Open access • Journal Article • DOI:10.1007/BF00756990

# Relativistic theory of superpotentials for a nonhomogeneous spatially isotropic medium — Source link

T. R. Hoffend, R. K. Kaul

**Institutions:** University of Minnesota, State University of New York System

**Published on:** 01 Nov 1992 - General Relativity and Gravitation (Kluwer Academic Publishers-Plenum Publishers)

**Topics:** Electromagnetic wave equation, Riemann curvature tensor, Electromagnetic field, Curvature and Isotropy

# RELATIVISTIC THEORY OF SUPERPOTENTIALS FOR A NONHOMOGENEOUS, SPATIALLY ISOTROPIC MEDIUM

By

**T.R. Hoffend Jr.**

and

**R.K. Kaul**

**IMA Preprint Series # 893**

December 1991

# RELATIVISTIC THEORY OF SUPERPOTENTIALS FOR A NONHOMOGENEOUS, SPATIALLY ISOTROPIC MEDIUM

T. R. HOFFEND JR.* AND R. K. KAUL†

**Introduction.** In optics and electrodynamics, situations involving materials with nonhomogeneous electromagnetic material properties often arise. In those situations where relativistic effects are negligible, problem solutions are generally obtained via a theoretical framework that disregards the fact that the structure and properties of the electromagnetic field, constitutive laws, and field equations must be invariant with respect to the choice of coordinates in space-time. In this work, we initiate the idea that new insight can be gained through the development of a covariant theory of electromagnetic wave propagation in a nonhomogeneous medium. It is our hope that utilization of the power of relativistic continuum theories of materials will bring greater understanding of the optics of nonhomogeneous media.

In this paper, we establish the existence of a new superpotential for the electromagnetic field and derive the governing field equation for the superpotential in covariant form. Several significant results are found along the path towards this goal, some of which are simply alternate but heretofore unstated ways of understanding the relativistic theory, and some of which are brand new.

We adopt the axiomatic approach of postulating the general covariant formulation of the Maxwell equations with the appropriate and tensorially consistent constitutive laws for linear media. Natural decomposition of these constitutive laws and the field tensors with respect to the Fermi frames immediately leads to the standard constitutive relations for the proper fields, including the optical rotation tensor. The notion that existence and consistence of the optical rotation tensor is a direct result of natural decomposition of the correct constitutive laws for the field tensors appears to be new.

For a spatially isotropic medium (with repect to the genalized 4-velocity), the constitutive laws can be written in terms of three scalars, the spatial projector, the generalized 4-velocity, and one of the spatial Levi-Civita tensor $\Delta$-densities (see §1). For optically inactive materials, one of the scalars can be dropped. This results in a simple relation, adopted sequentially from §2.

The theory is then developed in terms of the electromagnetic potential and a new gauge condition which is a generalization of the Lorentz gauge condition commonly used

*Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, MN 55455
†Department of Electrical and Computer Engineering, State University of New York at Buffalo, Buffalo, NY 14260.

in relativistic electrodynamics in vacuum. In the limiting case of vacuum, the new gauge condition takes the exact form of the classical Lorentz condition. We put the governing field equation for the electromagnetic potential in a form that reveals the nature and contributional origins of the various terms. If, for example, we consider some, any, or all of the specific cases that the motion of the material is rigid, rotation-free, or acceleration-free, that the medium is completely isotropic, or that the medium is homogeneous, certain terms in the special form of the governing field equation are readily seen to vanish.

The governing field equation for the electromagnetic potential forms a consistent set of coupled partial differential equations whose solutions exist only if the new gauge condition is satisfied simultaneously. To alleviate the problem of admissiblity, we introduce superpotentials which satisfy the new gauge condition identically. With the help of two new tensor identities involving the curvature tensor, we derive the governing field equation for the superpotential in covariant form. The nature of this (hyperbolic) system of partial differential equations is then seen to be quite simple and beautiful.

§1. **Preliminaries.** We carry out our analysis within the framework of the (general) relativistic Eulerian theory of continuous media. In that respect we adhere primarily to the work of Synge [1] and Bressan [2] while for geometric principles, language, and notation we follow the work of Schouten [3] and Flanders [4]. We begin by recalling several definitions, conventions, and formulae important to the ensuing development.

After Schouten [3], a $V_n$ is defined to be an $n$-dimensional geometric manifold endowed with a symmetric linear connection $\Gamma^{\mu}_{\alpha\beta} = 2!\,\Gamma^{\mu}_{(\alpha\beta)}$ and a symmetric covariant constant[1] real fundamental tensor $g_{\alpha\beta}$. An $R_n$ is a $V_n$ in which the curvature tensor vanishes. We consider space-time to be a $V_4$ with an indefinite fundamental tensor, and the associated fundamental quadratic form, or *space-time metric*

$$ds^2 = -g_{\alpha\beta}dx^{\alpha}dx^{\beta} \tag{1.1}$$

has signature $-2$. The space-time $V_4$ is therefore not ordinary.

Henceforth we follow the standard convention that Greek indices run from 1 to 4 and Latin indices run from 1 to 3. We remark that this convention does not apply to fixed labels on indices.

After Synge [1] and Bressan [2], we consider a smoothly moving 3-dimensional continuous body $\Omega$ and exclude the possibility of irregular motions such as sliding, fracture, or collision with another body. In that case we can think of $\Omega$ as a set of material points $\Omega = \cup_{p^*}$ and the union $W_\Omega = \cup W_{p^*}$ of the world lines $W_{p^*}$ of the material points is called the *world tube* occupied by $\Omega$. $W_\Omega$ represents a (4-dimensional) domain in the space-time $V_4$.

[1]A tensor is a *covariant constant* if its covariant derivative is equal to zero.

The *generalized 4-velocity* $u^\alpha$ at the event $x^\alpha$ is defined to be the tangent to the world line $W_p$ parameterized by the arc-length $s$ defined by (1.1). In that case

$$u^\alpha = \frac{dx^\alpha}{ds}. \tag{1.2}$$

The absolute derivative of a differentiable tensor $T^{\alpha_1 \ldots \alpha_p}_{\beta_1, \ldots \beta_q}$ is given by[2]

$$\frac{\delta}{\delta s} T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = u^\gamma \nabla_\gamma T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}, \tag{1.3}$$

where the symbol $\nabla_\gamma$ denotes the covariant derivative. The *intrinsic acceleration* $a^\alpha$ is defined as

$$a^\alpha \stackrel{\text{def}}{=} \frac{\delta u^\alpha}{\delta s}. \tag{1.4}$$

The following trivial formulae arise in the analysis

$$u^\alpha u_\alpha = -1, \tag{1.5}$$

$$u^\alpha \nabla_\beta u_\alpha = 0, \tag{1.6}$$

$$u^\alpha a_\alpha = 0. \tag{1.7}$$

We introduce the usual *spatial projector*

$$\overset{\perp}{g}_{\alpha\beta} \stackrel{\text{def}}{=} g_{\alpha\beta} + u_\alpha u_\beta, \tag{1.8}$$

and write[3]

$$T^{\overset{\perp}{\alpha_1}\alpha_2 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = \overset{\perp}{g}{}^{\alpha_1}{}_{\cdot\,\gamma} T^{\gamma\alpha_2 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}, \tag{1.9a}$$

$$T^{\alpha_1 \ldots \alpha_p}_{\overset{\perp}{\beta_1}\beta_2 \ldots \beta_q} = \overset{\perp}{g}{}_{\beta_1}{}^{\cdot\,\gamma} T^{\alpha_1 \ldots \alpha_p}_{\gamma\beta_2 \ldots \beta_q}, \tag{1.9b}$$

$$\overset{\perp}{T}{}^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = T^{\overset{\perp}{\alpha_1} \ldots \overset{\perp}{\alpha_p}}_{\overset{\perp}{\beta_1} \ldots \overset{\perp}{\beta_q}}. \tag{1.9c}$$

The index $\alpha$ of the tensor $T^{\alpha\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}$ is called *spatial* if

$$u_\alpha T^{\alpha\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \equiv 0, \tag{1.10}$$

---

[2] Synge [1].

[3] We remark that both the metric tensor $g_{\alpha\beta}$ and the spatial projector $\overset{\perp}{g}_{\alpha\beta}$ are symmetric tensors and the use of dots when raising or lowing indices is therefore optional. For example $\overset{\perp}{g}{}_{\alpha}{}^{\cdot\,\beta} = \overset{\perp}{g}{}^{\beta}{}_{\cdot\,\alpha}$ and one can write $\overset{\perp}{g}{}^{\beta}_{\alpha}$ without causing any confusion. We further note that $g^\beta_\alpha = \delta^\beta_\alpha$, where $\delta^\beta_\alpha$ is the Kronecker delta.

and any geometric quantity is called *spatial* whenever all of its indices are spatial. Note that $\overset{\perp}{g}_{\alpha\beta}$ is a spatial tensor and relation (1.10) is equivalent to the statement $T^{\alpha\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = \overset{\perp}{T}{}^{\alpha_1\alpha_2\ldots\alpha_p}_{\beta_1\ldots\beta_q}$.

The *natural decomposition of a tensor* $T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$ *with respect to the index* $\alpha_1$ is defined by

$$(1.11) \qquad T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = T^{\overset{\perp}{\alpha_1}\ldots\alpha_p}_{\beta_1\ldots\beta_q} - u^{\alpha_1}u_\gamma T^{\gamma\alpha_2\ldots\alpha_p}_{\beta_1\ldots\beta_q}.$$

A similar procedure can be applied to repeatedly decompose the tensors on the right hand side of (1.11) until all indices of $T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$ are exhausted. The representation of $T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$ as the sum of the $2^{p+q}$ individual terms thus obtained comprises the *natural decomposition of the tensor* $T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$.

Finally, two important geometric quantities, the *spatial alternating tensor $\Delta$-densities or spatial Levi-Civita tensor $\Delta$-densities*[4] of weight +1 and −1, respectively, are defined by

$$(1.12a) \qquad \overset{\perp}{\tilde{\varepsilon}}{}^{\alpha\beta\gamma} \overset{\text{def}}{=} u_\rho \tilde{\varepsilon}^{\rho\alpha\beta\gamma},$$

$$(1.12b) \qquad \overset{\perp}{\underset{\sim}{\varepsilon}}{}_{\alpha\beta\gamma} \overset{\text{def}}{=} u^\rho \underset{\sim}{\varepsilon}_{\rho\alpha\beta\gamma},$$

where the *Levi-Civita tensor $\Delta$-densities* $\tilde{\varepsilon}^{\alpha\beta\gamma\mu}$ and $\underset{\sim}{\varepsilon}_{\alpha\beta\gamma\mu}$ are alternating tensor $\Delta$-densities of weight +1 and −1, respectively, with $\tilde{\varepsilon}^{1234} = 1 = \underset{\sim}{\varepsilon}_{1234}$. We adopt the convention that a tilde ($\sim$) directly over a kernel denotes $\Delta$-density of weight +1 and a tilde directly under a kernel denotes $\Delta$-density of weight −1.[5] The following important properties[6] arise in the ensuing analysis

$$(1.13) \qquad \overset{\perp}{\tilde{\varepsilon}}{}^{\alpha\beta\gamma\mu} \equiv 0, \qquad \overset{\perp}{\underset{\sim}{\varepsilon}}{}_{\alpha\beta\gamma\mu} \equiv 0,$$

$$(1.14a) \qquad \overset{\perp}{\tilde{\varepsilon}}{}^{\alpha\beta\gamma}\,\overset{\perp}{\underset{\sim}{\varepsilon}}{}_{\sigma\mu\gamma} = \overset{\perp}{g}{}_\sigma^{\cdot\alpha}\overset{\perp}{g}{}_\mu^{\cdot\beta} - \overset{\perp}{g}{}_\sigma^{\cdot\beta}\overset{\perp}{g}{}_\mu^{\cdot\alpha},$$

$$(1.14b) \qquad \overset{\perp}{\tilde{\varepsilon}}{}^{\alpha\beta\gamma}\,\overset{\perp}{\underset{\sim}{\varepsilon}}{}_{\alpha\beta\mu} = 2!\,\overset{\perp}{g}{}_\mu^{\cdot\gamma},$$

$$(1.14c) \qquad \overset{\perp}{\tilde{\varepsilon}}{}^{\alpha\beta\gamma}\,\overset{\perp}{\underset{\sim}{\varepsilon}}{}_{\alpha\beta\gamma} = 3!\ .$$

---

[4] Bressan calls these quantities the spatial Ricci tensor $\Delta$-densities. We refrain from this appelation in order to avoid confusion with the standard nomonclature designating the Ricci tensor as $R_{\lambda\mu} \overset{\text{def}}{=} R_{\beta\lambda\mu}^{\cdot\,\cdot\,\cdot\,\beta}$, where $R_{\alpha\beta\gamma}^{\cdot\,\cdot\,\cdot\,\mu}$ is the curvature tensor of the $V_4$.

[5] Under a coordinate transformation $x^{\kappa'} = x^{\kappa'}(x^\kappa)$, a $\Delta$-density of weight $w$ transforms with a weight factor $\Delta^{-w}$, where $\Delta$ is the Jacobian, $\Delta \overset{\text{def}}{=} \det\left[\partial x^{\kappa'}/\partial x^\kappa\right]$, and the sign of the determinant is retained. For example $\tilde{\varepsilon}^{\alpha'\beta'\gamma'\mu'} = \Delta^{-1}\,\tilde{\varepsilon}^{\alpha\beta\gamma\mu}$. See Schouten [3], p. 12.

[6] Bressan [2], p. 51.

§2. **Maxwell Field Equations and Polarization Tensors.** In terms of the electromagnetic field bivectors $F_{\alpha\beta}$ and $G_{\alpha\beta}$ and the generalized current $\Delta$-density $\tilde{j}^{\alpha}$, the Maxwell equations in the space-time $V_4$ take the form[7]

$$\partial_{[\alpha} F_{\beta\gamma]} = 0, \tag{2.1}$$

$$\partial_{\beta} \widetilde{G}^{\alpha\beta} = \tilde{j}^{\alpha}, \tag{2.2}$$

where the process of *alternation* over the indices is indicated by a pair of square brackets and where $\widetilde{G}^{\alpha\beta}$ denotes the *dual representation* of $G_{\alpha\beta}$ in terms of a contravariant bivector $\Delta$-density of weight $+1$. The tensorial homeomorphism between the two sets of quantities $G_{\alpha\beta}$ and $\widetilde{G}^{\alpha\beta}$ is given by[8]

$$\widetilde{G}^{\alpha\beta} = \frac{1}{2!} \widetilde{\varepsilon}^{\alpha\beta\gamma\mu} G_{\gamma\mu}. \tag{2.3}$$

The *standard dual representation* $*G_{\alpha\beta}$, of $G_{\alpha\beta}$ is defined by

$$*G_{\alpha\beta} \stackrel{\text{def}}{=} \frac{1}{\sqrt{g}} \frac{1}{2!} \widetilde{\varepsilon}_{\dot{\alpha}\dot{\beta}}{}^{\kappa\sigma} G_{\kappa\sigma}, \tag{2.4}$$

where $g$ denotes the determinant of the metric tensor $g_{\alpha\beta}$ and is a scalar $\Delta$-density, and can thus be written

$$*G_{\alpha\beta} = \frac{1}{\sqrt{g}} g_{\alpha\gamma} g_{\beta\mu} \widetilde{G}^{\gamma\mu}. \tag{2.5}$$

Consider the natural decomposition of the field tensor $F_{\alpha\beta}$

$$F_{\alpha\beta} = \overset{\perp}{F}_{\alpha\beta} + 2!\, u_{[\alpha} \underset{(1)}{F}_{\beta]}, \tag{2.6}$$

where

$$\overset{\perp}{F}_{\alpha\beta} = \overset{\perp}{g}_{\dot{\alpha}}{}^{\gamma} \overset{\perp}{g}_{\dot{\beta}}{}^{\mu} F_{\gamma\mu}, \tag{2.7a}$$

$$\underset{(1)}{F}_{\beta} = \overset{\perp}{g}_{\dot{\beta}}{}^{\gamma} F_{\gamma\mu} u^{\mu}. \tag{2.7b}$$

Similarly

$$G_{\alpha\beta} = \overset{\perp}{G}_{\alpha\beta} + 2!\, u_{[\alpha} \underset{(1)}{G}_{\beta]}, \tag{2.8}$$

---

[7] Barut [5], p. 93 ff.

[8] Schouten [3], p. 26.

and

$$\tilde{j}^{\alpha} = \overset{\perp}{\tilde{j}}{}^{\alpha} - u^{\alpha} u_{\gamma} \tilde{j}^{\gamma}. \tag{2.9}$$

Clearly $\overset{\perp}{F}_{\alpha\beta}$ and $\overset{\perp}{G}_{\alpha\beta}$ are spatial bivectors, $\underset{(1)}{F}_{\beta}$ and $\underset{(1)}{G}_{\beta}$ are spatial vectors, $\overset{\perp}{\tilde{j}}{}^{\alpha}$ is a spatial vector $\Delta$-density of weight $+1$ and $u_{\gamma}\tilde{j}^{\gamma}$ is a scalar $\Delta$-density of weight $+1$. We associate with these the geometric quantities

$$\begin{aligned} cB_{\alpha\beta} &\overset{\text{def}}{=} \overset{\perp}{F}_{\alpha\beta}, \quad E_{\beta} \overset{\text{def}}{=} \underset{(1)}{F}_{\beta}, \\ cD_{\alpha\beta} &\overset{\text{def}}{=} \overset{\perp}{G}_{\alpha\beta}, \quad H_{\beta} \overset{\text{def}}{=} -\underset{(1)}{G}_{\beta}, \\ \tilde{J}^{\alpha} &\overset{\text{def}}{=} \overset{\perp}{\tilde{j}}{}^{\alpha}, \qquad c\tilde{\rho} \overset{\text{def}}{=} -u_{\gamma}\tilde{j}^{\gamma}, \end{aligned} \tag{2.10}$$

where $c$ is the speed of light in vacuum. The kernels $B, E, D, H, J$ and $\rho$ denote in MKS units the proper (i.e. measured with respect to an observer moving with the material point $P$) magnetic induction, electric field, electric displacement, magnetic field, true electric current, and true electric charge, respectively.

We remark that there exists a tensorial homeomorphism between *spatially dual* sets of spatial quantities, given by

$$\begin{aligned} E_{\alpha} &= \frac{1}{2!} \underset{\sim}{\overset{\perp}{\varepsilon}}_{\alpha\beta\gamma} \tilde{E}^{\beta\gamma}, \quad \tilde{E}^{\alpha\beta} = \overset{\perp}{\tilde{\varepsilon}}{}^{\alpha\beta\gamma} E_{\gamma}, \\ B_{\alpha\beta} &= \underset{\sim}{\overset{\perp}{\varepsilon}}_{\alpha\beta\gamma} \tilde{B}^{\gamma}, \quad \tilde{B}^{\alpha} = \frac{1}{2!} \overset{\perp}{\tilde{\varepsilon}}{}^{\alpha\beta\gamma} B_{\beta\gamma}, \\ D_{\alpha\beta} &= \overset{\perp}{\tilde{\varepsilon}}_{\alpha\beta\gamma} \tilde{D}^{\gamma}, \quad \tilde{D}^{\alpha} = \frac{1}{2!} \overset{\perp}{\tilde{\varepsilon}}{}^{\alpha\beta\gamma} D_{\beta\gamma}, \\ H_{\alpha} &= \frac{1}{2!} \underset{\sim}{\overset{\perp}{\varepsilon}}_{\alpha\beta\gamma} \tilde{H}^{\beta\gamma}, \quad \tilde{H}^{\alpha\beta} = \overset{\perp}{\tilde{\varepsilon}}{}^{\alpha\beta\gamma} H_{\gamma}, \\ J_{\alpha\beta} &= \underset{\sim}{\overset{\perp}{\varepsilon}}_{\alpha\beta\gamma} \tilde{J}^{\gamma}, \quad \tilde{J}^{\alpha} = \frac{1}{2!} \overset{\perp}{\tilde{\varepsilon}}{}^{\alpha\beta\gamma} J_{\beta\gamma}. \end{aligned} \tag{2.11}$$

We may further note that *integrability* of the Maxwell equations requires the condition

$$\partial_{\alpha} \tilde{j}^{\alpha} = 0, \tag{2.12}$$

which in terms of the proper true current and charge densities comprises a statement of the punctual law of conservation of electric charge.

We introduce the *polarization tensor* $P_{\alpha\beta}$ via the relation

$$z_0 G_{\alpha\beta} = *(F_{\alpha\beta} + P_{\alpha\beta}), \tag{2.13}$$

where $z_0$ is the *intrinsic impedance of the vacuum*. In the absence of matter it follows that

$$z_0 G_{\alpha\beta} = *F_{\alpha\beta} = \frac{1}{2!}\frac{1}{\sqrt{g}}\, g_{\alpha\gamma}\, g_{\beta\mu}\, \widetilde{\varepsilon}^{\gamma\mu\kappa\sigma} F_{\kappa\sigma}. \tag{2.14}$$

Taking the natural decompositions of $\widetilde{\varepsilon}^{\gamma\mu\kappa\sigma}$ and $F_{\kappa\sigma}$ and using (2.10) and (2.11) we find

$$\begin{aligned} \frac{1}{2!}\widetilde{\varepsilon}^{\gamma\mu\kappa\sigma} F_{\kappa\sigma} &= -\left(u^{[\gamma}\overset{\perp}{\widetilde{\varepsilon}}{}^{\mu]\kappa\sigma} - u^{[\kappa}\overset{\perp}{\widetilde{\varepsilon}}{}^{\sigma]\gamma\mu}\right)\left(cB_{\kappa\sigma} + 2!u_{[\kappa}E_{\sigma]}\right) \\ &= \left(\widetilde{E}^{\gamma\mu} - 2!cu^{[\gamma}\widetilde{B}^{\mu]}\right). \end{aligned} \tag{2.15}$$

Substituting this in (2.14) and using (2.8) and (2.9), we obtain the free-space relations

$$z_0 c D_{\alpha\beta} = E_{\alpha\beta}, \tag{2.16}$$

$$z_0 H_\alpha = cB_\alpha. \tag{2.17}$$

We define the *permittivity* $\varepsilon_0$ and *permeability* $\mu_0$ of the vacuum by

$$\varepsilon_0 \overset{\text{def}}{=} \frac{1}{z_0 c}, \quad \mu_0 \overset{\text{def}}{=} \frac{z_0}{c}, \tag{2.18}$$

and thus obtain the familiar relations

$$\mu_0\varepsilon_0 = \frac{1}{c^2}, \quad z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}. \tag{2.19}$$

In the presence of (polarizable) matter, we take the natural decomposition of the polarization bivector $P_{\alpha\beta}$ (compare with 2.6, 2.7a,b)

$$P_{\alpha\beta} = \overset{\perp}{P}_{\alpha\beta} + 2!u_{[\alpha}\underset{(1)}{P}_{\beta]}. \tag{2.20}$$

The tensor $\overset{\perp}{P}_{\alpha\beta}$ is a spatial bivector and $\underset{(1)}{P}_\beta$ is a spatial vector, and we associate these with the geometric quantities

$$M_{\alpha\beta} \overset{\text{def}}{=} -\frac{1}{c}P_{\alpha\beta}, \tag{2.21a}$$

$$\underset{(e)}{P}_\beta \overset{\text{def}}{=} \varepsilon_0 \underset{(1)}{P}_\beta. \tag{2.21b}$$

The kernel $M$ denotes the proper magnetic polarization, or *magnetization* and *the kernel* $\underset{(e)}{P}$ denotes the proper electric polarization.

§3. **Constitutive Relations for a Linear, Memoryless Material.** Even with the generalized velocity $u^\alpha$ prescribed, eqns. (2.1), (2.2) and (2.13) for the field and polarization tensors remain underdetermined. We thus consider the simplest practical example and complete the system of equations with the following linear constitutive relation for a memoryless material with no permanent polarization

$$P_{\alpha\beta} = \chi_{\dot{\alpha}\dot{\beta}}{}^{\gamma\mu} F_{\gamma\mu} \,. \tag{3.1}$$

Equation (2.13) then takes the form

$$z_0 G_{\alpha\beta} = * \left[ Q_{\dot{\alpha}\dot{\beta}}{}^{\gamma\mu} F_{\gamma\mu} \right] , \tag{3.2}$$

where

$$Q_{\dot{\alpha}\dot{\beta}}{}^{\gamma\mu} \overset{\text{def}}{=} g_{\alpha}{}^{\cdot[\gamma} g^{\mu]}_{\cdot\;\beta} + \chi_{\dot{\alpha}\dot{\beta}}{}^{[\gamma\mu]} \,. \tag{3.3}$$

Using (2.4) and the fact that $** \, T_{\alpha\beta} = -T_{\alpha\beta}$ in a $V_4$, eqn. (3.2) can be written as

$$z_0 \widetilde{G}^{\alpha\beta} = -\widetilde{Q}^{\alpha\beta\gamma\mu} F_{\gamma\mu}, \tag{3.4}$$

where

$$\widetilde{Q}^{\alpha\beta\gamma\mu} = \sqrt{g}\, g^{\alpha\kappa} g^{\beta\sigma} Q_{\dot{\kappa}\dot{\sigma}}{}^{\gamma\mu} \,. \tag{3.5}$$

Due to the antisymmetry property of the bivectors $\widetilde{G}^{\alpha\beta}$ and $F_{\alpha\beta}$ it is obvious that the tensor $\Delta$-density $\widetilde{Q}^{\alpha\beta\gamma\mu}$ is antisymmetric with respect to the pairs of indices $\alpha\beta$ and $\gamma\mu$. Furthermore $z_0 \widetilde{G}^{\alpha\beta} F_{\alpha\beta} = -\widetilde{Q}^{\alpha\beta\gamma\mu} F_{\gamma\mu} F_{\alpha\beta} = \widetilde{Q}^{\gamma\mu\alpha\beta} F_{\alpha\beta} F_{\gamma\mu}$ and upon factoring $F_{\alpha\beta}$ we find that (3.4) can be written in the form

$$z_0 \widetilde{G}^{\alpha\beta} = -\widetilde{Q}^{\alpha\beta\gamma\mu} F_{\gamma\mu} = \widetilde{Q}^{\gamma\mu\alpha\beta} F_{\gamma\mu}.$$

From these considerations it follows that the tensor $\Delta$-density $\widetilde{Q}^{\alpha\beta\gamma\mu}$ must have the symmetry properties

$$\widetilde{Q}^{\alpha\beta\gamma\mu} = \widetilde{Q}^{\gamma\mu\alpha\beta} = -\widetilde{Q}^{\mu\gamma\alpha\beta}. \tag{3.6}$$

These symmetry properties can also be derived from Green's thermodynamic principle.

Taking the natural decomposition and using (3.6) the quantity $\widetilde{Q}^{\alpha\beta\gamma\mu}$ can be expressed in the form

$$\widetilde{Q}^{\alpha\beta\gamma\mu} = \overset{\perp}{\widetilde{Q}}{}^{\alpha\beta\gamma\mu} + 2! \left( u^{[\alpha} \underset{(1)}{\widetilde{Q}}{}^{\beta]\gamma\mu} + u^{[\gamma} \underset{(1)}{\widetilde{Q}}{}^{\mu]\alpha\beta} \right) + 4 u^{[\alpha} \underset{(2)}{\widetilde{Q}}{}^{\beta][\gamma} u^{\mu]}, \tag{3.7}$$

where

$$\underset{(1)}{\widetilde{Q}}{}^{\alpha\beta\gamma} \overset{\text{def}}{=} -u_\sigma \widetilde{Q}^{\sigma\overset{\perp}{\alpha}\overset{\perp}{\beta}\overset{\perp}{\gamma}}, \tag{3.8}$$

$$\underset{(2)}{\widetilde{Q}}{}^{\alpha\beta} \overset{\text{def}}{=} -u_\kappa u_\sigma \widetilde{Q}^{\kappa\overset{\perp}{\alpha}\sigma\overset{\perp}{\beta}}. \tag{3.9}$$

Using the decomposition (3.7) and the fact that

$$\widetilde{G}^{\alpha\beta} = \frac{1}{2!}\widetilde{\varepsilon}^{\alpha\beta\gamma\mu}G_{\gamma\mu} = -(\widetilde{H}^{\alpha\beta} + 2!cu^{[\alpha}\widetilde{G}^{\beta]}), \tag{3.10}$$

(cf. 2.10) eqn. (3.4) can be written in terms of the proper fields as

$$z_0\widetilde{H}^{\alpha\beta} = \overset{\perp}{\widetilde{Q}}{}^{\alpha\beta\gamma\mu}(cB_{\gamma\mu}) + 2!u^{[\gamma}\underset{(1)}{\widetilde{Q}}{}^{\mu]\alpha\beta}(2!u_{[\gamma}E_{\mu]}), \tag{3.11a}$$

$$cz_0\widetilde{D}^{\beta} = 2\underset{(2)}{\widetilde{Q}}{}^{\beta[\gamma}u^{\mu]}(2!u_{[\gamma}E_{\mu]}) + \underset{(1)}{\widetilde{Q}}{}^{\beta\gamma\mu}(cB_{\gamma\mu}). \tag{3.11b}$$

A simple form of a complete and consistent relation can be obtained by dropping the cross-coupling terms, i.e. by setting the *optical rotation tensor* $\Delta$*-density*

$$\underset{(1)}{\widetilde{Q}}{}^{\alpha\beta\gamma} \equiv 0. \tag{3.12}$$

In that case, using (2.18), the relations (3.11a,b) take the form

$$\mu_0\widetilde{H}^{\alpha\beta} = \overset{\perp}{\widetilde{Q}}{}^{\alpha\beta\gamma\mu}B_{\gamma\mu}, \tag{3.13a}$$

$$\widetilde{D}^{\beta} = \varepsilon_0(2\underset{(2)}{\widetilde{Q}}{}^{\beta\gamma})E_\gamma. \tag{3.13b}$$

We take the spatial dual of the relation (3.13a) and use (2.10) to get

$$\mu_0 H_\alpha = \frac{1}{2!}\underset{\sim}{\overset{\perp}{\varepsilon}}_{\alpha\beta\gamma}\overset{\perp}{\widetilde{Q}}{}^{\beta\gamma\kappa\sigma}\underset{\sim}{\overset{\perp}{\varepsilon}}_{\kappa\sigma\mu}\widetilde{B}^{\mu} = \underset{(r)}{\underset{\sim}{\nu}}{}_{\alpha\mu}\widetilde{B}^{\mu}, \tag{3.14}$$

where

$$\underset{(r)}{\underset{\sim}{\nu}}{}_{\alpha\mu} \overset{\text{def}}{=} \frac{1}{2!}\underset{\sim}{\overset{\perp}{\varepsilon}}_{\alpha\beta\gamma}\overset{\perp}{\widetilde{Q}}{}^{\beta\gamma\kappa\sigma}\underset{\sim}{\overset{\perp}{\varepsilon}}_{\kappa\sigma\mu}. \tag{3.15}$$

The relation (3.13a) and its spatial dual (3.14) are required to be quasi-invertible in the sense that there exists a symmetric, spatial tensor $\Delta$-density $\underset{(r)}{\widetilde{\mu}}{}^{\alpha\beta}$ of weight $+1$ such that

$$(3.16) \qquad \underset{(r)}{\underset{\sim}{\nu}}{}_{\alpha\beta}\, \underset{(r)}{\widetilde{\mu}}{}^{\beta\gamma} = \overset{\perp}{g}{}_{\alpha}{}^{\cdot\gamma}\,.$$

A second symmetric, spatial tensor $\Delta$-density of weight $+1$ is defined by

$$(3.17) \qquad \underset{(r)}{\widetilde{\varepsilon}}{}^{\alpha\beta} \stackrel{\text{def}}{=} 2\underset{(2)}{\widetilde{Q}}{}^{\alpha\beta}\,.$$

Relations (3.13a,b) can then be written in the form

$$(3.18a) \qquad \widetilde{B}^{\alpha} = \underset{(r)}{\widetilde{\mu}}{}^{\alpha\beta}\mu_0 H_{\beta},$$

$$(3.18b) \qquad \widetilde{D}^{\alpha} = \underset{(r)}{\widetilde{\varepsilon}}{}^{\alpha\beta}\varepsilon_0 E_{\beta}\,.$$

The quantity $\underset{(r)}{\widetilde{\mu}}{}^{\alpha\beta}$ is called the *relative permeability tensor $\Delta$-density* and the quantity $\underset{(r)}{\widetilde{\varepsilon}}{}^{\alpha\beta}$ is called the *relative permittivity tensor $\Delta$-density.*

We remark that we could have postulated (3.18a,b) as the constitutive relations for the proper fields at the outset and worked backwards to (3.1)–(3.4). However the relations (3.1)–(3.4) better facilitate the succeeding analysis and they immediately reveal the possibility of materials exhibiting an optical rotation tensor[9] and the consistency of such a model. In addition, macroscopic theories of more complicated effects and materials almost always require general constitutive relations in terms of the polarization that cannot be reduced to simple linear relations between the proper fields.[10]

A medium characterized by the constitutive law (3.4) is called *(electromagnetically) spatially isotropic* if $\widetilde{Q}^{\alpha\beta\gamma\mu}$ remains invariant for spatial rotations relative to the generalized 4-velocity $u^{\alpha}$. In that case, by considering the natural decomposition (3.7), one can readily show that

$$(3.19a) \qquad \overset{\perp}{\widetilde{Q}}{}^{\alpha\beta\gamma\mu} = \underset{(0)}{\tilde{q}}\,(\overset{\perp}{g}{}^{\alpha\gamma}\overset{\perp}{g}{}^{\beta\mu} - \overset{\perp}{g}{}^{\alpha\mu}\overset{\perp}{g}{}^{\beta\gamma}),$$

$$(3.19b) \qquad \underset{(1)}{\widetilde{Q}}{}^{\alpha\beta\gamma} = \underset{(1)}{q}\,\overset{\perp}{\widetilde{\varepsilon}}{}^{\alpha\beta\gamma},$$

$$(3.19c) \qquad \underset{(2)}{\widetilde{Q}}{}^{\alpha\beta} = \underset{(2)}{\tilde{q}}\,\overset{\perp}{g}{}^{\alpha\beta},$$

[9] Optical rotation tensors are discussed in Azzam and Bashara [6].

[10] Many examples occur in the theory of nonlinear optics and include second-harmonic generation, parametric oscillation and frequency tuning, the electro-optic effect, and the Faraday and Kerr effects (see for example Bloembergen [7], Yariv [8], and Toupin [9]).

where

(3.20a) $$\underset{(0)}{\tilde{q}} = \frac{1}{3!}\overset{\perp}{\tilde{Q}}{}^{\alpha\beta}{}_{\cdot\cdot\alpha\beta},$$

(3.20b) $$\underset{(1)}{q} = \frac{1}{3!}\overset{\perp}{\underset{\sim}{\varepsilon}}_{\alpha\beta\gamma}\underset{(1)}{\tilde{Q}}{}^{\alpha\beta\gamma},$$

(3.20c) $$\underset{(2)}{\tilde{q}} = \frac{1}{3}\underset{(2)}{\tilde{Q}}{}^{\alpha}{}_{\cdot\alpha},$$

and by convention (3.12) we take $\underset{(1)}{q} \equiv 0$. We remark that the factors of 3 rather than factors of 4 arise in (3.20a,b,c) due the spatial nature of $\overset{\perp}{\tilde{Q}}{}^{\alpha\beta\gamma\mu}$, $\underset{(1)}{\tilde{Q}}{}^{\alpha\beta\gamma}$, and $\underset{(2)}{\tilde{Q}}{}^{\alpha\beta}$. In addition $\underset{(0)}{\tilde{q}}$ and $\underset{(2)}{\tilde{q}}$ are required to be positive.

We henceforth consider only spatially isotropic media. In that case $\tilde{Q}^{\alpha\beta\gamma\mu}$ takes the form

(3.21) $$\tilde{Q}^{\alpha\beta\gamma\mu} = \underset{(0)}{\tilde{q}}\,(\overset{\perp}{g}{}^{\alpha\gamma}\overset{\perp}{g}{}^{\beta\mu} - \overset{\perp}{g}{}^{\alpha\mu}\overset{\perp}{g}{}^{\beta\mu}) - \underset{(2)}{\tilde{q}}\,4u^{[\alpha}\overset{\perp}{g}{}^{\beta]\ [\gamma}u^{\mu]}.$$

From (3.3) it follows that in vacuum

(3.22) $$\underset{(0)}{\tilde{q}} = \underset{(2)}{\tilde{q}} = \frac{1}{2}\sqrt{g}.$$

We have an additional relation between the proper current density and proper electric field

(3.23) $$\tilde{\jmath}^{\alpha} = \underset{(0)}{\sigma}\,\underset{(r)}{\tilde{\sigma}}{}^{\alpha\beta}E_{\beta},$$

where the base *conductivity* $\underset{(0)}{\sigma}$ and the *relative conductivity tensor* $\Delta$*-density* $\underset{(r)}{\tilde{\sigma}}{}^{\alpha\beta}$ are related to the electronic charge and to the density, effective mass, and mean free time between collisions of the charge carriers. This linear relation, called *Ohm's Law*, describes the effect of drift of charge carriers under the proper electric field[11] and is derived by considering the generalized Lorentz force on the charge carriers when they are slowly moving and $B_{\alpha\beta} \approx 0$. More general relations arise through consideration of the contribution of the magnetic induction and charge carrier velocity (not necessarily $u_\alpha$) to the Lorentz force.

For perfect dielectrics $\underset{(0)}{\sigma} \equiv 0$. In that case, using the fact that $\tilde{\jmath}^{\alpha}$ is a contravariant vector $\Delta$-density of weight $+1$, substituting (3.23) in eqn. (2.12), and using (2.10) we get

(3.24) $$\nabla_{\alpha}(u^{\alpha}\tilde{\rho}) = 0.$$

[11] More appropriately, it describes the displacement of the Fermi sphere in momentum space. See Kittel [10], p. 140ff.

If the motion of matter is rigid,[12] expansion of (3.24) and application of (1.6) and (1.3) yields

$$\frac{\delta\tilde{\rho}}{\delta s} = 0, \tag{3.25}$$

everywhere in $W_\Omega$. Equation (3.15) is just a statement of the fact that the electric charge remains constant along the world lines.

The Maxwell equations for a linear, memoryless, spatially isotropic medium thus take the form

$$\partial_{[\alpha} F_{\beta\gamma]} = 0, \tag{3.26}$$

$$\partial_\beta(\widetilde{Q}^{\alpha\beta\gamma\mu} F_{\gamma\mu}) = -z_0 \tilde{j}^\alpha, \tag{3.27}$$

where $\widetilde{Q}^{\alpha\beta\gamma\mu}$ is given by (3.21) and (3.20 a,c).

§4. **Electromagnetic Potentials.** Equation (3.26) and the converse of Poincare's Lemma for a 2-form imply the existence of a $C^1$ vector $A_\beta$ such that

$$F_{\alpha\beta} = 2!\partial_{[\alpha} A_{\beta]}. \tag{4.1}$$

The vector $A_\beta$ is not unique since $F_{\alpha\beta}$ remains invariant under the gauge transformation

$$A_\beta \mapsto A_\beta + \partial_\beta \theta, \tag{4.2}$$

where $\theta$ is an arbitrary $C^2$ absolute scalar. The vector $A_\beta$ is called the *electromagnetic potential* associated with the electromagnetic field tensor $F_{\alpha\beta}$.

Substituting (4.1) in (3.27b) we arrive at the governing field equation for the electromagnetic potential

$$2!\,\partial_\beta\{\widetilde{Q}^{\alpha\beta\gamma\mu}\partial_{[\gamma} A_{\mu]}\} = -z_0 \tilde{j}^\alpha. \tag{4.3}$$

To take advantage of the gauge freedom, we expand eqn. (4.3) and seek a coordinate invariant condition that leaves the field equation in a tractable and enlightening form.

Using the facts that the bracketed expression in eqn. (4.3) represents a bivector $\Delta$-density of weight +1 and that the space-time manifold is a $V_4$, the equation can be written

$$2!\nabla_\beta\{\widetilde{Q}^{\alpha\beta\gamma\mu}\nabla_{[\gamma} A_{\mu]}\} = -z_0 \tilde{j}^\alpha. \tag{4.4}$$

[12] The motion of matter is called *rigid* if the strain rate tensor $e_{\alpha\beta} \overset{\text{def}}{=} \nabla_{\underset{(\alpha}{\perp}} u_{\beta)} \equiv 0$ everywhere in $W_\Omega$. The process of *symmetrization* over the indices is indicated by a pair of parentheses. See Bressan [2], pp. 142 ff.

The following definitions facilitate the analysis

$$\underset{(1)}{\widetilde{T}}^{\alpha\beta} \overset{\text{def}}{=} \underset{(0)}{\tilde{q}}\,(\overset{\perp}{g}{}^{\alpha\gamma}\overset{\perp}{g}{}^{\beta\mu} - \overset{\perp}{g}{}^{\alpha\mu}\overset{\perp}{g}{}^{\beta\gamma})\nabla_{[\gamma}A_{\mu]} = 2!\,\underset{(0)}{\tilde{q}}\,\overset{\perp}{g}{}^{\mu[\beta}\nabla^{\overset{\perp}{\alpha}]}A_\mu, \tag{4.5a}$$

$$\underset{(2)}{\widetilde{T}}^{\alpha\beta} \overset{\text{def}}{=} 2!\,\underset{(2)}{\tilde{q}}\,u^{[\alpha}\overset{\perp}{g}{}^{\beta]\gamma}u^\mu\nabla_\gamma A_\mu, \tag{4.5b}$$

$$\underset{(3)}{\widetilde{T}}^{\alpha\beta} \overset{\text{def}}{=} -\,2!\,\underset{(2)}{\tilde{q}}\,u^{[\alpha}\overset{\perp}{g}{}^{\beta]\gamma}u^\mu\nabla_\mu A_\gamma. \tag{4.5c}$$

Equation (4.4) can evidently be written as

$$\nabla_\beta(\underset{(1)}{\widetilde{T}}^{\alpha\beta} + \underset{(2)}{\widetilde{T}}^{\alpha\beta} + \underset{(3)}{\widetilde{T}}^{\alpha\beta}) = -\frac{z_0}{2!}\,\tilde{j}^\alpha. \tag{4.6}$$

We now examine each term in eqn. (4.6) separately.

The first term in eqn. (4.6) can be expanded as

$$\nabla_\beta\underset{(1)}{\widetilde{T}}^{\alpha\beta} = 2!\nabla_\beta\{\overset{\perp}{g}{}^{\mu[\beta}\nabla^{\overset{\perp}{\alpha}]}(\underset{(0)}{\tilde{q}}\,A_\mu) - \underset{(0)}{q}{}^{[\overset{\perp}{\alpha}}\underset{(1)}{\tilde{A}}{}^{\beta]}\}, \tag{4.7}$$

where

$$\underset{(0)}{q}{}^{\alpha} \overset{\text{def}}{=} \frac{1}{\underset{(0)}{\tilde{q}}}\nabla^\alpha \underset{(0)}{\tilde{q}}\,, \tag{4.8}$$

and

$$\underset{(1)}{\tilde{A}}{}^{\beta} \overset{\text{def}}{=} \underset{(0)}{\tilde{q}}\,\overset{\perp}{g}{}^{\beta\mu}A_\mu\,. \tag{4.9}$$

For any covariant vector $\Delta$-density $\widetilde{V}_\gamma$, the identity

$$\overset{\perp}{g}{}^{\beta\gamma}\nabla^{\overset{\perp}{\alpha}}\widetilde{V}_\gamma \equiv \nabla^{\overset{\perp}{\alpha}}\widetilde{V}^{\overset{\perp}{\beta}} - \widetilde{V}_\gamma\nabla^{\overset{\perp}{\alpha}}(u^\beta u^\gamma) \tag{4.10}$$

arises trivially from the definition of the spatial projector. Using this identity and definition (4.9), eqn. (4.7) can be written as

$$\nabla_\beta\underset{(1)}{\widetilde{T}}^{\alpha\beta} = \nabla_\beta\{\nabla^{\overset{\perp}{\alpha}}\underset{(1)}{\tilde{A}}{}^{\beta} - \nabla^{\overset{\perp}{\beta}}\underset{(1)}{\tilde{A}}{}^{\alpha} - 2!(\underset{(0)}{q}\,A_\gamma\nabla^{[\overset{\perp}{\alpha}}(u^{\beta]}u^\gamma) + \underset{(0)}{q}{}^{[\overset{\perp}{\alpha}}\underset{(1)}{\tilde{A}}{}^{\beta]})\}. \tag{4.11}$$

Upon adding and subtracting $\nabla^\alpha\nabla_\beta\underset{(1)}{\tilde{A}}{}^{\beta}$, (4.11) takes the form

$$\begin{aligned}\nabla_\beta\underset{(1)}{\widetilde{T}}^{\alpha\beta} = {} & \{(\nabla_\beta\nabla^{\overset{\perp}{\alpha}} - \nabla_\alpha\nabla_\beta)\underset{(1)}{\tilde{A}}{}^{\beta} + \nabla^\alpha\nabla_\beta\underset{(1)}{\tilde{A}}{}^{\beta} - \nabla_\beta\nabla^{\overset{\perp}{\beta}}\underset{(1)}{\tilde{A}}{}^{\alpha}\} \\ & -2!\nabla_\beta\{\underset{(0)}{q}\,A_\gamma\nabla^{[\overset{\perp}{\alpha}}(u^{\beta]}u^\gamma) + \underset{(0)}{q}{}^{[\overset{\perp}{\alpha}}\underset{(1)}{\tilde{A}}{}^{\beta]}\}.\end{aligned} \tag{4.12}$$

Again using the definition of the spatial projector, we write

$$\nabla_\beta \overset{\perp}{\nabla}{}^{\alpha} \underset{(1)}{\tilde{A}}{}^{\beta} = \nabla_\beta \nabla^\alpha \underset{(1)}{\tilde{A}}{}^{\beta} + \nabla_\beta (u^\alpha u^\gamma \nabla_\gamma \underset{(1)}{\tilde{A}}{}^{\beta}),$$

and

$$\nabla_\beta \overset{\perp}{\nabla}{}^{\beta} \underset{(1)}{\tilde{A}}{}^{\alpha} = \nabla_\beta \nabla^\beta \underset{(1)}{\tilde{A}}{}^{\alpha} + \nabla_\beta (u^\beta u^\gamma \nabla_\gamma \underset{(1)}{\tilde{A}}{}^{\alpha}).$$

Substituting these in (4.12) we get

$$\begin{aligned} \nabla_\beta \underset{(1)}{\widetilde{T}}{}^{\alpha\beta} &= R_{\beta}^{\cdot\,\alpha} \underset{(1)}{\tilde{A}}{}^{\beta} + \nabla^\alpha \nabla_\beta \underset{(1)}{\tilde{A}}{}^{\beta} - \nabla_\beta \nabla^\beta \underset{(1)}{\tilde{A}}{}^{\alpha} \\ &\quad + 2! \nabla_\beta \{ u^\gamma u^{[\alpha} \nabla_\gamma \underset{(1)}{\tilde{A}}{}^{\beta]} - \underset{(0)}{\tilde{q}}\, A_\gamma \overset{\perp}{\nabla}{}^{[\alpha} (u^{\beta]} u^\gamma) - \underset{(0)}{q}{}^{\overset{\perp}{[\alpha}} \underset{(1)}{\tilde{A}}{}^{\beta]} \}, \end{aligned} \tag{4.13}$$

where $R_{\alpha\beta} \overset{\text{def}}{=} R_{\gamma\alpha\beta}^{\cdot\,\cdot\,\cdot\,\gamma}$ is the Ricci tensor of the space-time $V_4$.

Expansion of the second term in (4.6) yields

$$\begin{aligned} \nabla_\beta \underset{(2)}{\widetilde{T}}{}^{\alpha\beta} &= \nabla_\beta \{ \underset{(2)}{\tilde{q}} (u^\alpha \overset{\perp}{g}{}^{\beta\gamma} u^\mu - u^\beta \overset{\perp}{g}{}^{\alpha\gamma} u^\mu) \nabla_\gamma A_\mu \}, \\ &= \nabla_\beta \{ \underset{(2)}{\tilde{q}} (u^\alpha u^\mu \overset{\perp}{\nabla}{}^{\beta} A_\mu - u^\beta u^\mu \overset{\perp}{\nabla}{}^{\alpha} A_\mu) \}, \\ &= \nabla_\beta \{ \overset{\perp}{\nabla}{}^{\beta} \underset{(2)}{\tilde{A}}{}^{\alpha} - \overset{\perp}{\nabla}{}^{\alpha} \underset{(2)}{\tilde{A}}{}^{\beta} + 2!\, \underset{(2)}{q}{}^{\overset{\perp}{[\alpha}} \underset{(2)}{\tilde{A}}{}^{\beta]} + 2!\, \underset{(2)}{\tilde{q}}\, A_\mu \overset{\perp}{\nabla}{}^{[\alpha} (u^{\beta]} u^\mu) \}, \end{aligned} \tag{4.14}$$

where

$$\underset{(2)}{\tilde{A}}{}^{\alpha} \overset{\text{def}}{=} \underset{(2)}{\tilde{q}}\, u^\alpha u^\mu A_\mu, \tag{4.15}$$

and

$$\underset{(2)}{q}{}^{\alpha} \overset{\text{def}}{=} \frac{1}{\underset{(2)}{\tilde{q}}} \nabla^\alpha \underset{(2)}{\tilde{q}}\,. \tag{4.16}$$

Adding and subtracting $\nabla^\alpha \nabla_\beta \underset{(2)}{\tilde{A}}{}^{\beta}$, (4.14) takes the form

$$\begin{aligned} \nabla_\beta \underset{(2)}{\widetilde{T}}{}^{\alpha\beta} &= -(\nabla_\beta \overset{\perp}{\nabla}{}^{\alpha} - \nabla^\alpha \nabla_\beta) \underset{(2)}{\tilde{A}}{}^{\beta} + \nabla_\beta \overset{\perp}{\nabla}{}^{\beta} \underset{(2)}{\tilde{A}}{}^{\alpha} - \nabla^\alpha \nabla_\beta \underset{(2)}{\tilde{A}}{}^{\beta} \\ &\quad + 2! \nabla_\beta \{ \underset{(2)}{q}{}^{\overset{\perp}{[\alpha}} \underset{(2)}{\tilde{A}}{}^{\beta]} + \underset{(2)}{\tilde{q}}\, A_\mu \overset{\perp}{\nabla}{}^{[\alpha} (u^{\beta]} u^\mu) \}. \end{aligned} \tag{4.17}$$

In a manner similar to that used in (4.12), eqn. (4.17) can be rewritten as

$$\begin{aligned}\nabla_\beta \underset{(2)}{\widetilde{T}}{}^{\alpha\beta} &= -R_\beta{}^{\cdot\alpha} \underset{(2)}{\tilde{A}}{}^\beta - \nabla^\alpha \nabla_\beta \underset{(2)}{\tilde{A}}{}^\beta + \nabla_\beta \nabla^\beta \underset{(2)}{\tilde{A}}{}^\alpha \\ &\quad - 2!\nabla_\beta\{u^\gamma u^{[\alpha}\nabla_\gamma \underset{(2)}{\tilde{A}}{}^{\beta]} - \underset{(2)}{\tilde{q}}\, A_\gamma \nabla^{[\overset{\perp}{\alpha}} u^{\beta]} u^\gamma - \underset{(2)}{q}{}^{[\overset{\perp}{\alpha}} \underset{(2)}{\tilde{A}}{}^{\beta]}\}.\end{aligned} \tag{4.18}$$

Finally, expanding the last term in eqn. (4.6) yields the result

$$\begin{aligned}\nabla_\beta \underset{(3)}{\widetilde{T}}{}^{\alpha\beta} &= \nabla_\beta\{\, \underset{(2)}{\tilde{q}}\,(u^\beta \overset{\perp}{g}{}^{\alpha\gamma} u^\mu - u^\alpha \overset{\perp}{g}{}^{\beta\gamma} u^\mu)\nabla_\mu A_\gamma\}, \\ &= \nabla_\beta\{\, \underset{(2)}{\tilde{q}}\,(u^\beta u^\mu g^{\alpha\gamma} - u^\alpha u^\mu g^{\beta\gamma})\nabla_\mu A_\gamma\}, \\ &= 2!\nabla_\beta\{u^\mu \underset{(2)}{q}{}_\mu u^{[\alpha} \underset{(2)}{\tilde{q}}\, A^{\beta]} - u^\gamma u^{[\alpha}\nabla_\gamma(\underset{(2)}{\tilde{q}}\, A^{\beta]})\}.\end{aligned} \tag{4.19}$$

We define a new contravariant vector Δ-density of weight +1

$$\underset{(3)}{\tilde{A}}{}^\beta \overset{\text{def}}{=} \underset{(1)}{\tilde{A}}{}^\beta - \underset{(2)}{\tilde{A}}{}^\beta = \underset{(0)}{\tilde{q}}\, A^{\overset{\perp}{\beta}} - \underset{(2)}{\tilde{q}}\, u^\beta u_\gamma A^\gamma. \tag{4.20}$$

Using this definition and (4.6), (4.13), (4.18) and (4.19), the field equation (4.6) can be written

$$\begin{aligned}&\nabla_\beta \nabla^\beta \underset{(3)}{\tilde{A}}{}^\alpha - R_\beta{}^{\cdot\alpha} \underset{(3)}{\tilde{A}}{}^\beta - \nabla^\alpha \nabla_\beta \underset{(3)}{\tilde{A}}{}^\beta + 2!\nabla_\beta\{(\underset{(0)}{\tilde{q}} - \underset{(2)}{\tilde{q}})A_\gamma \nabla^{[\overset{\perp}{\alpha}}(u^{\beta]}u^\gamma)\} \\ &\quad - 2!\nabla_\beta\{u^\gamma u^{[\alpha}\nabla_\gamma \underset{(3)}{\tilde{A}}{}^{\beta]} - u^\gamma u^{[\alpha}\nabla_\gamma(\underset{(2)}{\tilde{q}}\, A^{\beta]})\} \\ &\quad - 2!\nabla_\beta\{\underset{(2)}{q}{}^{[\overset{\perp}{\alpha}} \underset{(2)}{\tilde{A}}{}^{\beta]} - \underset{(0)}{q}{}^{[\overset{\perp}{\alpha}} \underset{(1)}{\tilde{A}}{}^{\beta]} + u_\mu \underset{(2)}{q}{}^\mu u^{[\alpha} \underset{(2)}{\tilde{q}}\, A^{\beta]}\} = \frac{z_0}{2!}\, j^\alpha .\end{aligned} \tag{4.21}$$

After some lengthy algebraic details and division by $\sqrt{g}$, eqn. (4.21) can be shown to have the coordinate-invariant form

$$\begin{aligned}&\nabla_\beta \nabla^\beta \underset{(3)}{A}{}^\alpha - R_\beta{}^{\cdot\alpha} \underset{(3)}{A}{}^\beta - \nabla^\alpha \nabla_\beta \underset{(3)}{A}{}^\beta + \nabla_\beta\{2!\, \underset{(2)}{q}{}^{[\alpha} \underset{(3)}{A}{}^{\beta]}\} + 2!\nabla_\beta\left\{(\underset{(0)}{q}{}^{[\alpha} - \underset{(2)}{q}{}^{[\alpha})\underset{(1)}{A}{}^{\beta]}\right\} \\ &\quad + 2!\nabla_\beta\left\{(\underset{(0)}{q} - \underset{(2)}{q})\left(A_\gamma \nabla^{[\alpha}(u^{\beta]}u^\gamma) - u^{[\alpha}\frac{\delta}{\delta s} A^{\overset{\perp}{\beta}]}\right)\right\} = \frac{z_0}{2!}\, j^\alpha,\end{aligned} \tag{4.22}$$

where $\underset{(3)}{A}{}^\beta = g^{-1/2}\underset{(3)}{\tilde{A}}$, $\underset{(1)}{A}{}^\beta = g^{-1/2}\underset{(1)}{\tilde{A}}{}^\beta$, etc.

At first glance eqn. (4.22) appears involved, but further analysis reveals its simplicity. In the case when $\widetilde{Q}^{\alpha\beta\gamma\mu}$ is an isotropic tensor Δ-density of weight +1 rather than a spatially isotropic tensor Δ-density of weight +1 we have

$$\underset{(0)}{q} \equiv \underset{(2)}{q} \overset{\text{def}}{=} q(x^\alpha), \tag{4.23}$$

$$\underset{(0)}{q}{}^\alpha \equiv \underset{(0)}{q}{}^\alpha \overset{\text{def}}{=} q^\alpha, \tag{4.24}$$

$$\widetilde{Q}^{\alpha\beta\gamma\mu} = \sqrt{g}\, q\,(g^{\alpha\gamma}g^{\beta\mu} - g^{\alpha\mu}g^{\beta\gamma}), \tag{4.25}$$

and

(4.26) $$\underset{(3)}{A}^{\beta} = q\,(\overset{\perp}{g}^{\beta\gamma}A_{\gamma} - u^{\beta}u^{\gamma}A_{\gamma}) = qA^{\beta}.$$

Equation (4.22) therefore immediately reduces to

(4.27) $$\nabla_{\beta}\nabla^{\beta}\underset{(3)}{A}^{\alpha} - R_{\beta}^{\cdot\,\alpha}\underset{(3)}{A}^{\beta} - \nabla^{\alpha}\nabla_{\beta}\underset{(3)}{A}^{\beta} + 2!\nabla_{\beta}\{q^{[\alpha}\underset{(3)}{A}^{\beta]}\} = \frac{z_0}{2!}\,j^{\alpha}.$$

This can be verified by substituting (4.25) in (4.4) to get (after dividing by $\sqrt{g}$)

$$2!\nabla_{\beta}\{q\nabla^{[\alpha}A^{\beta]}\} = -\frac{z_0}{2!}\,j^{\alpha},$$

or

(4.28) $$-2!\nabla_{\beta}\{\nabla^{[\alpha}(qA^{\beta]}) - q^{[\alpha}A^{\beta]}\} = \frac{z_0}{2!}\,j^{\alpha}.$$

Adding and subtracting $\nabla^{\alpha}\nabla_{\beta}(qA^{\beta})$ and using (4.26), (4.28) can be written as

$$\nabla_{\beta}\nabla^{\beta}\underset{(3)}{A}^{\alpha} - R_{\beta}^{\cdot\,\alpha}\underset{(3)}{A}^{\beta} - \nabla^{\alpha}\nabla_{\beta}\underset{(3)}{A}^{\beta} + 2!\nabla_{\beta}\{q^{[\alpha}\underset{(3)}{A}^{\beta]}\} = \frac{z_0}{2!}j^{\alpha},$$

which is in accord with eqn. (4.27). If the medium is homogeneous as well as isotropic, $q^{\alpha}$ vanishes, $q$ is a covariant constant, and eqn. (4.22) simplifies to

(4.29) $$\nabla_{\beta}\nabla^{\beta}\underset{(3)}{A}^{\alpha} - R_{\beta}^{\cdot\,\alpha}\underset{(3)}{A}^{\beta} - \nabla^{\alpha}\nabla_{\beta}\underset{(3)}{A}^{\beta} = \frac{z_0}{2!}\,j^{\alpha}.$$

In vacuum, $q = 1/2$ and thus eqn. (4.29) takes the familiar form

(4.30) $$\nabla_{\beta}\nabla^{\beta}A^{\alpha} - R_{\beta}^{\cdot\,\alpha}A^{\beta} - \nabla^{\alpha}\nabla_{\beta}A^{\beta} = z_0 j^{\alpha}.$$

Note that in most normalized systems of units (not MKS) $z_0 \equiv 1$.

We remark that the analysis of this section is independent of any assumptions concerning rigidity, angular velocity and/or intrinsic acceleration. Consider the sixth term on the left hand side of eqn. (4.22)

(4.31) $$2!\nabla_{\beta}\{(\underset{(0)}{q} - \underset{(2)}{q})A_{\gamma}\nabla^{[\overset{\perp}{\alpha}}(u^{\beta]}u^{\gamma})\} = \nabla_{\beta}\{(\underset{(0)}{q} - \underset{(2)}{q})A_{\gamma}(u^{\gamma}\omega^{\alpha\beta} - 2!u^{[\alpha}e^{\beta]\gamma} - 2!u^{[\alpha}\omega^{\beta]\gamma})\},$$

where $\omega_{\alpha\beta}$ is the *local angular velocity (or average rate of rotation)*,

(4.32) $$\omega_{\alpha\beta} \overset{\text{def}}{=} \nabla_{[\overset{\perp}{\alpha}}u_{\beta]},$$

and $e_{\alpha\beta}$ is the *strain rate tensor*

(4.33) $$e_{\alpha\beta} \overset{\text{def}}{=} \nabla_{(\overset{\perp}{\alpha}}u_{\beta)}.$$

Thus if the motion of matter is both rigid ($e_{\alpha\beta} \equiv 0$) and rotation free[13] ($\omega_{\alpha\beta} \equiv 0$), then the term (4.31) in eqn. (4.22) vanishes.

[13]The physical interpretation of the statement $\omega_{\alpha\beta} \equiv 0$ is that the motion of matter is such that in a neighborhood of a world line of a material point the medium does not rotate with respect to an orthonormal tetrad carried along the world line by Fermi-Walker transport. See Synge [1], p. 173.

§5. **Generalized Lorentz Gauge.** Equation (4.22) can be simplified by fixing the *generalized Lorentz Gauge*

$$\nabla_\beta \underset{(3)}{A}^\beta = 0. \tag{5.1}$$

This coordinate invariant condition immediately reduces to the classical Lorentz condition in the case of isotropic homogeneous media. In terms of tensor $\Delta$-densities, the generalized Lorentz Gauge can be written

$$\frac{1}{\sqrt{g}} \partial_\beta \underset{(3)}{\tilde{A}}^\beta = 0, \tag{5.2}$$

and therefore in terms of the dual representation

$$\underset{(3)}{A}_{\alpha\beta\gamma} \stackrel{\text{def}}{=} \underset{\sim}{\varepsilon}_{\alpha\beta\gamma\mu} \underset{(3)}{\tilde{A}}^\mu, \tag{5.3}$$

it has the alternate coordinate invariant form

$$\partial_{[\alpha} A_{\beta\gamma\mu]} = 0. \tag{5.4}$$

We remark that fixing the generalized Lorentz gauge does not uniquely specify the vector $A_\beta$. In fact, form invariance of (5.1) requires that in the gauge transformation (4.2) the additive scalar $\theta$ may be any solution of the equation

$$\nabla_\beta ( \underset{(0)}{q} \overset{\perp}{\nabla}{}^\beta \theta) - \nabla_\beta \left( \underset{(2)}{q}\, u^\beta \frac{\delta \theta}{\delta s} \right) = 0. \tag{5.5}$$

By considering eqn. (5.5) in a locally natural and proper frame and using the fact that $\underset{(0)}{q}$ and $\underset{(2)}{q}$ are positive, one can easily see that the equation is hyperbolic. This means that the solution depends only on the initial conditions for $\theta$, which are completely arbitrary.

Substituting (5.1) in eqn. (4.22), the governing field equation for the electromagnetic potential takes the form

$$\begin{aligned} &\nabla_\beta \nabla^\beta \underset{(3)}{\tilde{A}}^\alpha - R_\beta{}^\alpha \underset{(3)}{\tilde{A}}^\beta + 2! \nabla_\beta \left\{ \underset{(0)}{q}{}^{[\alpha} \underset{(1)}{A}{}^{\beta]} - \underset{(2)}{q}{}^{[\alpha} \underset{(2)}{A}{}^{\beta]} \right\} \\ &\quad + 2! \nabla_\beta \left\{ ( \underset{(0)}{q} - \underset{(2)}{q} ) \left( A_\gamma \overset{\perp}{\nabla}{}^{[\alpha} (u^{\beta]} u^\gamma) - u^{[\alpha} \frac{\delta}{\delta s} \overset{\perp}{A}{}^{\beta]} \right) \right\} = \frac{z_0}{2!} j^\alpha, \end{aligned} \tag{5.6}$$

subject to the condition (5.1), and where we have used the fact that

$$\underset{(2)}{q}{}^{[\alpha} \underset{(3)}{A}{}^{\beta]} + ( \underset{(0)}{q}{}^{[\alpha} - \underset{(2)}{q}{}^{[\alpha} ) \underset{(1)}{A}{}^{\beta]} = \underset{(0)}{q}{}^{[\alpha} \underset{(1)}{A}{}^{\beta]} - \underset{(2)}{q}{}^{[\alpha} \underset{(2)}{A}{}^{\beta]}. \tag{5.7}$$

For rigid motion that is locally rotation-free, eqn. (5.6) simplifies to

$$(5.8)\quad \nabla_\beta \nabla^\beta \underset{(3)}{A}^\alpha - R_\beta^{\cdot\,\alpha} \underset{(3)}{A}^\beta + 2!\nabla_\beta \left\{ \underset{(0)}{q}^{[\alpha} \underset{(1)}{A}^{\beta]} - \underset{(2)}{q}^{[\alpha} \underset{(2)}{A}^{\beta]} - (\underset{(0)}{q} - \underset{(2)}{q}) u^{[\alpha} \frac{\delta}{\delta s} \overset{\perp}{A}^{\beta]} \right\} = \frac{z_0}{2!} j^\alpha .$$

For homogeneous isotropic media, equation (5.1) immediately reduces to the classical Lorentz gauge condition and equation (5.6) becomes

$$(5.9)\quad \nabla_\beta \nabla^\beta \underset{(3)}{A}^\alpha - R_\beta^{\cdot\,\alpha} \underset{(3)}{A}^\beta = \frac{z_0}{2!} j^\alpha ,$$

which in vacuum takes the standard form

$$(5.10)\quad \nabla_\beta \nabla^\beta A^\alpha - R_\beta^{\cdot\,\alpha} A^\beta = z_0 j^\alpha .$$

§**6. Superpotentials.** The governing field equation (5.6) for the electromagnetic potential $A_\gamma$ represents a coupled system of partial differential equations which have to be solved in conjunction with the generalized Lorentz condition (5.1). Equation (5.6) forms a consistent set of partial differential equations and existence of a solution of this system is guaranteed only if (5.1) holds simultaneously. Finding a solution is therefore difficult because only those solutions of eqn. (5.6) which simultaneously satisfy condition (5.1) are admissible. To alleviate this problem, we now introduce superpotentials which satisfy the generalized Lorentz condition identically.

Equation (5.4) and the converse of Poincaré's Lemma for a 3-form in $V_4$ imply that locally there exists a $C^2$ covariant bivector $\Pi_{\alpha\beta}$ such that

$$(6.1)\quad \underset{(3)}{A}_{\alpha\beta\gamma} = 3!\partial_{[\alpha}\Pi_{\beta\gamma]} .$$

The bivector $\Pi_{\alpha\beta}$ is not unique since the representation (6.1) admits the gauge transformation

$$(6.2)\quad \Pi_{\alpha\beta} \mapsto \Pi_{\alpha\beta} + 2!\partial_{[\alpha}\psi_{\beta]} ,$$

where $\psi_\beta$ is an arbitrary $C^3$ covariant vector. In terms of the dual representations

$$(6.3)\quad \underset{(3)}{\tilde{A}}^\alpha = \frac{1}{3!} \tilde{\varepsilon}^{\alpha\beta\gamma\mu} A_{\beta\gamma\mu} ,$$

$$(6.4)\quad \widetilde{\Pi}^{\alpha\beta} = \frac{1}{2!} \tilde{\varepsilon}^{\alpha\beta\gamma\mu} \Pi_{\gamma\mu} ,$$

eqn. (6.1) can be written as

$$(6.5)\quad \underset{(3)}{\tilde{A}}^\alpha = 2!\nabla_\beta \widetilde{\Pi}^{\alpha\beta} .$$

The bivector $\Pi_{\alpha\beta}$ is called the *(electromagnetic) superpotential associated with the electromagnetic potential* $A_\gamma$.

For the sake of simplicity in deriving the governing field equation for the superpotential $\Pi_{\alpha\beta}$, we assume that the motion of the material is locally rotation-free ($\omega_{\alpha\beta} = \nabla_{\overset{\perp}{[\alpha}} u_{\beta]} \equiv 0$) and that the medium is a *perfect (spatially isotropic) dielectric*, that is

$$\underset{(0)}{\sigma} \equiv 0, \quad \underset{(0)}{q}{}^{\alpha} \equiv 0, \quad e_{\alpha\beta} = \nabla_{\overset{\perp}{(\alpha}} u_{\beta)} \equiv 0. \tag{6.6}$$

In the event that $\omega_{\alpha\beta} \not\equiv 0$ and/or any part of (6.6) does not hold, derivation of the governing field equation for $\Pi_{\alpha\beta}$ follows from our analysis in an obvious and straightforward manner.

Since $\omega_{\alpha\beta} \equiv 0$, it follows that

$$u_{[\alpha} \nabla_\beta u_{\gamma]} = u_{[\alpha} u_\beta a_{\gamma]} = 0, \tag{6.7}$$

and therefore due to the time-like nature of the world lines (meaning the true square $-u^\alpha u_\alpha > 0$), the vector field $u_\alpha$ is *$V_3$-forming;*[14] in other words, there exists a family of $V_3$'s to which the world lines are orthogonal. Thus the stream-lines form a *normal congruence.*[15] The family of $V_3$'s represent cross-sections of the world tube $W_\Omega$ and can be parameterized by $s$. We assume that there is a cross-section indicated by $s = s_0$ for which $\tilde{\rho}(x^\alpha(s_0)) \equiv 0$. Then (6.6) and eqn. (3.25) imply that $\tilde{\rho} \equiv 0$ everywhere in $W_\Omega$, and hence $\tilde{j}^\alpha \equiv 0$ everywhere in $W_\Omega$.

Substituting the representation (6.5) in eqn. (5.6) and using the above simplifying assumptions and identity (A1.1) from the Appendix we readily obtain

$$\begin{aligned} \frac{1}{\sqrt{g}} \nabla_\beta \Big\{ & \nabla_\sigma \nabla^\sigma \tilde{\Pi}^{\alpha\beta} - 2!(1-h) u^{[\alpha} \frac{\delta}{\delta s} (\overset{\perp}{g}{}^{\beta]}_{\cdot\,\gamma} \nabla_\sigma \tilde{\Pi}^{\gamma\sigma}) \\ & - 2! \underset{(2)}{q}{}^{[\alpha} u^{\beta]} u_\gamma \nabla_\sigma \tilde{\Pi}^{\gamma\sigma} - 3 R_{\gamma\sigma}{}^{\cdot\,\cdot\,[\alpha\beta} \tilde{\Pi}^{\gamma]\sigma} \Big\} = 0, \end{aligned} \tag{6.8}$$

where

$$h \overset{\text{def}}{=} \underset{(2)}{q} \,/\, \underset{(0)}{q}. \tag{6.9}$$

Evidently, the quantity within the braces is a contravariant bivector $\Delta$-density of weight $+1$, and the covariant derivative can thus be replaced by a partial derivative. It immediately follows from the converse of Poincare's Lemma for a 2-form in $V_4$ that there exists a $C^1$ covariant vector $\phi_\beta$ such that

$$\begin{aligned} \nabla_\sigma \nabla^\sigma \tilde{\Pi}^{\alpha\beta} - 2!(1-h) u^{[\alpha} \frac{\delta}{\delta s} (\overset{\perp}{g}{}^{\beta]}_{\cdot\,\gamma} \nabla_\sigma \tilde{\Pi}^{\gamma\sigma}) - 2! \underset{(2)}{q}{}^{[\alpha} u^{\beta]} u_\gamma \nabla_\sigma \tilde{\Pi}^{\gamma\sigma} \\ - 3 R_{\kappa\sigma}{}^{\cdot\,\cdot\,[\alpha\beta} \tilde{\Pi}^{\kappa]\sigma} = \tilde{\varepsilon}^{\alpha\beta\gamma\mu} \partial_{[\gamma} \phi_{\mu]}. \end{aligned} \tag{6.10}$$

[14] Schouten [3], p. 244 ff.
[15] Synge [1], p. 173.

The dual representation of the gauge transformation (6.2) can be written as

$$\widetilde{\Pi}^{\alpha\beta} \mapsto \widetilde{\Pi}^{\alpha\beta} + \widetilde{\varepsilon}^{\alpha\beta\gamma\mu}\partial_{[\gamma}\psi_{\mu]} = \widetilde{\Pi}^{\alpha\beta} + 2!\partial_\gamma\tilde{\psi}^{\alpha\beta\gamma}, \tag{6.11}$$

where

$$\tilde{\psi}^{\alpha\beta\gamma} = \widetilde{\varepsilon}^{\alpha\beta\gamma\mu}\psi_\mu. \tag{6.12}$$

Using this in (6.10) and applying identity (A1.6) from the Appendix we get

$$\begin{aligned} \nabla_\sigma\nabla^\sigma\widetilde{\Pi}^{\alpha\beta} &- 2!(1-h)u^{[\alpha}\frac{\delta}{\delta s}(\overset{\perp}{g}{}^{\beta}_{\cdot\gamma}\nabla_\sigma\widetilde{\Pi}^{\gamma\sigma}) - 2!\underset{(2)}{q}{}^{[\alpha}u^{\beta]}u_\gamma\nabla_\sigma\widetilde{\Pi}^{\gamma\sigma} \\ &- 3R_{\kappa\sigma}^{\cdot\cdot[\alpha\beta}\widetilde{\Pi}^{\kappa]\sigma} = -2!\nabla_\gamma\left\{\nabla_\sigma\nabla^\sigma\tilde{\psi}^{\alpha\beta\gamma} - 3R_\sigma^{\cdot[\alpha}\tilde{\psi}^{\beta\gamma]\sigma}\right. \\ &\left. - 3R_{\kappa\sigma}^{\cdot\cdot[\alpha\beta}\tilde{\psi}^{\gamma]\kappa\sigma} - \tilde{\phi}^{\alpha\beta\gamma}\right\}, \end{aligned} \tag{6.13}$$

where

$$\tilde{\phi}^{\alpha\beta\gamma} = \widetilde{\varepsilon}^{\alpha\beta\gamma\mu}\phi_\mu. \tag{6.14}$$

The freedom to choose $\psi_\beta$ (or $\tilde{\psi}^{\alpha\beta\gamma}$) and any attached initial conditions then allows us to suppress the unknown $\phi_\beta$ (or $\tilde{\phi}^{\alpha\beta\gamma}$) from the preceding analysis simply by noting that a solution of the equation

$$\nabla_\sigma\nabla^\sigma\tilde{\psi}^{\alpha\beta\gamma} - 3R_\sigma^{\cdot[\alpha}\tilde{\psi}^{\beta\gamma]\sigma} - 3R_{\kappa\sigma}^{\cdot\cdot[\alpha\beta}\tilde{\psi}^{\gamma]\kappa\sigma} = \tilde{\phi}^{\alpha\beta\gamma} \tag{6.15}$$

exists. The ensuing field equation for the superpotential $\widetilde{\Pi}^{\alpha\beta}$ thus takes the simple form

$$\begin{aligned} \nabla_\sigma\nabla^\sigma\widetilde{\Pi}^{\alpha\beta} &- 2!\,(1-h)u^{[\alpha}\frac{\delta}{\delta s}(\overset{\perp}{g}{}^{\beta}_{\cdot\gamma}\nabla_\sigma\widetilde{\Pi}^{\gamma\sigma}) \\ &- 2!\,\underset{(2)}{q}{}^{[\alpha}u^{\beta]}u_\gamma\nabla_\sigma\widetilde{\Pi}^{\gamma\sigma} - 3R_{\kappa\sigma}^{\cdot\cdot[\alpha\beta}\widetilde{\Pi}^{\kappa]\sigma} = 0. \end{aligned} \tag{6.16}$$

In the case of homogeneous, isotropic media, eqn. (6.16) reduces to

$$\nabla_\sigma\nabla^\sigma\widetilde{\Pi}^{\alpha\beta} - 3R_{\kappa\sigma}^{\cdot\cdot[\alpha\beta}\widetilde{\Pi}^{\kappa]\sigma} = 0. \tag{6.17}$$

We remark that in classical, non-relativistic theory, the curvature tensor vanishes and $\nabla_\alpha u_\beta \equiv 0$, and eqn. (6.17) thus takes the limiting form

$$\nabla_\sigma\nabla^\sigma\widetilde{\Pi}^{\alpha\beta} = 0. \tag{6.18}$$

By taking the natural decomposition of $\widetilde{\Pi}^{\alpha\beta}$

$$\widetilde{\Pi}^{\alpha\beta} = \underset{(m)}{\widetilde{\Pi}}{}^{\alpha\beta} + 2!\,\sqrt{g}\,u^{[\alpha}\underset{(e)}{\Pi}{}^{\beta]}, \tag{6.19}$$

where

$$\underset{(m)}{\widetilde{\Pi}}{}^{\alpha\beta} \stackrel{\text{def}}{=} \overset{\perp}{\widetilde{\Pi}}{}^{\alpha\beta} = \overset{\perp}{\tilde{\varepsilon}}{}^{\alpha\beta\gamma}\underset{(m)}{\Pi}{}_{\gamma}, \tag{6.20a}$$

$$\underset{(e)}{\Pi}{}^{\alpha} \stackrel{\text{def}}{=} \frac{1}{\sqrt{g}}\overset{\perp}{g}{}^{\alpha}_{\cdot\beta}\widetilde{\Pi}^{\beta\gamma}u_{\gamma}, \tag{6.20b}$$

eqn. (6.18) yields one equation for the spatial covariant vector $\underset{(m)}{\Pi}{}_{\gamma}$ and one for the spatial contravariant vector $\underset{(e)}{\Pi}{}^{\gamma}$, namely

$$\nabla_{\sigma}\nabla^{\sigma}\underset{(m)}{\Pi}{}_{\gamma} = 0, \tag{6.21}$$

$$\nabla_{\sigma}\nabla^{\sigma}\underset{(e)}{\Pi}{}^{\gamma} = 0. \tag{6.22}$$

Evidently, these two spatial vectors are related to the Hertz vectors of classical electrodynamics[16], and the superpotential $\widetilde{\Pi}^{\alpha\beta}$ is therefore of the Hertzian type. It is interesting to note that natural decomposition of the gauge transformation (6.2) gives rise to a form which is at variance with that of the standard gauge transformations for the classical Hertz vectors.[17] This strongly leads us to the conclusion that the standard gauge transformations used in classical electrodynamics are geometrically and relativistically inconsistent, and will be the topic of a forthcoming paper.

**Appendix 1.** In this appendix, two tensor identities invoked in §6 are proved. Derivation of each identity is motivated by the desire to write a certain geometric quantity as the divergence of another quantity that contains only the Laplace-Beltrami operator[18] acting on a contravariant $p$-vector $\Delta$-density of weight $+1$ plus transvections of the curvature tensor with the same $p$-vector $\Delta$-density.

IDENTITY 1. *Let $\widetilde{\Pi}^{\alpha\beta}$ be a contravariant bivector $\Delta$-density of weight $+1$ in a $V_n$. Then*

$$\nabla_{\beta}\nabla^{\beta}\nabla_{\sigma}\widetilde{\Pi}^{\alpha\sigma} - R_{\beta}^{\cdot\,\alpha}\nabla_{\sigma}\widetilde{\Pi}^{\beta\sigma} \equiv \nabla_{\beta}\left\{\nabla_{\sigma}\nabla^{\sigma}\widetilde{\Pi}^{\alpha\beta} - 3R_{\kappa\sigma}^{\cdot\,\cdot\,[\alpha\beta}\widetilde{\Pi}^{\kappa]\sigma}\right\}, \tag{A1.1}$$

*where $R_{\alpha\beta\gamma}^{\cdot\,\cdot\,\cdot\,\mu}$ is the curvature tensor of the $V_n$ and $R_{\beta}^{\cdot\,\alpha}$ is the Ricci tensor*

$$R_{\beta}^{\cdot\,\alpha} \stackrel{\text{def}}{=} R_{\gamma\beta}^{\cdot\,\cdot\,\alpha\gamma}. \tag{A1.2}$$

[16]Stratton [11], p. 28 ff.
[17]Born and Wolf [12], p. 81.
[18]Note that the operator is actually of the D'Alembertian type due to the indefinite nature of the metric tensor.

*Proof.* Expanding the left-hand side of (A1.1) and using the fact that $\tilde{\Pi}^{\alpha\beta}$ is a contravariant bivector $\Delta$-density of weight $+1$ in a $V_n$, we get

$$
\begin{aligned}
\nabla_\beta\nabla^\beta\nabla_\sigma\tilde{\Pi}^{\alpha\sigma} - R_\beta^{\cdot\,\alpha}\nabla_\sigma\tilde{\Pi}^{\beta\sigma} &= \nabla_\beta(\nabla^\beta\nabla_\sigma\tilde{\Pi}^{\alpha\sigma} - \nabla^\alpha\nabla_\sigma\tilde{\Pi}^{\beta\sigma}) \\
&= \nabla_\beta\Big\{(\nabla^\beta\nabla_\sigma - \nabla_\sigma\nabla^\beta)\tilde{\Pi}^{\alpha\sigma} - (\nabla^\alpha\nabla_\sigma - \nabla_\sigma\nabla^\alpha)\tilde{\Pi}^{\beta\sigma} \\
&\qquad +\nabla_\sigma\nabla^\beta\tilde{\Pi}^{\alpha\sigma} - \nabla_\sigma\nabla^\alpha\tilde{\Pi}^{\beta\sigma}\Big\}, \\
&= \nabla_\beta\Big\{g^{\beta\gamma}\,2!\,\nabla_{[\gamma}\nabla_{\sigma]}\tilde{\Pi}^{\alpha\sigma} - g^{\alpha\gamma}\,2!\,\nabla_{[\gamma}\nabla_{\sigma]}\tilde{\Pi}^{\beta\sigma} \\
&\qquad +\nabla_\sigma\nabla^\sigma\tilde{\Pi}^{\alpha\beta} - \nabla_\sigma(\nabla^\alpha\tilde{\Pi}^{\beta\sigma} + \nabla^\beta\tilde{\Pi}^{\sigma\alpha} - \nabla^\sigma\tilde{\Pi}^{\beta\alpha}\Big\}, \\
&= \nabla_\beta\Big\{g^{\beta\gamma}\,2!\,\nabla_{[\gamma}\nabla_{\sigma]}\tilde{\Pi}^{\alpha\sigma} - g^{\alpha\gamma}\,2!\,\nabla_{[\gamma}\nabla_{\sigma]}\tilde{\Pi}^{\beta\sigma} \\
&\qquad +\nabla_\sigma\nabla^\sigma\tilde{\Pi}^{\alpha\beta}\Big\} - 3\nabla_\beta\nabla_\sigma(\nabla^{[\alpha}\tilde{\Pi}^{\beta\sigma]}).
\end{aligned} \tag{A1.3}
$$

Evidently, since $\nabla^{[\alpha}\tilde{\Pi}^{\beta\sigma]}$ is a contravariant trivector $\Delta$-density of weight $+1$, we have

$$
\nabla_\beta\nabla_\sigma(\nabla^{[\alpha}\tilde{\Pi}^{\beta\sigma]}) = \nabla_\beta\partial_\sigma(\nabla^{[\alpha}\tilde{\Pi}^{\beta\sigma]}) = \partial_\beta\partial_\sigma(\nabla^{[\alpha}\tilde{\Pi}^{\beta\sigma]}) \equiv 0. \tag{A1.4}
$$

Also

$$
\begin{aligned}
&g^{\beta\gamma}\,2!\,\nabla_{[\gamma}\nabla_{\sigma]}\tilde{\Pi}^{\alpha\sigma} - g^{\alpha\gamma}\,2!\,\nabla_{[\gamma}\nabla_{\sigma]}\tilde{\Pi}^{\beta\sigma} \\
&\quad = R^{\beta\,\cdot\,\cdot\,\alpha}_{\cdot\,\sigma\kappa}\tilde{\Pi}^{\kappa\sigma} + R^{\beta\,\cdot\,\cdot\,\sigma}_{\cdot\,\sigma\kappa}\tilde{\Pi}^{\alpha\kappa} - R^{\alpha\,\cdot\,\cdot\,\beta}_{\cdot\,\sigma\kappa}\tilde{\Pi}^{\kappa\sigma} - R^{\alpha\,\cdot\,\cdot\,\sigma}_{\cdot\,\sigma\kappa}\tilde{\Pi}^{\beta\kappa}, \\
&\quad = (R^{\beta\,\cdot\,\cdot\,\alpha}_{\cdot\,\sigma\kappa} - R^{\alpha\,\cdot\,\cdot\,\beta}_{\cdot\,\sigma\kappa})\tilde{\Pi}^{\kappa\sigma} + R^{\beta\,\cdot\,\cdot\,\sigma}_{\cdot\,\sigma\kappa}\tilde{\Pi}^{\alpha\kappa} - R^{\alpha\,\cdot\,\cdot\,\sigma}_{\cdot\,\sigma\kappa}\tilde{\Pi}^{\beta\kappa}, \\
&\quad = -R_{\kappa\sigma}^{\cdot\,\cdot\,\alpha\beta}\tilde{\Pi}^{\kappa\sigma} + R_{\kappa\sigma}^{\cdot\,\cdot\,\sigma\beta}\tilde{\Pi}^{\alpha\kappa} - R_{\kappa\sigma}^{\cdot\,\cdot\,\alpha\sigma}\tilde{\Pi}^{\beta\kappa}, \\
&\quad = -R_{\kappa\sigma}^{\cdot\,\cdot\,\alpha\beta}\tilde{\Pi}^{\kappa\sigma} - R_{\kappa\sigma}^{\cdot\,\cdot\,\kappa\beta}\tilde{\Pi}^{\alpha\sigma} + R_{\kappa\sigma}^{\cdot\,\cdot\,\alpha\kappa}\tilde{\Pi}^{\beta\sigma}, \\
&\quad = -3R_{\kappa\sigma}^{\cdot\,\cdot\,[\alpha\beta}\tilde{\Pi}^{\kappa]\sigma},
\end{aligned} \tag{A1.5}
$$

where we have used the symmetry properties of the curvature tensor[19] and relabelled various dummy indices where required.

Substituting (A1.4) and (A1.5) in (A1.3), identity (A1.1) results immediately. ∎

IDENTITY 2. *Let $\tilde{\psi}^{\alpha\beta\gamma}$ be a contravariant trivector $\Delta$-density of weight $+1$ in a $V_n$. Then*

$$
\begin{aligned}
&\nabla_\sigma\nabla^\sigma\nabla_\gamma\tilde{\psi}^{\alpha\beta\gamma} - 3R_{\kappa\sigma}^{\cdot\,\cdot\,[\alpha\beta}\nabla_\gamma\tilde{\psi}^{\kappa]\sigma\gamma} \\
&\qquad \equiv \nabla_\gamma\Big\{\nabla_\sigma\nabla^\sigma\tilde{\psi}^{\alpha\beta\gamma} - 3R_\sigma^{\cdot\,[\alpha}\tilde{\psi}^{\beta\gamma]\sigma} - 3R_{\kappa\sigma}^{\cdot\,\cdot\,[\alpha\beta}\tilde{\psi}^{\gamma]\kappa\sigma}\Big\}.
\end{aligned} \tag{A1.6}
$$

[19] Schouten [3], pp. 144 ff.

*Proof.* Expansion of the left-hand side of (A1.6) yields

$$
\begin{aligned}
&\nabla_\sigma \nabla^\sigma \nabla_\gamma \tilde{\psi}^{\alpha\beta\gamma} - 3R_{\kappa\sigma}^{\cdot\ \cdot\ [\alpha\beta}\nabla_\gamma \tilde{\psi}^{\kappa]\sigma\gamma} \\
&\qquad = \nabla_\gamma\{\nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma}\} + \nabla^\sigma\{2!\nabla_{[\sigma}\nabla_{\gamma]}\tilde{\psi}^{\alpha\beta\gamma}\} \\
&\qquad\qquad + 2!\, g^{\sigma\mu}\nabla_{[\mu}\nabla_{\gamma]}\nabla_\sigma \psi^{\alpha\beta\gamma} - 3R_{\kappa\sigma}^{\cdot\ \cdot\ [\alpha\beta}\nabla_{|\gamma|}\tilde{\psi}^{\kappa]\sigma\gamma}, \\
&\qquad = \nabla_\gamma\{\nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} + g^{\gamma\mu}\, 2!\, \nabla_{[\mu}\nabla_{\sigma]}\tilde{\psi}^{\alpha\beta\sigma}\} \\
&\qquad\qquad + 2!\, g^{\sigma\mu}\nabla_{[\mu}\nabla_{\gamma]}\nabla_\sigma \psi^{\alpha\beta\gamma} - 3R_{\kappa\sigma}^{\cdot\ \cdot\ [\alpha\beta}\nabla_{|\gamma|}\tilde{\psi}^{\kappa]\sigma\gamma},
\end{aligned} \tag{A1.7}
$$

where a pair of bars around an index indicate that the index is to be excluded from the process of alternation or symmetrization. For the sake of economy, in order to obviate extensive algebra, we define

$$
\widetilde{\Psi}^{\alpha\beta} \overset{\text{def}}{=} 2!\, g^{\sigma\mu}\nabla_{[\mu}\nabla_{\gamma]}\nabla_\sigma \tilde{\psi}^{\alpha\beta\gamma} - 3R_{\kappa\sigma}^{\cdot\ \cdot\ [\alpha\beta}\nabla_{|\gamma|}\tilde{\psi}^{\kappa]\sigma\gamma}. \tag{A1.8}
$$

Expanding (A1.8), we get

$$
\begin{aligned}
\widetilde{\Psi}^{\alpha\beta} &= -R^{\sigma\ \cdot\ \cdot\ \kappa}_{\cdot\ \gamma\sigma}\nabla_\kappa \tilde{\psi}^{\alpha\beta\gamma} + R^{\sigma\ \cdot\ \cdot\ \alpha}_{\cdot\ \gamma\kappa}\nabla_\sigma \tilde{\psi}^{\kappa\beta\gamma} + R^{\sigma\ \cdot\ \cdot\ \beta}_{\cdot\ \gamma\kappa}\nabla_\sigma \tilde{\psi}^{\alpha\kappa\gamma} \\
&\quad + R^{\sigma\ \cdot\ \cdot\ \gamma}_{\cdot\ \gamma\kappa}\nabla_\sigma \tilde{\psi}^{\alpha\beta\kappa} - R_{\kappa\sigma}^{\cdot\ \cdot\ \alpha\beta}\nabla_\gamma \tilde{\psi}^{\kappa\sigma\gamma} - R_{\kappa\sigma}^{\cdot\ \cdot\ \kappa\alpha}\nabla_\gamma \tilde{\psi}^{\beta\sigma\gamma} + R_{\kappa\sigma}^{\cdot\ \cdot\ \kappa\beta}\nabla_\gamma \tilde{\psi}^{\alpha\sigma\gamma}.
\end{aligned}
$$

Upon relabelling dummy indices and collecting terms, this becomes

$$
\begin{aligned}
\widetilde{\Psi}^{\alpha\beta} &= R^{\gamma\ \cdot\ \cdot\ \alpha}_{\cdot\ \sigma\kappa}\nabla_\gamma \tilde{\psi}^{\kappa\beta\sigma} + R^{\gamma\ \cdot\ \cdot\ \beta}_{\cdot\ \sigma\kappa}\nabla_\gamma \tilde{\psi}^{\alpha\kappa\sigma} \\
&\quad - R_{\kappa\sigma}^{\cdot\ \cdot\ \alpha\beta}\nabla_\gamma \tilde{\psi}^{\kappa\sigma\gamma} - R_{\kappa\sigma}^{\cdot\ \cdot\ \kappa\alpha}\nabla_\gamma \tilde{\psi}^{\beta\sigma\gamma} + R_{\kappa\sigma}^{\cdot\ \cdot\ \kappa\beta}\nabla_\gamma \tilde{\psi}^{\alpha\sigma\gamma}.
\end{aligned} \tag{A1.9}
$$

Using the cyclic identity for the curvature tensor and the fact that $\tilde{\psi}^{\alpha\beta\gamma}$ is a completely antisymmetric quantity, we can write

$$
R^{\mu\ \cdot\ \cdot\ \nu}_{\cdot\ \sigma\kappa}\nabla_\gamma \tilde{\psi}^{\lambda\kappa\sigma} = (R^{\nu\ \cdot\ \cdot\ \mu}_{\cdot\ \sigma\kappa} + R_{\kappa\sigma}^{\cdot\ \cdot\ \mu\nu})\nabla_\gamma \tilde{\psi}^{\lambda\kappa\sigma} = (-R^{\mu\ \cdot\ \cdot\ \nu}_{\cdot\ \sigma\kappa} + R_{\kappa\sigma}^{\cdot\ \cdot\ \mu\nu})\nabla_\gamma \tilde{\psi}^{\lambda\kappa\sigma},
$$

and therefore

$$
R^{\mu\ \cdot\ \cdot\ \nu}_{\cdot\ \sigma\kappa}\nabla_\gamma \tilde{\psi}^{\lambda\kappa\sigma} = \frac{1}{2}R_{\kappa\sigma}^{\cdot\ \cdot\ \mu\nu}\nabla_\gamma \tilde{\psi}^{\lambda\kappa\sigma}. \tag{A1.10}
$$

This can be used to rewrite (A1.9) as

$$
\begin{aligned}
\widetilde{\Psi}^{\alpha\beta} &= \frac{1}{2}R_{\kappa\sigma}^{\cdot\ \cdot\ \gamma\alpha}\nabla_\gamma \tilde{\psi}^{\kappa\beta\sigma} - \frac{1}{2}R_{\kappa\sigma}^{\cdot\ \cdot\ \gamma\beta}\nabla_\gamma \tilde{\psi}^{\alpha\kappa\sigma} \\
&\quad - R_{\kappa\sigma}^{\cdot\ \cdot\ \alpha\beta}\nabla_\gamma \tilde{\psi}^{\gamma\kappa\sigma} - R_{\kappa\sigma}^{\cdot\ \cdot\ \kappa\alpha}\nabla_\gamma \tilde{\psi}^{\beta\sigma\gamma} + R_{\kappa\sigma}^{\cdot\ \cdot\ \kappa\beta}\nabla_\gamma \tilde{\psi}^{\alpha\sigma\gamma}.
\end{aligned} \tag{A1.11}
$$

Factoring out the covariant derivative in (A1.11) yields

$$
\begin{aligned}
\widetilde{\Psi}^{\alpha\beta} = \nabla_\gamma \Big\{ & \frac{1}{2} R_{\kappa\sigma}{}^{\gamma\alpha} \tilde{\psi}^{\kappa\beta\sigma} - \frac{1}{2} R_{\kappa\sigma}{}^{\gamma\beta} \tilde{\psi}^{\alpha\kappa\sigma} - R_{\kappa\sigma}{}^{\alpha\beta} \tilde{\psi}^{\gamma\kappa\sigma} - R_{\kappa\sigma}{}^{\kappa\alpha} \tilde{\psi}^{\beta\sigma\gamma} \\
& + R_{\kappa\sigma}{}^{\kappa\beta} \tilde{\psi}^{\alpha\sigma\gamma} \Big\} - \Big\{ \frac{1}{2} (\nabla_\gamma R_{\kappa\sigma}{}^{\gamma\alpha}) \tilde{\psi}^{\kappa\beta\sigma} + \frac{1}{2} (\nabla_\gamma R_{\kappa\sigma}{}^{\gamma\beta}) \tilde{\psi}^{\alpha\kappa\sigma} \\
& - (\nabla_\gamma R_{\kappa\sigma}{}^{\alpha\beta}) \tilde{\psi}^{\gamma\kappa\sigma} - (\nabla_\gamma R_{\kappa\sigma}{}^{\kappa\alpha}) \tilde{\psi}^{\beta\sigma\gamma} + (\nabla_\gamma R_{\kappa\sigma}{}^{\kappa\beta}) \tilde{\psi}^{\alpha\sigma\gamma} \Big\}
\end{aligned}
\tag{A1.12}
$$

Application of the Bianchi identity and the complete antisymmetry of $\tilde{\psi}^{\kappa\beta\sigma}$ gives (after relabelling dummy indices)

$$
\begin{aligned}
\frac{1}{2} (\nabla_\gamma R_{\kappa\sigma}{}^{\gamma\alpha}) \tilde{\psi}^{\kappa\beta\sigma} &= (\nabla_\gamma R_{\kappa\sigma}{}^{\kappa\alpha}) \tilde{\psi}^{\beta\sigma\gamma}, \\
\frac{1}{2} (\nabla_\gamma R_{\kappa\sigma}{}^{\gamma\beta}) \tilde{\psi}^{\alpha\kappa\sigma} &= -(\nabla_\gamma R_{\kappa\sigma}{}^{\kappa\beta}) \tilde{\psi}^{\alpha\sigma\gamma}, \\
(\nabla_\gamma R_{\kappa\sigma}{}^{\alpha\beta}) \tilde{\psi}^{\gamma\kappa\sigma} &= (\nabla_{[\gamma} R_{\kappa\sigma]}{}^{\alpha\beta}) \tilde{\psi}^{\gamma\kappa\sigma} \equiv 0.
\end{aligned}
\tag{A1.13}
$$

The second bracketed term in (A1.12) therefore vanishes and

$$
\begin{aligned}
\widetilde{\Psi}^{\alpha\beta} = \nabla_\gamma \Big\{ & \frac{1}{2} R_{\kappa\sigma}{}^{\gamma\alpha} \tilde{\psi}^{\kappa\beta\sigma} - \frac{1}{2} R_{\kappa\sigma}{}^{\gamma\beta} \tilde{\psi}^{\alpha\kappa\sigma} - R_{\kappa\sigma}{}^{\alpha\beta} \tilde{\psi}^{\gamma\kappa\sigma} \\
& - R_{\kappa\sigma}{}^{\kappa\alpha} \tilde{\psi}^{\beta\sigma\gamma} + R_{\kappa\sigma}{}^{\kappa\beta} \tilde{\psi}^{\alpha\sigma\gamma} \Big\}.
\end{aligned}
\tag{A1.14}
$$

Equation (A1.7) thus takes the form

$$
\begin{aligned}
\nabla_\sigma \nabla^\sigma \nabla_\gamma \tilde{\psi}^{\alpha\beta\gamma} - 3 R_{\kappa\sigma}{}^{[\alpha\beta} \nabla_\gamma \tilde{\psi}^{\kappa]\sigma\gamma} = \nabla_\gamma \Big\{ & \nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} + R^{\gamma}{}_{\sigma\kappa}{}^{\alpha} \tilde{\psi}^{\kappa\beta\sigma} \\
& + R^{\gamma}{}_{\sigma\kappa}{}^{\beta} \tilde{\psi}^{\alpha\kappa\sigma} + R^{\gamma}{}_{\sigma\kappa}{}^{\sigma} \tilde{\psi}^{\alpha\beta\kappa} + \frac{1}{2} R_{\kappa\sigma}{}^{\gamma\alpha} \tilde{\psi}^{\kappa\beta\sigma} - \frac{1}{2} R_{\kappa\sigma}{}^{\gamma\beta} \tilde{\psi}^{\alpha\kappa\sigma} \\
& - R_{\kappa\sigma}{}^{\alpha\beta} \tilde{\psi}^{\gamma\kappa\sigma} - R_{\kappa\sigma}{}^{\kappa\alpha} \tilde{\psi}^{\beta\sigma\gamma} + R_{\kappa\sigma}{}^{\kappa\beta} \tilde{\psi}^{\alpha\sigma\gamma} \Big\}.
\end{aligned}
$$

Using (A1.10) this can be written as

$$
\begin{aligned}
\nabla_\sigma \nabla^\sigma \nabla_\gamma \tilde{\psi}^{\alpha\beta\gamma} - 3 R_{\kappa\sigma}{}^{[\alpha\beta} \nabla_\gamma \tilde{\psi}^{\kappa]\sigma\gamma} &= \nabla_\gamma \Big\{ \nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} + R_{\kappa\sigma}{}^{\gamma\alpha} \tilde{\psi}^{\kappa\beta\sigma} + \\
& + R_{\kappa\sigma}{}^{\gamma\beta} \tilde{\psi}^{\alpha\kappa\sigma} - R_{\kappa\sigma}{}^{\alpha\beta} \tilde{\psi}^{\gamma\kappa\sigma} + R_{\kappa\sigma}{}^{\kappa\alpha} \tilde{\psi}^{\beta\gamma\sigma} - R_{\kappa\sigma}{}^{\kappa\beta} \tilde{\psi}^{\alpha\gamma\sigma} + R_{\kappa\sigma}{}^{\kappa\gamma} \tilde{\psi}^{\alpha\beta\sigma} \Big\}, \\
&= \nabla_\gamma \Big\{ \nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} + 3 R_{\kappa\sigma}{}^{\kappa[\alpha} \tilde{\psi}^{\beta\gamma]\sigma} - 3 R_{\kappa\sigma}{}^{[\alpha\beta} \tilde{\psi}^{\gamma]\kappa\sigma} \Big\}, \\
&= \nabla_\gamma \Big\{ \nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} - 3 R_{\sigma}{}^{[\alpha} \tilde{\psi}^{\beta\gamma]\sigma} - 3 R_{\kappa\sigma}{}^{[\alpha\beta} \tilde{\psi}^{\gamma]\kappa\sigma} \Big\},
\end{aligned}
$$

which is the required result. ∎

**References.**

[1] Synge, J.L.: *Relativity: The General Theory* (North-Holland, New York, NY, 1966).

[2] Bressan, A.: *Relativistic Theories of Materials* (Springer-Verlag, New York, NY, 1978).

[3] Schouten, J.A.: *Ricci Calculus, Second Ed.* (Springer-Verlag, New York, NY, 1954).

[4] Flanders, H.: *Differential Forms With Applications to the Physical Sciences* (Academic Press, New York, NY, 1963).

[5] Barut, A.O.: *Electrodynamics and Classical Theory of Fields and Particles* (Macmillan Co., New York, NY, 1964).

[6] Azzam, R.M.A. and Bashara, N.M.: *Ellipsometry and Polarized Light* (North-Holland, New York, NY, 1977).

[7] Bloembergen, N.: *Nonlinear Optics* (W.A. Benjamin, Reading, MA, 1977).

[8] Yariv, A.: *Optical Electronics* (Holt, Rinehart, Winston, New York, NY, 1985).

[9] Toupin, R.A.: *Int. J. Engng. Sci.*, **1**, 101(1963).

[10] Kittel, C. *Introduction to Solid State Physics, Sixth Ed.* (John Wiley and Sons, New York, NY, 1986).

[11] Stratton, J.A.: *Electromagnetic Theory.* (McGraw-Hill Book Company, Inc., New York, NY, 1941).

[12] Born, M and Wolf, E.: *Principles of Optics, Sixth (Corrected) Ed.* (Pergamon Press, New York, NY, 1980).

## Recent IMA Preprints

| # | Author/s | Title |
|---|---|---|
| 818 | **Avner Friedman and Bei Hu** | The Stefan problem with kinetic condition at the free boundary |
| 819 | **M.A. Grinfeld** | The stress driven instabilities in crystals: mathematical models and physical manifestations |
| 820 | **Bei Hu and Lihe Wang** | A free boundary problem arising in electrophotography: solutions with connected toner region |
| 821 | **Yongzhi Xu, T. Craig Poling, and Trent Brundage** | Direct and inverse scattering of time harmonic acoustic waves in an inhomogeneous shallow ocean |
| 822 | **Steven J. Altschuler** | Singularities of the curve shrinking flow for space curves |
| 823 | **Steven J. Altschuler and Matthew A. Grayson** | Shortening space curves and flow through singularities |
| 824 | **Tong Li** | On the Riemann problem of a combustion model |
| 825 | **L.A. Peletier & W.C. Troy** | Self-similar solutions for diffusion in semiconductors |
| 826 | **C.J. van Duijn, L.A. Peletier & R.J. Schotting** | On the analysis of brine transport in porous media |
| 827 | **Minkyu Kwak** | Finite dimensional description of convective reaction-diffusion equations |
| 828 | **Minkyu Kwak** | Finite dimensional inertial forms for the 2D Navier–Stokes equations |
| 829 | **Victor A. Galaktionov and Sergey A. Posashkov** | On some monotonicity in time properties for a quasilinear parabolic equation with source |
| 830 | **Victor A. Galaktionov** | Remark on the fast diffusion equation in a ball |
| 831 | **Hi Jun Choe and Lihe Wang** | A regularity theory for degenerate vector valued variational inequalities |
| 832 | **Vladimir I. Oliker and Nina N. Uraltseva** | Evolution of nonparametric surfaces with speed depending on curvature, II. The mean curvature case. |
| 833 | **S. Kamin and W. Liu** | Large time behavior of a nonlinear diffusion equation with a source |
| 834 | **Shoshana Kamin and Juan Luis Vazquez** | Singular solutions of some nonlinear parabolic equations |
| 835 | **Bernhard Kawohl and Robert Kersner** | On degenerate diffusion with very strong absorption |
| 836 | **Avner Friedman and Fernando Reitich** | Parameter identification in reaction-diffusion models |
| 837 | **E.G. Kalnins, H.L. Manocha and Willard Miller, Jr.** | Models of $q$-algebra representations I. Tensor products of special unitary and oscillator algebras |
| 838 | **Robert J. Sacker and George R. Sell** | Dichotomies for linear evolutionary equations in Banach spaces |
| 839 | **Oscar P. Bruno and Fernando Reitich** | Numerical solution of diffraction problems: a method of variation of boundaries |
| 840 | **Oscar P. Bruno and Fernando Reitich** | Solution of a boundary value problem for Helmholtz equation via variation of the boundary into the complex domain |
| 841 | **Victor A. Galaktionov and Juan L. Vazquez** | Asymptotic behaviour for an equation of superslow diffusion. The Cauchy problem |
| 842 | **Josephus Hulshof and Juan Luis Vazquez** | The Dipole solution for the porous medium equation in several space dimensions |
| 843 | **Shoshana Kamin and Juan Luis Vazquez** | The propagation of turbulent bursts |
| 844 | **Miguel Escobedo, Juan Luis Vazquez and Enrike Zuazua** | Source-type solutions and asymptotic behaviour for a diffusion-convection equation |
| 845 | **Marco Biroli and Umberto Mosco** | Discontinuous media and Dirichlet forms of diffusion type |
| 846 | **Stathis Filippas and Jong-Shenq Guo** | Quenching profiles for one-dimensional semilinear heat equations |
| 847 | **H. Scott Dumas** | A Nekhoroshev-like theory of classical particle channeling in perfect crystals |
| 848 | **R. Natalini and A. Tesei** | On a class of perturbed conservation laws |
| 849 | **Paul K. Newton and Shinya Watanabe** | The geometry of nonlinear Schrödinger standing waves |
| 850 | **S.S. Sritharan** | On the nonsmooth verification technique for the dynamic programming of viscous flow |
| 851 | **Mario Taboada and Yuncheng You** | Global attractor, inertial manifolds and stabilization of nonlinear damped beam equations |
| 852 | **Shigeru Sakaguchi** | Critical points of solutions to the obstacle problem in the plane |
| 853 | **F. Abergel, D. Hilhorst and F. Issard-Roch** | On a dissolution-growth problem with surface tension in the neighborhood of a stationary solution |
| 854 | **Erasmus Langer** | Numerical simulation of MOS transistors |
| 855 | **Haïm Brezis and Shoshana Kamin** | Sublinear elliptic equations in $\mathbb{R}^n$ |
| 856 | **Johannes C.C. Nitsche** | Boundary value problems for variational integrals involving surface curvatures |
| 857 | **Chao–Nien Chen** | Multiple solutions for a semilinear elliptic equation on $\mathbb{R}^N$ with nonlinear dependence on the gradient |
| 858 | **D. Brochet, X. Chen and D. Hilhorst** | Finite dimensional exponential atttractor for the phase field model |
| 859 | **Joseph D. Fehribach** | Mullins-Sekerka stability analysis for melting-freezing waves in helium-4 |
| 860 | **Walter Schempp** | Quantum holography and neurocomputer architectures |
| 861 | **D.V. Anosov** | An introduction to Hilbert's 21st problem |
| 862 | **Herbert E Huppert and M Grae Worster** | Vigorous motions in magma chambers and lava lakes |

863 **Robert L. Pego and Michael I. Weinstein**, A class of eigenvalue problems, with applications to instability of solitary waves
864 **Mahmoud Affouf**, Numerical study of a singular system of conservation laws arising in enhanced oil reservoirs
865 **Darin Beigie, Anthony Leonard and Stephen Wiggins**, The dynamics associated with the chaotic of tangles two dimensional quasiperiodic vector fields: theory and applications
866 **Gui–Qiang Chen and Tai–Ping Liu**, Zero relaxation and dissipation limits for hyperbolic conservation laws
867 **Gui–Qiang Chen and Jian–Guo Liu**, Convergence of second–order schemes for isentropic gas dynamics
868 **Aleksander M. Simon and Zbigniew J. Grzywna**, On the Larché–Cahn theory for stress-induced diffusion
869 **Jerzy Łuczka, Adam Gadomski and Zbigniew J. Grzywna**, Growth driven by diffusion
870 **Mitchell Luskin and Tsorng-Whay Pan**, Nonplanar shear flows for nonaligning nematic liquid crystals
871 **Mahmoud Affouf**, Unique global solutions of initial-boundary value problems for thermodynamic phase transitions
872 **Richard A. Brualdi, Keith L. Chavey and Bryan L. Shader**, Rectangular $L$-matrices
873 **Xinfu Chen, Avner Friedman and Bei Hu**, The thermistor problem with zero–one conductivity II
874 **Raoul LePage**, Controlling a diffusion toward a large goal and the Kelly principle
875 **Raoul LePage**, Controlling for optimum growth with time dependent returns
876 **Marc Hallin and Madan L. Puri**, Rank tests for time series analysis a survey
877 **V.A. Solonnikov**, Solvability of an evolution problem of thermocapillary convection in an infinite time interval
878 **Horia I. Ene and Bogdan Vernescu**, Viscosity dependent behaviour of viscoelastic porous media
879 **Kaushik Bhattacharya**, Self-accommodation in martensite
880 **D. Lewis, T. Ratiu, J.C. Simo and J.E. Marsden**, The heavy top: a geometric treatment
881 **Leonid V. Kalachev**, Some applications of asymptotic methods in semiconductor device modeling
882 **David C. Dobson**, Phase reconstruction via nonlinear least-squares
883 **Patricio Aviles and Yoshikazu Giga**, Minimal currents, geodesics and relaxation of variational integrals on mappings of bounded variation
884 **Patricio Aviles and Yoshikazu Giga**, Partial regularity of least gradient mappings
885 **Charles R. Johnson and Michael Lundquist**, Operator matrices with chordal inverse patterns
886 **B.J. Bayly**, Infinitely conducting dynamos and other horrible eigenproblems
887 **Charles M. Elliott and Stefan Luckhaus**, 'A generalised diffusion equation for phase separation of a multi-component mixture with interfacial free energy'
888 **Christian Schmeiser and Andreas Unterreiter**, The derivation of analytic device models by asymptotic methods
889 **LeRoy B. Beasley and Norman J. Pullman**, Linear operators that strongly preserve the index of imprimitivity
890 **Jerry Donato**, The Boltzmann equation with lie and cartan
891 **Thomas R. Hoffend Jr., Peter Smereka and Roger J. Anderson**, A method for resolving the laser induced local heating of moving magneto-optical recording media
892 **E.G. Kalnins, Willard Miller, Jr. and Sanchita Mukherjee**, Models of $q$-algebra representations: the group of plane motions
893 **T.R. Hoffend Jr. and R.K. Kaul**, Relativistic theory of superpotentials for a nonhomogeneous, spatially isotropic medium
894 **Reinhold von Schwerin**, Two metal deposition on a microdisk electrode
895 **Vladimir I. Oliker and Nina N. Uraltseva**, Evolution of nonparametric surfaces with speed depending on curvature, III. Some remarks on mean curvature and anisotropic flows
896 **Wayne Barrett, Charles R. Johnson, Raphael Loewy and Tamir Shalom**, Rank incrementation via diagonal perturbations
897 **Thierry Dombre, Alain Pumir, and Eric D. Siggia**, On the interface dynamics for convection in porous media
898 **Mingxiang Chen, Xu-Yan Chen and Jack K. Hale**, Structural stability for time-periodic one-dimensional parabolic equations
899 **Hong-Ming Yin**, Global solutions of Maxwell's equations in an electromagnetic field with the temperature-dependent electrical conductivity
900 **Robert Grone, Russell Merris and William Watkins**, Laplacian unimodular equivalence of graphs
901 **Miroslav Fiedler**, Structure-ranks of matrices
902 **Miroslav Fiedler**, An estimate for the nonstochastic eigenvalues of doubly stochastic matrices
903 **Miroslav Fiedler**, Remarks on eigenvalues of Hankel matrices
904 **Charles R. Johnson, D.D. Olesky, Michael Tsatsomeros and P. van den Driessche**, Spectra with positive elementary symmetric functions
905 **Pierre-Alain Gremaud**, Thermal contraction as a free boundary problem
906 **K.L. Cooke, Janos Turi and Gregg Turner**, Stabilization of hybrid systems in the presence of feedback delays
907 **Robert P. Gilbert and Yongzhi Xu**, A numerical transmutation approach for underwater sound propagation
908 **LeRoy B. Beasley, Richard A. Brualdi and Bryan L. Shader**, Combinatorial orthogonality
909 **Richard A. Brualdi and Bryan L. Shader**, Strong hall matrices
910 **Håkan Wennerström and David M. Anderson**, Difference versus Gaussian curvature energies; monolayer versus bilayer curvature energies applications to vesicle stability
911 **Shmuel Friedland**, Eigenvalues of almost skew symmetric matrices and tournament matrices