and commonly employed relativistic field theory is that of Dirac for particles of spin one-half; a formalism which is noted for its elegance and ease of manipulation. Indeed, KFH,[3] who present a treatment of the electron-positron plasma as well as the spin-zero pair plasma, employ the second quantized Dirac theory in their work, as was done before them by Delsante and Frankel[6] in their study of the relativistic electron plasma. However, there is a dearth of complete and self-consistent formalisms for particles of other than spin-one-half. Whilst the Klein-Gordon equation is appropriate for the description of spin-zero bosons, one must move to a two component wave function formalism (where the two degrees of freedom are those of *charge*), such as that proposed by Feshbach and Villars,[7] to find a theory which allows comparable ease of manipulation in the RPA treatment for plasma linear response theory. Both KFH[3] and WKH[4] employ this formalism in their spin-zero work.

The common features of both the Dirac theory and the Feshbach-Villars formalism which facilitate the use of the RPA method are those which we seek in an analogous spin-one theory. First, we require an explicit form for the Hamiltonian, which we employ to calculate the equations of motion of various bilinear products of creation and annihilation operators which will arise. (We work in the Heisenberg picture). Second, there must be a consistent and straightforward coupling of the matter fields to electromagnetic fields. Although we have no external fields, first order perturbative fields are present, arising from interactions within the plasma itself. Furthermore, we must have a second quantized four-current from which to extract the polarization tensor, and once this is done we require a sensible way to calculate the matrix elements which will constitute it. In the RPA treatment, this requires the free particle plane-wave solutions to be found. As we shall discuss, the Sakata-Taketani formalism admirably fulfills these requirements. First, however, we undertake a thorough review of spin-one formalisms, for the sake of comparison, and in order to demonstrate the development of the essential features of the Sakata-Taketani formalism.

B. *Description of spin-one field theories*

The work of Proca[8] was first to establish a consistent theory of spin-one particles. The field components $\phi_\nu (\nu=0, \cdots, 3)$ satisfy individually the covariant Klein-Gordon equation:

$$(\partial_\mu \partial^\mu - m^2)\phi_\nu = 0\,, \tag{2·1}$$

where m is the particle mass.

The components ϕ_ν are constrained by the Lorentz condition:

$$\partial^\nu \phi_\nu = 0\,. \tag{2·2}$$

The Lagrangian density of the system is the Proca Lagrangian density:

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi^\nu \partial^\mu \phi_\nu + m^2 \phi_\nu \phi^\nu)\,, \tag{2·3}$$

from which the field theories to be discussed evolve.

This treatment leads to a formalism which is similar to the standard theory for the photon field, which is the massless spin-one particle. The terminology used is that of the photon field, with the matter fields decomposed into longitudinal and transverse polarizations. It is the field theory most prevalent in the quantum field theory texts (see, for example, Ref. 9)), and has also been employed in studies of the anomalous magnetic moment of the spin-one particle. This has been done by Lee and

Yang,[10] Tzou,[11] Corben and Schwinger,[12] Kyriakopoulous,[13] Velo and Zwangzinger,[14] and Aronson.[15]

For our purposes, however, this formalism is of limited value. The four-current and free particle solutions are readily obtainable, but difficulties arise in the calculation of the interaction Hamiltonian, and the calculation of matrix elements—similar to the reasons for which Klein-Gordon theory is less suited to the RPA method for spin-zero plasmas than the two component Feshbach-Villars theory.

Another, but related, approach is the generalization of the Dirac theory of electrons to particles of higher spin, leading to a variety of multicomponent wave function theories. Typical of these are the sixteen-component single-rank spinor theories, as described by Corson.[16]

For example, the first rank spinor theory would involve wave functions which obey a Dirac-type equation:

$$(\beta^{\nu}\partial_{\nu}-i\mu)\psi=0\,, \tag{2·4}$$

where β^{ν} and μ are respectively the appropriate generalizations of the γ^{ν} and mI matrices in Dirac spin-one-half theory.

The main disadvantage of such a theory is that the single rank spinors are the subject of constraining equations, similar to those in Proca theory, and would thus contain dynamically redundant components, introducing difficulties of both a conceptual and calculational nature. Also proposed is a second-rank spinor theory, such as that by Belinfante[17] and Tsai (et al.).[18] However, it is by no means obvious how one should proceed in the determination of an explicit Hamiltonian with field couplings, nor in the calculation of the free-particle solution of the wave equation.

Foldy[20] is able to overcome the problems of constraining equations and dynamically redundant components in the wave function by employing the Foldy-Wouthuysen transformation upon the Proca wave functions satisfying Eq. (2·1). The wave equation which he develops is in the canonical Foldy-Wouthuysen diagonalized form:

$$i\frac{\partial\chi}{\partial t}=\beta E\chi\,, \tag{2·5}$$

where in the spin-one case

$$\beta=\begin{pmatrix} I_{(3\times3)} & 0 \\ 0 & -I_{(3\times3)} \end{pmatrix}, \tag{2·6}$$

and E is the energy eigenvalue.

Good (et al.)[21)~23] pursue this avenue further, deriving explicit forms for the single particle Hamiltonian. The main disadvantage of this particular formalism is that complicated transformations are induced upon the wave function and the four-current, rendering the calculation of the polarization tensor, and the subsequent evaluation of the matrix elements, extremely tedious.

An attempt is made by Tsai (et al.)[18] to reconcile the vector (Proca), multi-spinor and Foldy six-component theories, and in particular a detailed investigation of the energy eigenvalues of each is given, with certain discrepancies found. This work is later continued by Vijayalakshmi, Seetharaman and Mathews,[24] who use the Foldy formalism to study the anomalous magnetic moment of the spin-one particle.

For a propagator approach to spin-one theory, the paper by Weinberg[19] is of great value. Aside from giving a general algebraic overview of any-spin relativistic field theories, Weinberg also presents the Feynman rules for particles of any spin. The massive spin-one boson propagator is the Proca propagator, and is that employed by Williams and Melrose[5] in their ensemble averaged propagator approach to the spin-one pair plasma.

Ultimately, the formalism we choose to employ is that developed by Sakata and Taketani,[1] and reviewed at length by Heitler.[25] The formalism encompasses within it descriptions of both spin-one and spin-zero particles. Indeed, the spin becomes a parameter which is simply set to zero when one wishes to study scalar bosons, whence it becomes precisely the two-column formalism of Feshbach and Villars.[7]

Heitler[25] gives a clear description of the procedure for the derivation of the full six-component theory for spin-one particles, and we give here a brief account of this, as it is instructional in reconciling the formalism we employ with the more familiar Proca formalism for spin-one particles.

The field equations for Proca theory are:

$$\partial_\mu \phi_\nu - \partial_\nu \phi_\mu = m\chi_{\mu\nu} \tag{2·7}$$

and

$$\partial^\nu \chi_{\mu\nu} = m\phi_\mu \,. \tag{2·8}$$

The Duffin-Kemmer equation (Kemmer[26] and Duffin[27])

$$\beta^\mu p_\mu \psi = im\psi \,, \tag{2·9}$$

is obtained from (2·7) and (2·8) via the assignments

$$\begin{aligned} &\chi_{01},\ \chi_{02},\ \chi_{03} \to \psi_1,\ \psi_2,\ \psi_3\,, \\ &\chi_{23},\ \chi_{31},\ \chi_{12} \to -\psi_4,\ -\psi_5,\ -\psi_6\,, \\ &\phi_1,\ \phi_2,\ \phi_3,\ -\phi_0 \to \psi_7,\ \psi_8,\ \psi_9,\ \psi_{10}\,, \end{aligned} \tag{2·10}$$

which thus constitute the ten-component wave function ψ.

The matrices β_μ, which are 10×10 matrices, obey the algebra of the Duffin-Kemmer ring:

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = \delta_{\mu\nu}\beta_\lambda + \delta_{\lambda\nu}\beta_\mu \,. \tag{2·11}$$

As Heitler[25] demonstrates, the Duffin-Kemmer equation (2·9) has the dynamically redundant field quantities χ_{ik} and ϕ_0. Motivated by the need to eliminate these, Sakata and Taketani[1] proceed to reduce the Duffin-Kemmer formalism to a six-component formalism, which we shall employ in our work. The details of the manipulation of the Duffin-Kemmer ring which are required to achieve this (which, in Heitler's exposition, he credits to Schrödinger) are intricate and lengthy, and will not be discussed here. The six-component formalism itself, however, will be clearly described in the ensuing section, and developed to some extent beyond the level given by Heitler. Of value in this work is a paper by Weaver[28] on spin-one matrix algebra. Specifically, we second quantize the theory, and find the free-particle solutions explicitly.

§ 3. Six-component theory for spin-one particles

A. *Basic definitions*

The six-component Sakata-Taketani wave function for spin-one particles satisfies the familiar wave equation:

$$\mathcal{H}\Psi = E\Psi\,, \tag{3·1}$$

where $\mathcal{H}$ is the Hamiltonian for a particle interacting with electromagnetic fields:

$$\mathcal{H} = e\boldsymbol{\Phi} + \rho_3 mc^2 + \rho_0\left(\frac{1}{2m}\boldsymbol{\pi}^2 - \frac{e\hbar}{2mc}(\boldsymbol{\sigma}\cdot\boldsymbol{B})\right) - i\rho_2\frac{1}{m}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\,, \tag{3·2}$$

where

$\boldsymbol{A}$ and $\boldsymbol{\Phi}$ are the vector and scalar potentials respectively,

$\boldsymbol{\pi} = \boldsymbol{p} - (e/c)\boldsymbol{A}$ is the canonical momentum,

and $\boldsymbol{B}$ is the magnetic field strength.

The matrices ρ_i are 2×2 Pauli (super)matrices. We choose the following representation:

$$\rho_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad i\rho_2 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

$$\rho_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{3·3}$$

where I is the 3×3 unit matrix, and where we also define for convenience $\rho_0 = \rho_3 + i\rho_2$.

The matrix operators σ_i are the spin-one angular momentum matrices. It is to be noted that these matrices are closely related to the matrices β_μ appearing in the Duffin-Kemmer equation (2·9). Indeed, one may define 10×10 spin matrices σ_i', where

$$\sigma_i' = -i\varepsilon_{ijk}\beta_j\beta_k\,, \tag{3·4}$$

and in the process of the reduction of the ten-component formalism to the six-component form, these σ'-matrices reduce directly to the familiar 3×3 spin-one angular momentum matrices.

In our work, we employ the representation:

$$\sigma_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3·5}$$

The group properties for these operators are well known, and are most easily derived from Eq. (3·4) and the properties of the Duffin-Kemmer ring (2·11). They are

$$\text{(i)} \quad \sigma_i\sigma_j\sigma_k + \sigma_k\sigma_j\sigma_i = \delta_{ij}\sigma_k + \delta_{jk}\sigma_i ,$$

$$\text{(ii)} \quad \sigma_i\sigma_j - \sigma_j\sigma_i = i\varepsilon_{ijk}\sigma_k , \tag{3·6}$$

the latter being the standard commutation relation for angular momentum matrices.

The Hamiltonian (3·2) has several features which solicit comment. First, at an initial glance, it appears to be extraordinarily similar to familiar forms for non-relativistic Hamiltonians. However, we stress that it is covariant and relativistic. Of interest also is the explicit coupling of the spin to the magnetic field in the next to last term of the Hamiltonian. The final term in the Hamiltonian, $-i\rho_2(1/m)(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2$ indicates a coupling of the spin to the momentum, and is a gyroscopic term expected of particles of finite spin. Finally, by putting $\boldsymbol{\sigma}\equiv 0$ in the Hamiltonian, one recovers the Hamiltonian given by Feshbach and Villars[7] for spin-zero particles:

$$\mathcal{H}_{(\text{spin zero})} = e\Phi + \rho_3 mc^2 + \rho_0 \frac{1}{2m}\boldsymbol{\pi}^2 \tag{3·7}$$

with the added condition, naturally, that all 6×6 matrices reduce to 2×2 forms.

Whilst the Sakata-Taketani formalism explicitly involves six-component wave functions, they may be treated as two component forms, each component itself having three components. The approach is thus much like Dirac theory, and we may write for the wave functions Ψ:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{3·8}$$

B. *Free-particle solutions*

The relationship between ψ_1 and ψ_2 for the free particle case is given by Heitler,[25] and is

$$\psi_2{}^{\epsilon} = \psi_1{}^{\epsilon}\left(\frac{2(\boldsymbol{p}\cdot\boldsymbol{\sigma})^2c^2 - \boldsymbol{p}^2c^2}{(\epsilon E(p)+mc^2)^2}\right), \tag{3·9}$$

where

$$E(p) = (p^2c^2 + m^2c^4)^{1/2} ,$$

and ϵ takes a + or − as a symbol, and the value +1 or −1 in expressions for positive energy solutions (bosons) and negative energy solutions (antibosons).

The plane-wave solutions to the wave equation are, in the Heisenberg picture:

$$\Psi^{\epsilon}(p, s) = \varphi^{\epsilon}(p, s)e^{(i/\hbar)\epsilon \boldsymbol{p}\cdot\boldsymbol{r}} , \tag{3·10}$$

and all that is now required is an appropriate normalization of these wave functions to obtain the free-field solutions. As both Heitler[25] and Feshbach and Villars[7] point out, the only sensible way to normalize the wave functions is to *charge*. This is to overcome the problem found in the interpretation of the possibility of a negative current density, for it may be now considered as a charge (rather than probability) current. Thus, in a manner proposed by Heitler, and Feshbach and Villars, we normalize the positive and negative energy solutions of the wave equation to positive and negative charge, respectively.

First, the expectation value of an operator $\mathcal{O}$ is given by

$$\bar{\mathcal{O}}=\int d^3\boldsymbol{r}\Psi^*\rho_3\mathcal{O}\,\Psi\,, \tag{3·11}$$

where Ψ^* is the complex conjugate transpose of Ψ. This follows from the general definition of a matrix element:

$$\langle\Psi'|\mathcal{O}|\Psi\rangle\equiv\int d^3\boldsymbol{r}\Psi'^*\rho_3\mathcal{O}\,\Psi\,. \tag{3·12}$$

Thus, the normalization is

$$(\varphi_1{}^\epsilon)^2-(\varphi_2{}^\epsilon)^2=\epsilon\,, \tag{3·13}$$

leading to a solution of the wave equation:

$$\Psi^\epsilon(p,s)=\frac{1}{2(E(p)mc^2V)^{1/2}}\begin{pmatrix}(\epsilon E(p)+mc^2)\chi_s\\ \dfrac{2(\boldsymbol{p}\cdot\boldsymbol{\sigma})^2c^2-\boldsymbol{p}^2c^2}{(\epsilon E(p)+mc^2)}\chi_s\end{pmatrix}e^{(i/\hbar)\epsilon\boldsymbol{p}\cdot\boldsymbol{r}}\,, \tag{3·14}$$

where V is the volume of the system.

The three-columns χ_s, s being the spin projection quantum number, become in the representation of the σ matrices given by (3·5):

$$\chi_{+1}=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \chi_0=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad \chi_{-1}=\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{3·15}$$

In this aspect, the similarity to Dirac theory is evident. For example, spin ladder operators may be defined as follows:

$$\sigma_\pm=\sigma_x\pm i\sigma_y\,,$$

so that

$$\sigma_+\chi_s=\chi_{s+1}\,,\quad s\leq 0\,,$$
$$\sigma_-\chi_s=\chi_{s-1}\,,\quad s\geq 0\,. \tag{3·16}$$

Parameterizing our wave functions with momentum and spin projection quantum numbers, and with the sign of the energy eigenvalue (ϵ), we have

$$\langle\epsilon',p',s'|\epsilon,p,s\rangle=\epsilon\delta_{\epsilon,\epsilon'}\delta_{p,p'}\delta_{s,s'}\,, \tag{3·17}$$

from which we obtain the completeness relation:

$$\sum_{\epsilon,p,s}\epsilon|\epsilon,p,s\rangle\langle\epsilon,p,s|=1\,. \tag{3·18}$$

C. *Charge and current densities*

The single particle charge and current densities are given by Heitler,[25] and we display them below.

The charge density is

$$\rho = e\Psi^* \rho_3 \Psi \,, \tag{3·19}$$

and the current density $\boldsymbol{j}$ is given by

$$\begin{aligned}\frac{2m}{e}\boldsymbol{j} &= \Psi^* \rho_3 \rho_0 \left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)\Psi \\ &\quad - \Psi^* \rho_3 (i\rho_2) \left\{ \boldsymbol{\sigma}\left[\left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)\cdot \boldsymbol{\sigma}\right] + \left[\left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)\cdot \boldsymbol{\sigma}\right]\boldsymbol{\sigma}\right\}\Psi \\ &\quad + i\Psi^* \left[\left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)\times \boldsymbol{\sigma}\right]\Psi\end{aligned} \tag{3·20}$$

plus the same terms with $\boldsymbol{p}$ acting to the left upon Ψ^* instead of Ψ, and i replaced by $-i$ in the last term.

It is to be noted that these are derived by Heitler[25] (following a procedure for a general operator) via the reduction of the corresponding Duffin-Kemmer four-current

$$j_\mu = e\phi^\dagger \beta_\mu \phi \,, \tag{3·21}$$

to the appropriate six-component equivalents shown above.

D. *Second quantization*

We employ the canonical technique to obtain the second quantized fields, and, as a check, calculate the normal ordered free particle Hamiltonian operator. We expect the standard diagonal form.

The second quantized spin-one field is

$$\hat{\Psi} = \sum_{p,s} \{ b_{p,s}(t)\Psi^+_{p,s}(\boldsymbol{r}) + d^\dagger_{p,s}(t)\Psi^-_{p,s}(\boldsymbol{r}) \} \,. \tag{3·22}$$

The operators $b_{p,s}(t)$ and $d_{p,s}(t)$ are, respectively, the destruction operators for a particle and antiparticle state. They obey the standard commutation relations for boson operators:

$$[b_{p,s}, b^\dagger_{p,s}] = 1 \quad \text{and} \quad [d_{p,s}, d^\dagger_{p,s}] = 1 \tag{3·23}$$

with all others vanishing.

The second quantization for the free-field Hamiltonian

$$\mathcal{H}_0 = \rho_3 mc^2 + \rho_0 \frac{\boldsymbol{p}^2}{2m} - i\rho_2 \frac{1}{m}(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 \,, \tag{3·24}$$

is done via the canonical procedure:

$$\hat{H}_0 = \int d^3\boldsymbol{r}\, \hat{\Psi}^\dagger \rho_3 \mathcal{H}_0 \hat{\Psi} \,. \tag{3·25}$$

The normal ordered free-field Hamiltonian operator becomes, after some calculation:

$$:\hat{H}_0:=\sum_{p,s}E(p)(b^\dagger_{p,s}b_{p,s}+d^\dagger_{p,s}d_{p,s})\,, \tag{3·26}$$

which is the standard diagonal form.

As we work with an unrenormalized vacuum, we note here that we will not use normal order operators in our response theory calculation.

E. *Traces of products of spin matrices*

Given the fact that the free-field solutions (3·14) bear a resemblance to Dirac equation free-field solutions, it is no surprise that the response theory calculation requires the calculation of traces of products of spin matrices, in analogy to Dirac theory requiring traces of products of Dirac γ matrices. We list here, for later reference, the results for the products of spin-one matrices.

First, inspection of (3·6ii) immediately implies

$$\mathrm{Tr}(\sigma_i)=0\,. \tag{3·27}$$

Furthermore, inspection of the σ matrices (3·5) allows the evaluation of the trace of a bilinear product

$$\mathrm{Tr}(\sigma_i\sigma_j)=2\delta_{ij}\,. \tag{3·28}$$

We then employ the group properties (3·6i) and (3·6ii) to develop a recursion relation for the traces of higher order.

Consider

$$\mathrm{Tr}(\sigma_1\sigma_2\cdots\sigma_{n-3}\sigma_{n-2}\sigma_{n-1}\sigma_n)\,. \tag{3·29}$$

Employing (3·6ii) on the last two matrices $\sigma_{n-1}\sigma_n$, (3·29) becomes

$$\frac{1}{2}\mathrm{Tr}\{(\sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}\sigma_n)+(\sigma_1\sigma_2\cdots\sigma_{n-3}\sigma_{n-2}\sigma_n\sigma_{n-1})+i\varepsilon_{n-1,n,q}(\sigma_1\sigma_2\cdots\sigma_{n-3}\sigma_{n-2}\sigma_q)\}\,. \tag{3·30}$$

This step is repeated until (3·6ii) can be employed on the last three σ matrices appearing in the first two terms of (3·30) to reduce them to the sum of two single σ matrices. Thus (3·30) becomes

$$\frac{1}{2}\mathrm{Tr}\{(\sigma_1\sigma_2\cdots\sigma_{n-3}\sigma_n)\delta_{n-2,n-1}+(\sigma_1\sigma_2\cdots\sigma_{n-3}\sigma_{n-2})\delta_{n-1,n}+i\varepsilon_{n-1,n,q}(\sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_q)$$
$$+i\varepsilon_{n-2,n,t}(\sigma_1\sigma_2\cdots\sigma_{n-3}\sigma_t\sigma_{n-1})+i\varepsilon_{n-2,n-1,r}(\sigma_1\sigma_2\cdots\sigma_{n-3}\sigma_n\sigma_r)\}\,. \tag{3·31}$$

We give the first few explicitly:

$$\mathrm{Tr}(\sigma_i\sigma_j\sigma_k)=i\varepsilon_{ijk}\,, \tag{3·32}$$

$$\mathrm{Tr}(\sigma_i\sigma_j\sigma_k\sigma_l)=\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk}\,, \tag{3·33}$$

$$\mathrm{Tr}(\sigma_i\sigma_j\sigma_k\sigma_l\sigma_m)=\frac{i}{2}\{\varepsilon_{ijk}\delta_{lm}+\varepsilon_{ijm}\delta_{lk}+\varepsilon_{kli}\delta_{jm}+\varepsilon_{kmj}\delta_{il}+\varepsilon_{lmk}\delta_{ij}+\varepsilon_{lmi}\delta_{jk}\}\,. \tag{3·34}$$

§ 4. Linear response theory

A. *Basic definitions*

Before we begin our response theory calculation, it is important to set out the basic definitions and conventions employed in our work. First, we work in the metric with signature $(+, -, -, -)$, which was used by WKH[4] in their work. However, in contrast to those authors, who employ S. I. units, we choose to work in Gaussian (c. g. s.) units.

The polarization tensor is defined via the relationship between the Fourier transformed four-current density and the four-potential.

$$J^{\nu}(k)=\sum_{k'}\Pi^{\nu}{}_{\mu}(k, k')A^{\mu}(k'), \tag{4·1}$$

where k is the wave number four-vector

$$k\equiv\left(\frac{\omega}{c}, \boldsymbol{k}\right). \tag{4·2}$$

(We note here that $\omega=\omega'$ for a stationary plasma.)

The current density four-vector is

$$J\equiv(c\rho, \boldsymbol{j}), \tag{4·3}$$

and the potential four-vector is given by

$$A\equiv(\Phi, \boldsymbol{A}). \tag{4·4}$$

We note that Eq. (4·1) is a linear response theory relationship, and the A^{μ} are thus the electromagnetic potentials inducing the perturbations within the plasma.

The three-current density $\boldsymbol{j}$ is related to the perturbing electric field $\boldsymbol{E}$ via

$$\boldsymbol{j}(\boldsymbol{k}, \omega)=\sum_{k'}\overleftrightarrow{\boldsymbol{\sigma}}(\boldsymbol{k}, \boldsymbol{k}', \omega)\cdot\boldsymbol{E}(\boldsymbol{k}, \omega), \tag{4·5}$$

where $\overleftrightarrow{\boldsymbol{\sigma}}$ is the conductivity three-tensor.

We shall derive the polarization tensor in a gauge invariant way. However, to find the modes of oscillation, we shall employ the Coulomb gauge

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}=0 \tag{4·6}$$

as this gauge is useful in separating the longitudinal and transverse parts of the polarization tensor.

The components of the polarization tensor are related to the conductivity tensor in the following way:

$$\sigma_{ij}=i\Pi_{i0}\frac{k_j}{k^2}+\frac{ic}{\omega}\Pi_{ir}\left(\delta_{rj}-\frac{k_r k_j}{k^2}\right). \tag{4·7}$$

In turn, the conductivity three-tensor is related to the dielectric tensor thus:

$$\overleftrightarrow{\varepsilon}=\overleftrightarrow{I}+\frac{4\pi i}{\omega}\overleftrightarrow{\sigma}\,, \tag{4·8}$$

from which we obtain the longitudinal and transverse dielectric response functions of the plasma. These are given by

$$\varepsilon_L=1-\frac{4\pi}{\omega}\frac{k_i\Pi_{i0}}{k^2} \tag{4·9}$$

and

$$\varepsilon_T=1-\frac{4\pi c}{\omega^2}\Pi_{ij}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right), \tag{4·10}$$

respectively. The longitudinal and transverse modes of oscillation are found by the following equations:

$$\varepsilon_L=0 \tag{4·11}$$

and

$$\varepsilon_T=\frac{k^2c^2}{\omega^2}, \tag{4·12}$$

respectively.

B. *Interaction Hamiltonian and current*

Our next task is to linearize and second quantize the interaction Hamiltonian and four-current. The linearization is achieved by expanding the four-potential in the following manner:

$$A^\mu(\boldsymbol{r},t)=A_0{}^\mu(\boldsymbol{r},t)+A_1{}^\mu(\boldsymbol{r},t)\,. \tag{4·13}$$

Above, $A_0{}^\mu$ is the four-potential due to external fields, and in our present work is set to zero. The first order perturbative potential, arising from interactions within the plasma, is $A_1{}^\mu$. This linearization yields the single-particle interaction Hamiltonian:

$$\begin{aligned}\mathcal{H}_1=e\Phi_1-\rho_0\frac{e}{2mc}\{(\boldsymbol{p}\cdot\boldsymbol{A}_1)+(\boldsymbol{A}_1\cdot\boldsymbol{p})+\hbar\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}_1)\}\\ +i\rho_2\frac{e}{mc}\{(\boldsymbol{\sigma}\cdot\boldsymbol{A}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{p})+(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{A}_1)\}\,.\end{aligned} \tag{4·14}$$

We decompose the perturbative potential into its Fourier components, recalling that Eq. (4·1) for the tensor is in Fourier space,

$$A_1{}^\mu(\boldsymbol{r},t)=\sum_{k'}e^{i\boldsymbol{k}'\cdot\boldsymbol{r}}A_1{}^\mu(\boldsymbol{k}',t)\,. \tag{4·15}$$

Substituting (4·15) into (4·14), and then second quantizing the interaction Hamiltonian leads to its operator form:

$$\hat{H}_1=\sum_{k'}\sum_{p,p'}\sum_{s,s'}\{b^\dagger_{p',s'}b_{p,s}\beta(+,+)+b^\dagger_{p',s'}d^\dagger_{p,s}\beta(+,-)$$
$$+d_{p',s'}b_{p,s}\beta(-,+)+d_{p',s'}d^\dagger_{p,s}\beta(-,-)\}, \tag{4·16}$$

where $\beta(\epsilon',\epsilon)$ is defined as

$$\begin{aligned}\beta(\epsilon',\epsilon)&=\beta(\epsilon',\epsilon,p',p,s',s,\boldsymbol{k}')\\
&\equiv\langle\epsilon',p',s'|e^{i\boldsymbol{k}'\cdot\boldsymbol{r}}|\epsilon,p,s\rangle e\Phi_1\\
&\quad-\langle\epsilon',p',s'|\rho_0e^{i\boldsymbol{k}'\cdot\boldsymbol{r}}\left(\boldsymbol{p}+\frac{1}{2}\hbar\boldsymbol{k}'\right)|\epsilon,p,s\rangle\cdot\frac{e}{mc}\boldsymbol{A}_1\\
&\quad-\langle\epsilon',p',s'|\rho_0e^{i\boldsymbol{k}'\cdot\boldsymbol{r}}(\boldsymbol{\sigma}\times\hbar\boldsymbol{k}')|\epsilon,p,s\rangle\cdot\frac{ie}{2mc}\boldsymbol{A}_1\\
&\quad+\langle\epsilon',p',s'|i\rho_2e^{i\boldsymbol{k}'\cdot\boldsymbol{r}}[\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\boldsymbol{p})+(\boldsymbol{\sigma}\cdot(\boldsymbol{p}+\hbar\boldsymbol{k}'))\boldsymbol{\sigma}]|\epsilon,p,s\rangle\cdot\frac{e}{mc}\boldsymbol{A}_1\,.\end{aligned} \tag{4·17}$$

The charge density (3·19) is also second quantized and Fourier-transformed to obtain the charge density operator:

$$\hat{j}^0(\boldsymbol{k},t)=\frac{ec}{V}\sum_{p,p'}\sum_{s,s'}\{b^\dagger_{p',s'}b_{p,s}\zeta(+,+)+b^\dagger_{p',s'}d^\dagger_{p,s}\zeta(+,-)$$
$$+d_{p',s'}b_{p,s}\zeta(-,+)+d_{p',s'}d^\dagger_{p,s}\zeta(-,-)\}, \tag{4·18}$$

and similarly the three-current operator:

$$\hat{\boldsymbol{j}}(\boldsymbol{k},t)=\frac{e}{2mV}\sum_{p,p'}\sum_{s,s'}\{b^\dagger_{p',s'}b_{p,s}\boldsymbol{\gamma}(+,+)+b^\dagger_{p',s'}d^\dagger_{p,s}\boldsymbol{\gamma}(+,-)$$
$$+d_{p',s'}b_{p,s}\boldsymbol{\gamma}(-,+)+d_{p',s'}d^\dagger_{p,s}\boldsymbol{\gamma}(-,-)\} \tag{4·19}$$

with the definitions

$$\zeta(\epsilon',\epsilon)=\zeta(\epsilon',\epsilon,p',p,s',s,\boldsymbol{k})\equiv\langle\epsilon',p',s'|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}|\epsilon,p,s\rangle \tag{4·20}$$

and

$$\begin{aligned}\boldsymbol{\gamma}(\epsilon',\epsilon)&=\boldsymbol{\gamma}(\epsilon',\epsilon,p',p,s',s)\\
&\equiv\langle\epsilon',p',s'|\rho_0e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}(2\boldsymbol{p}-\hbar\boldsymbol{k})|\epsilon,p,s\rangle\\
&\quad+\langle\epsilon',p',s'|i\rho_2e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}[\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot(2\boldsymbol{p}-\hbar\boldsymbol{k}))+(\boldsymbol{\sigma}\cdot(2\boldsymbol{p}-\hbar\boldsymbol{k}))\boldsymbol{\sigma}]|\epsilon,p,s\rangle\\
&\quad-i\langle\epsilon',p',s'|\rho_3e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{\sigma}\times\hbar\boldsymbol{k})|\epsilon,p,s\rangle\\
&\quad+\frac{2e}{c}\sum_{k'}\{\langle\epsilon',p',s'|i\rho_2e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}}(\boldsymbol{\sigma}\sigma^j+\sigma^j\boldsymbol{\sigma})|\epsilon,p,s\rangle A_1{}^j\\
&\quad-\langle\epsilon',p',s'|\rho_0e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}}|\epsilon,p,s\rangle\boldsymbol{A}_1\}\,.\end{aligned} \tag{4·21}$$

C. *Equations of motion*

Upon inspection of the operator forms for the charge (4·18) and current (4·19) densities, we see that the following bilinear products of particle operators appear:

$$b^{\dagger}_{p',s'}b_{p,s}\,,\quad b^{\dagger}_{p',s'}d^{\dagger}_{p,s}\,,\quad d_{p',s'}b_{p,s}\,,\quad d_{p',s'}d^{\dagger}_{p,s}\,. \tag{4·22}$$

The equations of motion for these operator products are now found, and then ensemble averaged to find the equations of motion for the particle and antiparticle distribution functions.

In our linearized regime, the equation of motion for an operator $\hat{\mathcal{O}}$ is

$$i\hbar\frac{\partial\hat{\mathcal{O}}}{\partial t}=[\hat{\mathcal{O}},\hat{H}_0]+[\hat{\mathcal{O}},\hat{H}_1]\,. \tag{4·23}$$

Taking an ensemble average over all possible states of the system implies a concomitant quantum mechanical average, by virtue of working in the operator Fock space. For this purpose, we make the following definitions:

$$\begin{aligned}
F(a',a,t)&=\sum_{\alpha}P_{\alpha}\langle\alpha|b^{\dagger}_{a'}(t)b_{a}(t)|\alpha\rangle\,,\\
H(a',a,t)&=\sum_{\alpha}P_{\alpha}\langle\alpha|b^{\dagger}_{a'}(t)d^{\dagger}_{a}(t)|\alpha\rangle\,,\\
K(a',a,t)&=\sum_{\alpha}P_{\alpha}\langle\alpha|d_{a'}(t)b_{a}(t)|\alpha\rangle\,,\\
G(a',a,t)&=\sum_{\alpha}P_{\alpha}\langle\alpha|d_{a'}(t)d^{\dagger}_{a}(t)|\alpha\rangle\,,
\end{aligned} \tag{4·24}$$

where we have labelled momentum and spin projection as a single quantum number, and where P_{α} is the probability that the system is in the state $|\alpha\rangle$.

We obtain, thus, the equations of motion for the ensemble averages F, H, K and G. For example, the equation of motion for $F(a',a,t)$ is

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}F(a',a,t)=(E(a)-E(a'))F(a',a,t)&\\
+\sum_{\boldsymbol{k}'}\sum_{g}\{F(a',g,t)\beta(+,+,a,g,\boldsymbol{k}')&\\
+H(a',g,t)\beta(+,-,a,g,\boldsymbol{k}')&\\
-F(g,a,t)\beta(-,+,g,a',\boldsymbol{k}')&\\
-K(g,a,t)\beta(-,-,g,a',\boldsymbol{k}')\}\,.&
\end{aligned} \tag{4·25}$$

We now linearize the four ensemble averages F, G, H and K:

$$\begin{aligned}
F(a',a,t)&=F(a)\delta_{a,a'}+F_1(a',a,t)\,,\\
G(a',a,t)&=G(a)\delta_{a,a'}+G_1(a',a,t)\,,\\
H(a',a,t)&=H_1(a',a,t)\,,\\
K(a',a,t)&=K_1(a',a,t)\,,
\end{aligned} \tag{4·26}$$

The first two operator averages are now seen to be closely related to the distribution functions for bosons and antibosons, respectively. Indeed:

$$F(a)=N_+(a) \quad \text{and} \quad G(a)=1+N_-(a)\,, \tag{4·27}$$

where $N_+(a)$ and $N_-(a)$ are, respectively, the zeroth order distribution functions for bosons and antibosons, and are given by

$$N_+=\frac{1}{3}\frac{1}{e^{\frac{1}{\kappa T}(E(p)-\mu)}-1}$$

and $$N_-=\frac{1}{3}\frac{1}{e^{\frac{1}{\kappa T}(E(p)+\mu)}-1}\,, \tag{4·28}$$

where μ is the chemical potential, κ is Boltzmann's constant, and the factor of 1/3 arises from the three spin projections.

We note that the ensemble averages H and K have no zeroth order term, as they represent the mixing of boson and antiboson states, and arise as a result of pair production processes.

The first order perturbations in the distribution functions, and the four-potential, are taken to have the following time-dependent behaviour:

$$F_1,\, G_1,\, H_1,\, K_1,\, A_1{}^{\mu}\sim e^{-i\omega t+\eta t}\,, \tag{4·29}$$

consistent with the perturbative nature of the interactions in the linear regime. The positive infinitesimal η is introduced so that the perturbations vanish at $t=-\infty$, which is equivalent to the Landau prescription for the avoidance of singularities in the sums and integrals which will arise. This gives the same result as performing a Laplace (rather than a Fourier) transform in the time co-ordinate, as the $t=-\infty$ boundary condition is included, and the perturbations are resultantly causal.

The linearized distribution functions (4·26) are introduced into the distribution function equations of motion to obtain expressions for the first order distribution functions in terms of the first order potentials.

$$F_1(a',a,\omega)=\sum_{k'}\left\{\frac{F(a')-F(a)}{\hbar\omega-(E(a)-E(a'))+i\eta}\right\}\beta(+,+,a,a',\boldsymbol{k}')\,,$$

$$H_1(a',a,\omega)=-\sum_{k'}\left\{\frac{F(a')+G(a)}{\hbar\omega+(E(a')+E(a))+i\eta}\right\}\beta(-,+,a,a',\boldsymbol{k}')\,,$$

$$K_1(a',a,\omega)=\sum_{k'}\left\{\frac{F(a)+G(a')}{\hbar\omega-(E(a)+E(a'))+i\eta}\right\}\beta(+,-,a,a',\boldsymbol{k}')\,,$$

$$G_1(a',a,\omega)=\sum_{k'}\left\{\frac{G(a)-G(a')}{\hbar\omega-(E(a')-E(a))+i\eta}\right\}\beta(-,-,a,a',\boldsymbol{k}')\,. \tag{4·30}$$

The final step before the extraction of the polarization tensor is to ensemble average the current and charge densities. This is particularly straightforward—all of the creation and destruction operator products of (4·22) appear in (4·18) and (4·19). Thus, they become the appropriate distribution functions upon ensemble averaging. The current and charge densities are then linearized, and we consider only the first order perturbative terms, as the polarization tensor relates perturbations in the four-current to those in the four-potential.

The ensemble-averaged and linearized charge density is

$$\langle \hat{J}_1{}^0(\boldsymbol{k}, \omega)\rangle$$

$$=\frac{ec}{V}\sum_{a,a'}\{F_1(a', a, \omega)\langle +, a'|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}|+, a\rangle + H_1(a', a, \omega)\langle +, a'|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}|-, a\rangle$$

$$+K_1(a', a, \omega)\langle -, a'|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}|+, a\rangle + G_1(a', a, \omega)\langle -, a'|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}|-, a\rangle\}, \quad (4\cdot31)$$

and the current density becomes

$$\langle \hat{\boldsymbol{j}}_1(\boldsymbol{k}, \omega)\rangle$$

$$=\frac{e}{2mV}\sum_{a,a'}\{F_1(a', a, \omega)\boldsymbol{\tau}(+, +, a', a, \boldsymbol{k}) + H_1(a', a, \omega)\boldsymbol{\tau}(+, -, a', a, \boldsymbol{k})$$

$$+K_1(a', a, \omega)\boldsymbol{\tau}(-, +, a', a, \boldsymbol{k}) + G_1(a', a, \omega)\boldsymbol{\tau}(-, -, a', a, \boldsymbol{k})\}$$

$$+\frac{e^2}{mcV}\sum_a\sum_{k'}\{F_0(a)\xi(+, a, \boldsymbol{k}, \boldsymbol{k}') + G_0(a)\xi(-, a, \boldsymbol{k}, \boldsymbol{k}')\}, \quad (4\cdot32)$$

where

$$\boldsymbol{\tau}(\epsilon', \epsilon, a', a, \boldsymbol{k})$$

$$\equiv\langle \epsilon'\, a'|\rho_0 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}(2\boldsymbol{p}-\hbar\boldsymbol{k})|\epsilon, a\rangle - \langle \epsilon', a'|\rho_3 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{\sigma}\times\hbar\boldsymbol{k})|\epsilon, a\rangle$$

$$-\langle \epsilon', a'|i\rho_2 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}[\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot(2\boldsymbol{p}-\hbar\boldsymbol{k})) + (\boldsymbol{\sigma}\cdot(2\boldsymbol{p}-\hbar\boldsymbol{k}))\boldsymbol{\sigma}]|\epsilon, a\rangle \quad (4\cdot33)$$

and

$$\xi(\epsilon, a, \boldsymbol{k}, \boldsymbol{k}')$$

$$\equiv\langle \epsilon, a|i\rho_2 e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}}(\boldsymbol{\sigma}\sigma^j + \sigma^j\boldsymbol{\sigma})|\epsilon, a\rangle A_1{}^j + \langle \epsilon, a|\rho_0 e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}}|\epsilon, a\rangle \boldsymbol{A}_1. \quad (4\cdot34)$$

D. *Polarization tensor*

The components of the polarization tensor are readily obtained by writing the current and charge densities in terms of the vector and scalar potentials.

With the definitions

$$F_+(a)\equiv N_+(a) \text{ and } F_-(a)\equiv 1+N_-(a), \quad (4\cdot35)$$

we find that the components of the polarization tensor are

$$\Pi^0{}_0(\boldsymbol{k}, \boldsymbol{k}', \omega)=\frac{e^2c}{V}\sum_{\epsilon,\epsilon'}\sum_{a,a'}\left\{\frac{\epsilon F_{\epsilon'}(a')-\epsilon' F_\epsilon(a)}{\hbar\omega-(\epsilon E(a)-\epsilon' E(a'))+i\eta}\right.$$

$$\left.\times\langle \epsilon', a'|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}|\epsilon, a\rangle\langle \epsilon, a|e^{i\boldsymbol{k}'\cdot\boldsymbol{r}}|\epsilon', a'\rangle\right\}, \quad (4\cdot36)$$

$$\Pi^0{}_j(\boldsymbol{k}, \boldsymbol{k}', \omega)=-\frac{e^2}{mV}\sum_{\epsilon,\epsilon'}\sum_{a,a'}\left\{\frac{\epsilon F_{\epsilon'}(a')-\epsilon' F_\epsilon(a)}{\hbar\omega-(\epsilon E(a)-\epsilon' E(a'))+i\eta}\right.$$

$$\times\langle \epsilon', a'|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}|\epsilon, a\rangle\Big[\langle \epsilon, a|e^{i\boldsymbol{k}'\cdot\boldsymbol{r}}\Big(p_j+\frac{1}{2}\hbar k_j'\Big)|\epsilon', a'\rangle$$

$$+\frac{i}{2}\langle\epsilon,a|\rho_0 e^{ik'\cdot r}\varepsilon_{uvj}\sigma_u\hbar k_v|\epsilon',a'\rangle$$

$$-\langle\epsilon,a|i\rho_2 e^{ik'\cdot r}[\sigma_j(\boldsymbol{\sigma}\cdot\boldsymbol{p})+(\boldsymbol{\sigma}\cdot(\boldsymbol{p}+\hbar\boldsymbol{k}))\sigma_j]|\epsilon',a'\rangle\Big]\Big\}, \qquad (4\cdot37)$$

$$\Pi^i{}_0(\boldsymbol{k},\boldsymbol{k}',\omega)=\frac{e^2}{mV}\sum_{\epsilon,\epsilon'}\sum_{a,a'}\Big\{\frac{\epsilon F_{\epsilon'}(a')-\epsilon' F_\epsilon(a)}{\hbar\omega-(\epsilon E(a)-\epsilon' E(a'))+i\eta}$$

$$\times\Big[\langle\epsilon',a'|\rho_0 e^{-ik\cdot r}\Big(p^i-\frac{1}{2}\hbar k^i\Big)|\epsilon,a\rangle$$

$$-\frac{i}{2}\langle\epsilon',a'|\rho_3 e^{-ik\cdot r}\varepsilon_{mn}{}^i\sigma_m\hbar k_n|\epsilon,a\rangle$$

$$-\langle\epsilon',a'|i\rho_2 e^{-ik\cdot r}\Big[\sigma^i\Big(\boldsymbol{\sigma}\cdot\Big(\boldsymbol{p}-\frac{1}{2}\hbar\boldsymbol{k}\Big)\Big)+\Big(\boldsymbol{\sigma}\cdot\Big(\boldsymbol{p}-\frac{1}{2}\hbar\boldsymbol{k}\Big)\Big)\sigma^i\Big]|\epsilon,a\rangle\Big]$$

$$\times\langle\epsilon,a|e^{ik'\cdot r}|\epsilon',a'\rangle\Big\}, \qquad (4\cdot38)$$

$$\Pi^i{}_j(\boldsymbol{k},\boldsymbol{k}',\omega)=\frac{-e^2}{m^2cV}\sum_{\epsilon,\epsilon'}\sum_{a,a'}\Big\{\frac{\epsilon F_{\epsilon'}(a')-\epsilon' F_\epsilon(a)}{\hbar\omega-(\epsilon E(a)-\epsilon' E(a'))+i\eta}$$

$$\times\Big[\langle\epsilon',a'|\rho_0 e^{-ik\cdot r}\Big(p^i-\frac{1}{2}\hbar k^i\Big)|\epsilon,a\rangle$$

$$-\frac{i}{2}\langle\epsilon',a'|\rho_3 e^{-ik\cdot r}\varepsilon_{mn}{}^i\sigma_m\hbar k_n|\epsilon,a\rangle$$

$$-\langle\epsilon',a'|i\rho_2 e^{-ik\cdot r}\Big[\sigma^i\Big(\boldsymbol{\sigma}\cdot\Big(\boldsymbol{p}-\frac{1}{2}\hbar\boldsymbol{k}\Big)\Big)+\Big(\boldsymbol{\sigma}\cdot\Big(\boldsymbol{p}-\frac{1}{2}\hbar\boldsymbol{k}\Big)\Big)\sigma^i\Big]|\epsilon,a\rangle\Big]$$

$$\times\Big[\langle\epsilon,a|\rho_0 e^{ik'\cdot r}\Big(p_j+\frac{1}{2}\hbar k'_j\Big)|\epsilon',a'\rangle$$

$$+\frac{i}{2}\langle\epsilon,a|\rho_0 e^{ik'\cdot r}\varepsilon_{uvj}\sigma_u\hbar k_v|\epsilon',a'\rangle$$

$$-\langle\epsilon,a|i\rho_2 e^{ik'\cdot r}[\sigma_j(\boldsymbol{\sigma}\cdot\boldsymbol{p})+(\boldsymbol{\sigma}\cdot(\boldsymbol{p}+\hbar\boldsymbol{k}'))\sigma_j]|\epsilon',a'\rangle\Big]\Big\}$$

$$+\frac{e^2}{mcV}\sum_\epsilon\sum_a F_\epsilon(a)[\langle\epsilon,a|i\rho_2 e^{i(k-k')\cdot r}(\sigma^i\sigma_j+\sigma_j\sigma^i)|\epsilon,a\rangle$$

$$-\delta^i{}_j\langle\epsilon,a|\rho_0 e^{i(k-k')\cdot r}|\epsilon,a\rangle]. \qquad (4\cdot39)$$

We note that the polarization tensor has been derived in a manifestly gauge invariant way, whereas WKH[4] require some manipulations in the Coulomb gauge to render their polarization tensor for the spin-zero pair plasma gauge invariant. Indeed, a cursory application of the technique suggested in Ref. 4), § 3D shows, as required, that the polarization tensor above is gauge invariant, and current conserving. That is,

$$\Pi^\mu{}_\nu k'^\nu=0 \qquad (4\cdot40)$$

and $\quad k_\mu \Pi^\mu{}_\nu = 0\,.$ (4·41)

§ 5. Evaluation of the polarization tensor

Having obtained a formal result for the zero-field polarization tensor, we now proceed to evaluate it at zero temperature.

A. *Matrix elements and spin sums*

The procedure for the calculation of the matrix elements in (4·36)~(4·39) is conceptually straightforward, and very similar to analogous calculations in Dirac spin-one-half physics. In the zero field case, the distribution functions appearing in the polarization tensor depend upon the momenta p and p', but not upon the spin projections s and s'. Thus, we may sum the matrix element products appearing under the sums in the polarization tensor over the spin projections s and s' directly. Also, the evaluation of the spatial integrals in the matrix element products produces the Krönecker delta product:

$$\delta_{\epsilon \boldsymbol{p}, \epsilon' \boldsymbol{p}' + \hbar \boldsymbol{k}'} \delta_{\epsilon \boldsymbol{p}, \epsilon' \boldsymbol{p}' + \hbar \boldsymbol{k}} \tag{5·1}$$

which implies $\boldsymbol{k} = \boldsymbol{k}'$, as we expect for a homogeneous plasma.

Consider, as an example, the following spin sum:

$$\begin{aligned}
&\sum_{s,s'} \langle \epsilon', a' | \rho_0 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \left(\boldsymbol{p} - \frac{1}{2}\hbar\boldsymbol{k}\right) | \epsilon, a \rangle \langle \epsilon, a | \rho_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \left(\boldsymbol{p} + \frac{1}{2}\hbar\boldsymbol{k}\right) | \epsilon', a' \rangle \\
&= \frac{1}{16mc^2 E(p) E(p')} \delta_{\epsilon \boldsymbol{p}, \epsilon' \boldsymbol{p}' + \hbar \boldsymbol{k}} \left(\epsilon \boldsymbol{p} - \frac{1}{2}\hbar\boldsymbol{k}\right) \left(\epsilon' \boldsymbol{p}' + \frac{1}{2}\hbar\boldsymbol{k}\right) \\
&\times \sum_{s,s'} \left\{ \left(\chi_{s'}^\dagger (\epsilon' E(p') + mc^2) \;\; \chi_{s'}^\dagger \frac{2(\boldsymbol{p}'\cdot\boldsymbol{\sigma})^2 c^2 - \boldsymbol{p}'^2 c^2}{(\epsilon' E(p') + mc^2)} \right) \begin{pmatrix} I & I \\ I & I \end{pmatrix} \right. \\
&\times \begin{pmatrix} (\epsilon E(p) + mc^2) \chi_s \\ \dfrac{2(\boldsymbol{p}\cdot\boldsymbol{\sigma})^2 c^2 - \boldsymbol{p}^2 c^2}{(\epsilon E(p) + mc^2)} \chi_s \end{pmatrix} \left(\chi_s^\dagger (\epsilon E(p) + mc^2) \;\; \chi_s^\dagger \frac{2(\boldsymbol{p}\cdot\boldsymbol{\sigma})^2 c^2 - \boldsymbol{p}^2 c^2}{(\epsilon E(p) + mc^2)} \right) \\
&\times \left. \begin{pmatrix} I & I \\ I & I \end{pmatrix} \begin{pmatrix} (\epsilon' E(p') + mc^2) \chi_{s'} \\ \dfrac{2(\boldsymbol{p}'\cdot\boldsymbol{\sigma})^2 c^2 - \boldsymbol{p}'^2 c^2}{(\epsilon' E(p') + mc^2)} \chi_{s'} \end{pmatrix} \right\} .
\end{aligned} \tag{5·2}$$

The first step is to calculate the sum on s:

$$\begin{aligned}
&\sum_s \begin{pmatrix} (\epsilon E(p) + mc^2) \chi_s \\ \dfrac{2(\boldsymbol{p}\cdot\boldsymbol{\sigma})^2 c^2 - \boldsymbol{p}^2 c^2}{(\epsilon E(p) + mc^2)} \chi_s \end{pmatrix} \left(\chi_s^\dagger (\epsilon E(p) + mc^2) \;\; \chi_s^\dagger \frac{2(\boldsymbol{p}\cdot\boldsymbol{\sigma})^2 c^2 - \boldsymbol{p}^2 c^2}{(\epsilon E(p) + mc^2)} \right) \\
&= \begin{pmatrix} (\epsilon E(p) + mc^2)^2 & 2(\boldsymbol{p}\cdot\boldsymbol{\sigma})^2 c^2 - \boldsymbol{p}^2 c^2 \\ 2(\boldsymbol{p}\cdot\boldsymbol{\sigma})^2 c^2 - \boldsymbol{p}^2 c^2 & (\epsilon E(p) - mc^2)^2 \end{pmatrix} .
\end{aligned} \tag{5·3}$$

The remaining matrices are multiplied through, and we are finally left with the sum on s':

$$\sum_{s'}(\chi_{s'}{}^{\dagger}0)\begin{pmatrix} W & X \\ Y & Z \end{pmatrix}\begin{pmatrix} \chi_{s'} \\ 0 \end{pmatrix}=\mathrm{Tr}(W)\,, \tag{5·4}$$

where W, X, Y and Z are 3×3 matrices. We then employ the traces of § 3E to evaluate these s' sums.

The matrix element products for $\Pi^i{}_0$ and $\Pi^i{}_j$ are calculated for a zero temperature plasma. It can be seen from (4·9) and (4·10) that $\Pi^i{}_0$ will determine the longitudinal component of the conductivity tensor, and $\Pi^i{}_j$ the transverse component. Evaluation at zero temperature facilitates the task of calculation markedly, as in this case all of the bosons occupy the ground state, so that

$$N_+(p)=\frac{N}{3}\delta_{p,0}\,. \tag{5·5}$$

Furthermore, no antibosons exist in equilibrium with the condensed bosons. That is

$$N_-(p)=0\,. \tag{5·6}$$

We will also comment upon the general features of the longitudinal response function at finite temperature.

B. *Longitudinal response function*

The matrix elements for $\Pi^i{}_0$ (4·38) are calculated generally for all temperatures, and are given below:

$$\begin{aligned}
&\sum_{s,s'}\langle\epsilon', a'|\rho_0 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\left(p^i-\frac{1}{2}\hbar k^i\right)|\epsilon, a\rangle\langle\epsilon, a|e^{i\boldsymbol{k}\cdot\boldsymbol{r}}|\epsilon', a'\rangle \\
&\quad=\frac{1}{4mc^2E(p)E(p')}\left[(2E^2(p)+m^2c^4)\epsilon' E(p')+(2E^2(p')+m^2c^4)\epsilon E(p)\right] \\
&\qquad\times(\epsilon p^i+\epsilon' p'^i)\delta_{\epsilon \boldsymbol{p},\epsilon'\boldsymbol{p}'+\hbar \boldsymbol{k}}\,,
\end{aligned} \tag{5·7}$$

$$\sum_{s,s'}-\frac{i}{2}\langle\epsilon', a'|\rho_3 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\varepsilon_{mn}{}^i\sigma_m\hbar k_n|\epsilon, a\rangle\langle\epsilon, a|e^{i\boldsymbol{k}\cdot\boldsymbol{r}}|\epsilon', a'\rangle=0\,, \tag{5·8}$$

$$\begin{aligned}
&\sum_{s,s'}\Bigl\{-\langle\epsilon', a'|i\rho_2 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\Bigl[\sigma^i\Bigl(\boldsymbol{\sigma}\cdot\Bigl(\boldsymbol{p}-\frac{1}{2}\hbar\boldsymbol{k}\Bigr)\Bigr)+\Bigl(\boldsymbol{\sigma}\cdot\Bigl(\boldsymbol{p}-\frac{1}{2}\hbar\boldsymbol{k}\Bigr)\Bigr)\sigma^i\Bigr]|\epsilon, a\rangle \\
&\qquad\times\langle\epsilon, a|e^{i\boldsymbol{k}\cdot\boldsymbol{r}}|\epsilon', a'\rangle\Bigr\} \\
&\quad=-\frac{1}{2mc^2E(p)E(p')}\left[(E^2(p)-m^2c^4)\epsilon' E(p')\epsilon p^i+(E^2(p')-m^2c^4)\epsilon E(p)\epsilon' p'^i\right. \\
&\qquad\left.+(\epsilon\boldsymbol{p}\cdot\epsilon'\boldsymbol{p}'c^2)(\epsilon' E(p')\epsilon p^i+\epsilon E(p)\epsilon' p'^i)\right]\delta_{\epsilon \boldsymbol{p},\epsilon'\boldsymbol{p}'+\hbar \boldsymbol{k}}\,.
\end{aligned} \tag{5·9}$$

This yields for the longitudinal response function (4·9) at all temperatures:

$$\epsilon_L = 1 - \frac{4\pi e^2}{\omega m V} \sum_{p,p'} \sum_{\epsilon,\epsilon'} \left\{ \frac{\epsilon F_{\epsilon'}(p') - \epsilon' F_\epsilon(p)}{\hbar\omega - (\epsilon E(p) - \epsilon' E(p')) + i\eta} \right.$$

$$\times \frac{1}{2mc^2 E(p) E(p')} \left(\left[\left(\frac{3}{2} m^2 c^4 + (\epsilon \boldsymbol{p} \cdot \epsilon' \boldsymbol{p}') c^2 \right) \epsilon' E(p') \right. \right.$$

$$\left. + \frac{1}{2}(2E^2(p') + m^2 c^4) \epsilon E(p) \right] \frac{\epsilon \boldsymbol{p} \cdot \boldsymbol{k}}{k^2}$$

$$+ \left[\left(\frac{3}{2} m^2 c^4 + (\epsilon \boldsymbol{p} \cdot \epsilon' \boldsymbol{p}') c^2 \right) \epsilon E(p) \right.$$

$$\left. \left. \left. + \frac{1}{2}(2E^2(p) + m^2 c^4) \epsilon' E(p') \right] \frac{\epsilon' \boldsymbol{p}' \cdot \boldsymbol{k}}{k^2} \right) \delta_{\epsilon p, \epsilon' p' + \hbar k} \right\}. \qquad (5\cdot10)$$

At zero temperature, we employ the distribution functions (5·5) and (5·6) to obtain for the longitudinal response function:

$$\epsilon_L(T=0) = 1 - \frac{\hbar^2 \omega_p{}^2}{2E(\hbar k)} \left(\frac{(E(\hbar k) + E(0))}{\hbar^2 \omega^2 - (E(\hbar k) - E(0))^2} + \frac{(E(\hbar k) - E(0))}{\hbar^2 \omega^2 - (E(\hbar k) + E(0))^2} \right), \qquad (5\cdot11)$$

where

$$\omega_p = \left(\frac{4\pi N e^2}{mV} \right)^{1/2}$$

is the plasma frequency.

This is precisely the longitudinal response function found by KFH[3] and WKH[4] for the spin-zero pair plasma at zero temperature. We note that, at zero temperature, the coupling of the spin-one boson momentum to the intrinsic spin vanishes, as all of the bosons are in the ground state. Thus, the only true distinction between spin-one and spin-zero bosons is that the former couple to the perturbing magnetic field via their intrinsic spin. The result (5·11) indicates clearly that, at zero temperature, this coupling does not manifest itself in the longitudinal response of the system. The result is not surprising, as in the longitudinal case we are investigating the perturbations due to the first order *scalar* potential Φ_1. When all of the bosons are in the ground state, this must correspond purely to an electric field perturbation, and hence no effect due to the spin-magnetic field coupling of spin-one bosons manifests itself in this case.

Whilst we do not propose to explicitly evaluate the dielectric function for a finite temperature plasma, it is possible to make at least some qualitative remarks regarding the general expression for the dielectric function (5·10). First, it differs markedly from the result of KFH[3] and WFH[4] for the spin-zero plasma, which is

$$\varepsilon_{L(\text{spin zero})} = 1 - \frac{4\pi e^2}{\omega m V} \sum_{p,p'} \sum_{\epsilon,\epsilon'} \left\{ \frac{\epsilon F_{\epsilon'}(p') - \epsilon' F_\epsilon(p)}{\hbar\omega - (\epsilon E(p) - \epsilon' E(p')) + i\eta} \right.$$

$$\left. \times \frac{mc^2}{4E(p)E(p')} (\epsilon' E(p') + \epsilon E(p)) (\epsilon \boldsymbol{p} + \epsilon' \boldsymbol{p}') \cdot \frac{\boldsymbol{k}}{k^2} \delta_{\epsilon p, \epsilon' p' + \hbar k} \right\}. \qquad (5\cdot12)$$

We note, referring to the matrix elements (5·7)~(5·9), that the difference arises from the coupling of spin to momentum in the spin-one case. It is by no means clear that (5·10) reduces to (5·12) upon direct evaluation, and one concludes, *a priori,* that the longitudinal response of the spin-one pair plasma at finite temperature differs from that of the spin-zero pair plasma, the two agreeing only at zero temperature.

C. *Transverse response function*

For the transverse response function given by (4·10), we calculate the spin sums on the matrix elements for $\Pi^i{}_j$ (4·39) at zero temperature, and find that only five of these are non-zero upon contraction with the tensor $\overleftrightarrow{I}-(1/k^2)\boldsymbol{kk}$. These are

$$\sum_{s,s'}\frac{1}{4}\langle\epsilon', a'|\rho_3 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\varepsilon_{mni}\sigma_m\hbar k_n|\epsilon, a\rangle\langle\epsilon, a|\rho_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\varepsilon_{uvj}\sigma_u\hbar k_v|\epsilon', a'\rangle$$

$$=\frac{1}{8E(\hbar k)E(0)}(\epsilon E(\hbar k)+\epsilon' E(0))^2(\hbar^2k^2\delta_{ij}-\hbar k_i\hbar k_j)\,, \tag{5·13}$$

$$\sum_{s,s'}\left\{\frac{i}{2}\langle\epsilon', a'|\rho_3 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\varepsilon_{mni}\sigma_m\hbar k_n|\epsilon, a\rangle\right.$$

$$\left.\times\langle\epsilon, a|i\rho_2 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}[\sigma_j(\boldsymbol{\sigma}\cdot\boldsymbol{p})+(\boldsymbol{\sigma}\cdot(\boldsymbol{p}+\hbar\boldsymbol{k}))\sigma_j]|\epsilon', a'\rangle\right\}$$

$$=\sum_{s,s'}\left\{-\frac{i}{2}\langle\epsilon', a'|i\rho_2 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\left[\sigma_i\left(\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{1}{2}\hbar\boldsymbol{k}\right)\right)+\left(\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{1}{2}\hbar\boldsymbol{k}\right)\right)\sigma_i\right]|\epsilon, a\rangle\right.$$

$$\left.\times\langle\epsilon, a|\rho_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\varepsilon_{uvj}\sigma_u\hbar k_v|\epsilon', a'\rangle\right\}$$

$$=-\frac{1}{8E(\hbar k)E(0)}(E^2(\hbar k)-E^2(0))(\hbar^2k^2\delta_{ij}-\hbar k_i\hbar k_j)\,, \tag{5·14}$$

$$\sum_{s,s'}\left\{\langle\epsilon', a'|i\rho_2 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\left[\sigma_i\left(\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{1}{2}\hbar\boldsymbol{k}\right)\right)+\left(\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{1}{2}\hbar\boldsymbol{k}\right)\right)\sigma_i\right]|\epsilon, a\rangle\right.$$

$$\left.\times\langle\epsilon, a|i\rho_2 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}[\sigma_j(\boldsymbol{\sigma}\cdot\boldsymbol{p})+(\boldsymbol{\sigma}\cdot(\boldsymbol{p}+\hbar\boldsymbol{k}))\sigma_j]|\epsilon', a'\rangle\right\}$$

$$=\frac{1}{8E(\hbar k)E(0)}(\epsilon E(\hbar k)-\epsilon' E(0))^2(\hbar^2k^2\delta_{ij}-\hbar k_i\hbar k_j)\,, \tag{5·15}$$

$$\sum_{s}\langle\epsilon, a|\rho_0|\epsilon, a\rangle=3\,. \tag{5·16}$$

These combine to form Π_{ij} at zero temperature, and hence via (4·10) the zero temperature transverse dielectric response function:

$$\varepsilon_T=1-\frac{\omega_p{}^2}{\omega^2}\left\{1+\frac{1}{3}\frac{\hbar^2k^2c^2}{E(\hbar k)}\left(\frac{(E(\hbar k)+E(0))}{\hbar^2\omega^2-(E(\hbar k)+E(0))^2}+\frac{(E(\hbar k)-E(0))}{\hbar^2\omega^2-(E(\hbar k)-E(0))^2}\right)\right\}. \tag{5·17}$$

This expression may be simplified to

$$\varepsilon_T=1-\frac{\omega_p{}^2}{\omega^2}\left(1+\frac{2}{3}\frac{k^2(\omega^2-k^2)}{(\omega^2-k^2)^2-4m^2\omega^2}\right), \tag{5·18}$$

where we have put $\hbar=c=1$.

In (5·18), we have a new result which is of great interest. First, it must be noted that whilst the longitudinal response function (5·11) which we obtain agrees with that of Williams and Melrose [Ref. 5), Eq. (41)], the transverse response function (5·18) differs from the corresponding expression in their paper [Ref. 5), Eq. (42)], by a factor of $-2k^2/(k^2-\omega^2)$ in the final term. It is difficult to establish whether the cause for the discrepancy arises from the manipulation of the formalism employed by Williams and Melrose, or in their explicit evaluation of the response function at zero temperature. Certainly, it must be stated that the two results should agree, and that in the RPA equations of motion technique which we have employed in this paper, the origin of each matrix element product appearing in the polarization tensor is clear.

In the case of the transverse response function, the matrix elements appearing at zero temperature (5·13) to (5·16) are those which arise due to the coupling of the particle spin to the perturbing electromagnetic field. These form the final term in (5·18):

$$\frac{2}{3}\frac{\omega_p{}^2}{\omega^2}\frac{k^2(\omega^2-k^2)}{(\omega^2-k^2)^2-4m^2\omega^2}\cdot \tag{5·19}$$

The factor of 2/3 in this term strongly suggests, that if we consider the transverse modes of oscillation as the response of the plasma to electromagnetic waves impinging upon an ensemble of spin states, then only two-thirds of these ($s=1$ and $s=-1$) will couple to the (perturbative) magnetic field component of the wave. Now, (5·19) takes into account pair creation caused by the collective transverse plasma oscillations. One may postulate that the correlation between the pair-creation particles' charge and spin projection, and the coupling of the latter to the perturbative magnetic field and the former to the perturbative electric field (which induces the conventional transverse collective mode of oscillation) indicates that we have a phenomenon in the spin-one pair plasma that did not exist in the spin-zero case—the influence of single particle effects, via a magnetic field coupling, upon the transverse collective mode of oscillation. Indeed, for the spin-zero plasma, which is only distinguishable from the spin-one plasma at zero temperature by the lack of spin-field coupling, WKH[4] find the standard cold plasma result for the transverse response function:

$$\varepsilon_T=1-\frac{\omega_p{}^2}{\omega^2}. \tag{5·20}$$

Note that for $k^2=0$ (that is, infinite wavelength electromagnetic radiation), $\omega^2=k^2$ (the vacuum dispersion relation, equivalent to the plasma not existing at all), or $\omega_p{}^2=0$ (also equivalent to the "removal" of the plasma by setting $e=0$ or $N/V\equiv n=0$), (5·19) vanishes. Furthermore, if our interpretation of (5·18) is correct, then we expect the pair-creation effects to vanish in a non-relativistic limit, whence $\omega_p{}^2\ll 1$, $k^2\ll 1$ and $m^2\gg 1$. A cursory inspection of (5·18) shows this to be so, and we obtain for ε_T, in this limit, (5·20), the cold plasma result.

D. *Mode analysis*

Both KFH[3] and WKH[4] discuss in detail the longitudinal modes of oscillation for the spin-zero pair plasma, which we have shown to be those of the spin-one pair plasma as well. However, as we shall see from our discussion of the transverse modes, this analysis requires some revision following the manifestation of some subtleties which have been previously neglected.

The transverse response function (5·18) yields the transverse modes of oscillation via (4·12):

$$\frac{k^2}{\omega^2}=1-\frac{\omega_p{}^2}{\omega^2}\left(1+\frac{2}{3}\frac{k^2(\omega^2-k^2)}{(\omega^2-k^2)^2-4m^2\omega^2}\right). \tag{5·21}$$

It would be tempting to multiply both sides of (5·21) by $\omega^2((\omega^2-k^2)^2-4m^2\omega^2)$ to obtain a cubic in ω^2, and then solve for the functional dependence of ω^2 upon k^2. This indeed is the method pursued by Williams and Melrose, and they obtain three modes of oscillation for the plasma [Ref. 5), Eq. (45) to (47)]. Two of these are clearly spurious, as they do not yield the vacuum dispersion relation for electromagnetic waves when ω_p is set to zero. Apart from the fact that their transverse response function is in disagreement with our own, Williams and Melrose have incorrectly analysed the mode equation. This is compounded by the fact that they label one of their modes "roton-like". To give such an interpretation to a transverse mode of oscillation of a plasma is fallacious. They have also done this for the transverse modes of oscillation of the spin-zero pair plasma in the same paper [Ref. 5), Eq. (32)]. Whilst in this case they recover the correct transverse response function at zero temperature, which yields the cold plasma modes of oscillation ($\omega^2=\omega_p{}^2+k^2$), they invent a limiting procedure where they find the "dispersion relation where complete degeneracy is approached", and find two modes of oscillation (the "pair branch" and "roton-like" modes [Ref. 5), Eqs. (31) and (32)] which are clearly spurious, as again they do not yield the vacuum dispersion relation for electromagnetic radiation when ω_p is set to zero. Their employment of integrals over momentum states at zero temperature, the boson ground state distribution function as a Dirac delta function, and this questionable limiting procedure, yield these spurious transverse modes for both the spin-zero and spin-one pair plasma. The correct manner for dealing with the boson ground state is to leave the momentum sums as they appear, and employ for the distribution functions Krönecker delta functions on the ground state, as we have done (5·5). In the light of this, no "limit to complete degeneracy" is evident. Thus, no "pair branch" or "roton-like" modes appear.

Referring to the mode equation (5·21), one sees that by multiplying both sides by $((\omega^2-k^2)^2-4m^2\omega^2)$ would, in the limit that $\omega_p\to 0$, predict *three* modes of oscillation for electromagnetic radiation in a vacuum, these being

$$\omega^2=k^2 \tag{5·22}$$

and

$$\omega^2=2m^2\pm 2m(m^2+k^2)^{1/2}+k^2\,. \tag{5·23}$$

The two modes (5·23) indicate that we have introduced a situation where pair creation no longer occurs solely through the collective mode of oscillation of the plasma, and indeed persists when the plasma is removed. The dispersion relations

(5·23) can in no way represent electromagnetic radiation *in vacuo,* and hence are clearly spurious. Physically, the final term of (5·19) should influence a collective mode, not establish itself as a mode of oscillation of the plasma. Again, these spurious modes are introduced through careless algebraic manipulation of the mode equation.

Thus, the correct procedure for finding the modes of oscillation is to leave (5·21) in its present form, and to iterate for small values of $k^2 \ll 1$, in order to find small order expansions for the single collective mode, for various values of the scaling variables ω_p and m. Following this procedure, we obtain for the transverse mode of oscillation:

$$\frac{{\omega_p}^2}{4m^2}\to\infty: \quad \omega^2={\omega_p}^2+\frac{5}{3}k^2+O(k^4)+\cdots, \tag{5·24}$$

$$1<\frac{{\omega_p}^2}{4m^2}<\infty: \quad \omega^2={\omega_p}^2+\left[1+\frac{2}{3}\left(\frac{{\omega_p}^2}{{\omega_p}^2-4m^2}\right)\right]k^2+O(k^4)+\cdots, \tag{5·25}$$

$$\frac{{\omega_p}^2}{4m^2}=1: \quad \omega^2={\omega_p}^2+\left(\frac{2}{3}\right)^{\frac{1}{2}}\omega_p|\boldsymbol{k}|+\frac{3}{2}k^2+O(|\boldsymbol{k}|^3)+\cdots, \tag{5·26}$$

$$\frac{3}{5}<\frac{{\omega_p}^2}{4m^2}<1: \quad \omega^2={\omega_p}^2+\left[1-\frac{2}{3}\left(\frac{{\omega_p}^2}{4m^2-{\omega_p}^2}\right)\right]k^2+O(k^4)+\cdots, \tag{5·27}$$

$$\frac{{\omega_p}^2}{4m^2}=\frac{3}{5}: \quad \omega^2={\omega_p}^2+\frac{4k^4}{{\omega_p}^2}+O(k^6)+\cdots, \tag{5·28}$$

$$0<\frac{{\omega_p}^2}{4m^2}<\frac{3}{5}: \quad \omega^2={\omega_p}^2+\left[1-\frac{2}{3}\left(\frac{{\omega_p}^2}{4m^2-{\omega_p}^2}\right)\right]k^2+O(k^4)+\cdots, \tag{5·29}$$

$$\frac{{\omega_p}^2}{4m^2}\to 0: \quad \omega^2={\omega_p}^2+k^2\,. \tag{5·30}$$

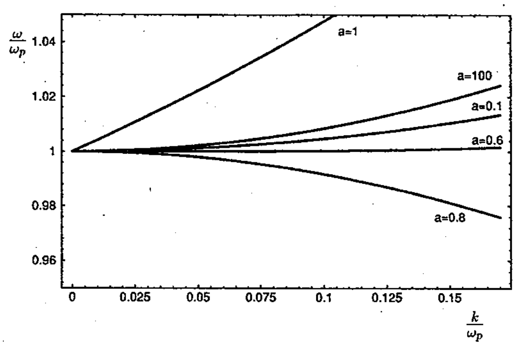

Fig. 1. The transverse mode of oscillation, given by Eqs. (5·24) to (5·30), is shown above with ω/ω_p as a function of k/ω_p, for different values of the scale parameter $a={\omega_p}^2/4m^2$. The curves are plotted for $k/\omega_p \ll 1$, as the validity of Eqs. (5·24) to (5·30) is limited to this asymptotic region. Note in particular the near linear behaviour for $a=1$, and the negative dispersion region demonstrated by the curve for $a=0.8$.

We have in Eqs. (5·24) to (5·30) a rich harvest of results for the various regions delineated by the value of ω_p relative to m. We note that for all of these regions, the cutoff frequency for the modes of oscillation is the plasma frequency, as one would expect in the case of zero field and zero temperature. The dispersion relation (5·24) is that of the ultra-relativistic spin-one pair plasma, and we see that in this limit, the pair creation term (5·19) is contributing fully (to order k^2) to the collective behaviour of the system. In the region of (5·25), the pair particle contribution to the k^2 term in the dispersion relation is weighted by the relative magnitude of the particle mass to the plasma frequency. The dispersion relation when $\omega_p^2/4m^2=1$ (5·26) is most interesting. First, it is here that the dispersion relation changes order, from being one where $\omega^2 \propto k^2$ to leading order, to one where the leading term in the expansion is of order $|\boldsymbol{k}|$. This must be essentially a resonance effect, as now the collective oscillation of the plasma is precisely tuned to the creation of pairs, altering quite dramatically the dispersion relation. Again, in the region of (5·27), the dispersion relation returns to being of order k^2. This region is also of great interest, as the group velocity of the electromagnetic wave now points in the opposite direction to the wave vector, as the coefficient of k^2 is always *negative*. Such a phenomenon is also apparent in the longitudinal modes of oscillation, as discussed by KFH[3] and WKH,[4] and they refer to this phenomenon as "negative dispersion". In the transverse case, this would correspond to the impinging electromagnetic wave being reflected by the plasma, a fascinating result. Equation (5·28) indicates that again, at $\omega_p^2/4m^2=(3/5)$, the dispersion relation changes order, with k^4 now becoming the leading term in the expansion. It appears, at least to order k^2, that the pair creation effects are countervailing the collective behaviour of the system. In the region of (5·29), the dispersion relation returns to that of (5·25). Finally, (5·30) is the exact expression in the non-relativistic limit, and here we obtain the standard cold plasma dispersion relation, as expected.

We re-iterate that the results of (5·24) to (5·30) differ significantly from the cold plasma result obtained by WKH[4] for the spin-zero pair plasma, which in that case applies from the non-relativistic through to the ultra-relativistic limits, and it is only in the non-relativistic limit that a spin-one and spin-zero boson plasma have the same transverse mode of oscillation.

This now leads to a re-examination of the longitudinal modes of oscillation, obtained from the substitution of (5·11) into (4·12). These modes were first analysed by KFH[3] and again by WKH[4] in their respective studies of the spin-zero pair plasma. The mode equation is

$$\frac{\omega_p^2(\omega^2-k^2-4m^2)}{(\omega^2-k^2)^2-4m^2\omega^2}=1\,. \tag{5·31}$$

One sees immediately that multiplying up by the pair denominator will introduce similar spurious modes into the dispersion relation as would occur if the transverse mode equation were manipulated into a cubic. However, as KFH[3] show, the quartic obtained in the longitudinal case factors into two quadratics in ω. However, this procedure is only valid if one recognizes the spurious mode (the "pair branch", as it is labelled in Refs. 3)~5)), in the various regions of solution delineated by the value of the parameter $\omega_p^2/4m^2$, and rejects it. Thus, there is only one longitudinal mode of oscillation at zero temperature, for the spin-zero and spin-one pair plasma. This is

$$\begin{aligned}\frac{\omega_p^2}{4m^2}>1:\ \omega^2&=\frac{1}{2}[4m^2+2k^2+\omega_p^2+((\omega_p^2-4m^2)^2+16m^2k^2)^{1/2}]\\ &=\omega_p^2+\frac{\omega_p^2}{\omega_p^2-4m^2}k^2+O(k^4)+\cdots\,,\end{aligned} \tag{5·32}$$

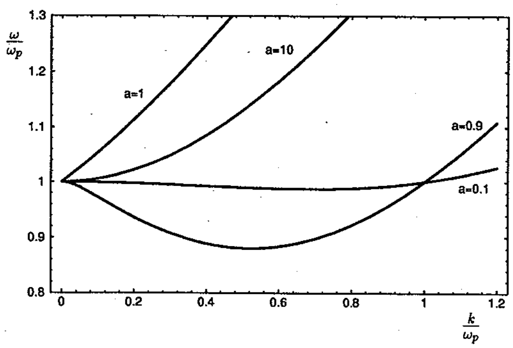

Fig. 2. The longitudinal mode of oscillation, described by Eqs. (5·32) to (5·35), is plotted above. The full form for the dispersion relation, valid for all k, has been used. Again, the curve for $a=1$ is near-linear, and the longitudinal negative dispersion region is shown by the curves for $a=0.9$ and $a=0.1$.

$$\frac{\omega_p^2}{4m^2}=1:\ \omega^2=4m^2+2m|\boldsymbol{k}|+k^2$$
$$=\omega_p^2+\omega_p|\boldsymbol{k}|+k^2, \qquad (5\cdot33)$$
$$\frac{\omega_p^2}{4m^2}<1:\ \omega^2=\frac{1}{2}[4m^2+2k^2+\omega_p^2$$
$$-((4m^2-\omega_p^2)^2+16m^2k^2)^{1/2}]$$
$$=\omega_p^2-\frac{\omega_p^2}{4m^2-\omega_p^2}k^2$$
$$+O(k^4)+\cdots, \qquad (5\cdot34)$$
$$\frac{\omega_p^2}{4m^2}\to0:\ \omega^2=\omega_p^2+\frac{k^4}{4m^2}. \qquad (5\cdot35)$$

Again, the pair creation resonance effect is apparent at $\omega_p^2=4m^2$ (5·33), altering the leading order of the dispersion relation from k^2 to $|\boldsymbol{k}|$. The relation (5·34) is the "negative dispersion" mode discussed by KFH[3] and WKH.[4] Equation (5·35) is the exact result for the longitudinal modes of a non-relativistic spin-zero or spin-one boson plasma at zero temperature. Finally, we note that Kowalenko and Frankel[29] are preparing a similar examination of the modes of oscillation of a spin-zero pair plasma with an external magnetic field present.

§ 6. Conclusion

In this paper, we have given a comprehensive treatment of the linear response theory of the spin-one pair plasma, evaluating a general expression for the polarization tensor in the case of no external fields, and explicitly calculating the longitudinal and transverse modes of oscillation at zero temperature. All of this work required a thorough and detailed study of the Sakata-Taketani equation for relativistic spin-one particles, which we have presented in § 3 of this paper.

The zero temperature results for the modes of oscillation were seen to be, in the longitudinal case, precisely those of the spin-zero pair plasma, which was studied by KFH[3] and WKH.[4] However, we found it necessary to alter the analysis of the modes by these authors, since there exist spurious modes of oscillation which should be discarded, leaving a single longitudinal mode of oscillation. We found, similarly, that there is only one transverse mode of oscillation, which disagrees with the result of Williams and Melrose[5] who find three modes, two of which do not exhibit the appropriate behaviour in the limit of $\omega_p\to0$. This disagreement also extended to their formal result for the transverse response function itself. Furthermore, it was seen that pair creation effects, via the coupling of spin to the perturbative magnetic field, caused the transverse mode of oscillation to differ from the cold plasma result obtained by WKH[4] in their spin-zero plasma work.

The most obvious extension of this work is to a spin-one pair plasma with an external magnetic field present. In this case, we expect that the spin-one plasma results will differ markedly from the corresponding spin-zero results investigated by KFH[3] and WKH,[4] due to the coupling of the particle spin to the external field, as well as to the perturbative magnetic field. This study will require a detailed treatment of the solutions of the Sakata-Taketani equation in the presence of external potentials, followed by the evaluation of the matrix elements comprising the polarization tensor for this magnetized plasma. We look to present this in a future paper.

Furthermore, a thorough renormalization program for the polarization tensor of the vacuum for both the free-field and magnetized case is required, and will also be discussed in a future paper. Another important extension of the work, to which we have alluded in § 5 of this paper, is to a finite temperature plasma. The statistical mechanics of a relativistic (pair) boson system has been given by Haber and Weldon.[30] It would be a most engaging study to incorporate this work into the study of a spin-one plasma, to yield the temperature dependent response functions and concomitant modes of oscillation, particularly about the appropriate Bose-Einstein condensation temperature. This would represent the relativistic generalization of the work of Hore and Frankel[31] for the non-relativistic spin-zero boson plasma. This, at present, is our most optimistic goal.

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