PASJ: Publ. Astron. Soc. Japan **61**, 367–374, 2009 April 25

Relativistic Variable Eddington Factor in a Relativistic Plane-Parallel Flow

Jun FUKUE
Astronomical Institute, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582
fukue@cc.osaka-kyoiku.ac.jp

(Received 2008 October 10; accepted 2008 December 5)

Abstract

We examine the behavior of the variable Eddington factor for a relativistically moving radiative flow in the vertical direction. We adopt the "one-tau photo-oval" approximation in the comoving frame. Namely, the comoving observer sees radiation coming from a closed surface where the optical depth measured from the observer is unity; such a surface is called a *one-tau photo-oval*. In general, the radiative intensity emitted by the photo-oval is non-uniform and anisotropic. Furthermore, the photo-oval surface has a relative velocity with respect to the comoving observer, and therefore the observed intensity suffers from the Doppler effect and aberration. In addition, the background intensity usually depends on the optical depth. All of these introduce *anisotropy* to the radiation field observed by the comoving observer. As a result, the relativistic Eddington factor, f, generally depends on the optical depth τ, the four velocity u, and the velocity gradient $du/d\tau$. In the case of a plane-parallel vertical flow, we found that the relativistic variable Eddington factor, f, generally decreases as the velocity gradient increases, but it increases as the velocity increases for some cases. When the comoving radiation field is uniform, it is well approximated by $3f \sim 1/[\,1 + (16/15)(-du/\gamma d\tau) + (-du/\gamma d\tau)^{1.6-2}]$. When the radiation field in the inertial frame is uniform, on the other hand, it is expressed as $f = (1+3\beta^2)/(3+\beta^2)$. These relativistic variable Eddington factors can be used in various relativistic radiatively-driven flows, such as black-hole accretion flows, relativistic astrophysical jets and outflows, and relativistic explosions like gamma-ray bursts.

Key words: accretion, accretion disks — astrophysical jets — gamma-ray bursts — radiative transfer — relativity

1. Relativistic Variable Eddington Factor

In the moment formalism of (relativistic) radiation hydrodynamics, in order to close the moment equations truncated at a finite order, we need a *closure relation*, such as the Eddington approximation or more complex expressions. In non-relativistic static atmospheres (Chandrasekhar 1960; Mihalas 1970; Rybicki & Lightman 1979; Mihalas & Mihalas 1984; Shu 1991; Peraiah 2002; Castor 2004), for example, many researchers have used the Eddington approximation and variable Eddington factors as closure relations (Milne 1921; Eddington 1926; Kosirev 1934; Chandrasekhar 1934; Hummer & Rybicki 1971; Wilson et al. 1972; Tamazawa et al. 1975; Unno & Kondo 1976; Masaki & Unno 1978). As is well known, the standard Eddington approximation with an Eddington factor of $1/3$ is valid when the radiation field is almost *isotropic*. In the spherically symmetric case, where the radiation field becomes anisotropic towards the outer optically thin region, they often used the variable Eddington factor, which depends on the optical depth (e.g., Tamazawa et al. 1975).

In relativistically moving flows (Mihalas et al. 1975, 1976a, b; Thorne 1981; Thorne et al. 1981; Flammang 1982, 1984; Mihalas & Mihalas 1984; Nobili et al. 1991, 1993; Kato et al. 1998, 2008; Castor 2004; Mihalas & Auer 2001; Park 2001, 2006; Takahashi 2007), on the other hand, the standard Eddington approximation is adopted in the comoving frame, and is then transformed to the inertial frame, if necessary (Castor 1972; Hsieh & Spiegel 1976; Fukue et al. 1985; Sen & Wilson 1993; Baschek et al. 1995, 1997; Kato et al. 1998, 2008). Even for a comoving observer, however, the comoving radiation field may become *anisotropic* due to various reasons, as shown below. Instead of the standard Eddington approximation in the comoving frame, for such a relativistic case, several types of a *variable Eddington factor*, which depends on the velocity and its gradient as well as the optical depth, were proposed (Fukue 2006; Fukue & Akizuki 2006, 2007; Akizuki & Fukue 2008; Fukue 2008a, b; Koizumi & Umemura 2008). However, the moment formalism of relativistic radiation hydrodynamics is yet *incomplete*, in the sense that we do not have a sufficiently adequate closure relation.

There are mainly three reasons that cause the anisotropy of the comoving radiation field in the relativistically moving flow. (1) The optical depth effect. In the outer region where the flow becomes optically thin, similar to the non-relativistic case, the comoving radiation field would be anisotropic. (2) The velocity gradient effect. When the radiative flow is accelerated up to the relativistic regime, and there is a strong velocity gradient in the direction of the flow, the velocity fields as well as the density distribution are no longer uniform, even in the comoving frame. Then, the comoving radiation field also becomes non-uniform. Furthermore, due to the relative speed between the observer and the radiation field, the observed radiation field suffers from the Doppler effect. As a result, the comoving radiation field becomes anisotropic. (3) The effect of the relativistic speed itself. Although we do not know the radiative intensity in the inertial frame, in the highly relativistic regime, it is generally affected by a strong aberration effect when it is converted to the comoving frame. As a result, except for some special cases, the comoving radiative intensity generally has strong anisotropy.

Hence, the relativistic variable Eddington factor (RVEF) generally depends on the optical depth, the flow velocity, and the velocity gradient. In Fukue (2006) and Akizuki and Fukue (2008), they proposed such variable Eddington factors from physical viewpoints in the plane-parallel and spherical cases, respectively. In Fukue (2008a, b), on the other hand, he derived semi-analytically variable Eddington factors for the plane-parallel flows in the vertical direction, and found that the Eddington factor decreases as the velocity-gradient increases. In these studies (Fukue 2008a, b), however, the treatments were rather restrictive; e.g., the comoving radiation field was assumed to be uniform. Thus, in the present paper we extend the previous work under the similar procedure to examine the above three reasons more extensively.

In the next section we describe the shape of the surface (one-tau photo-oval), where the optical depth measured by the comoving observer is unity, and derive an expression of the one-tau photo-oval. In section 3, we discuss how we numerically calculated the comoving radiation field within the photo-oval, and the relativistic variable Eddington factor for several cases. The final section is devoted to concluding remarks.

2. One-Tau Photo-Oval and Photo-Vessel

Let us suppose a relativistic radiative flow, which is accelerated in the vertical (z) direction, and a comoving observer, who moves upward with the flow (figure 1).

Sufficiently deep inside the flow, where the optical depth is very large, the mean free path of photons is very short. In such a case, within the range of the mean free path, for the comoving observer the flow is seen to be almost uniform; the velocity gradient and the resultant density gradient are negligible. Hence, the mean free path is the same in all directions, and the shape of the surface where the optical depth measured from the comoving observer is unity, is almost a sphere; we call it a *one-tau photo-sphere* (a dashed circle in figure 1). As a result, the comoving radiation field is nearly isotropic, and the usual Eddington approximation in the comoving frame is valid.

On the other hand, if the velocity gradient becomes very large, the behavior of the radiation field may be affected by changes in the hydrodynamics of the flow, even on a length-scale comparable to that of the photon mean free path. Hence, the mean free path becomes longer in the downstream direction than in the upstream and other directions, and the shape of the one-tau range elongates in the downstream direction; we call it a *one-tau photo-oval* (a dashed oval in figure 1). As a result, the comoving radiation field becomes *anisotropic* due to various reasons, as described in the introduction, and we should modify the usual Eddington approximation.

In addition, when the optical depth is sufficiently small and/or the velocity gradient is suffiently large, the mean free path of photons in the downstream direction become less than unity. Hence, the shape of the one-tau range is open in the downstream direction; we call it a *one-tau photo-vessel* (a dashed hemi-circle in figure 1).

In order to obtain an appropriate form of the RVEF in these regimes, we thus carefully treat and examine the radiation field in a comoving frame.

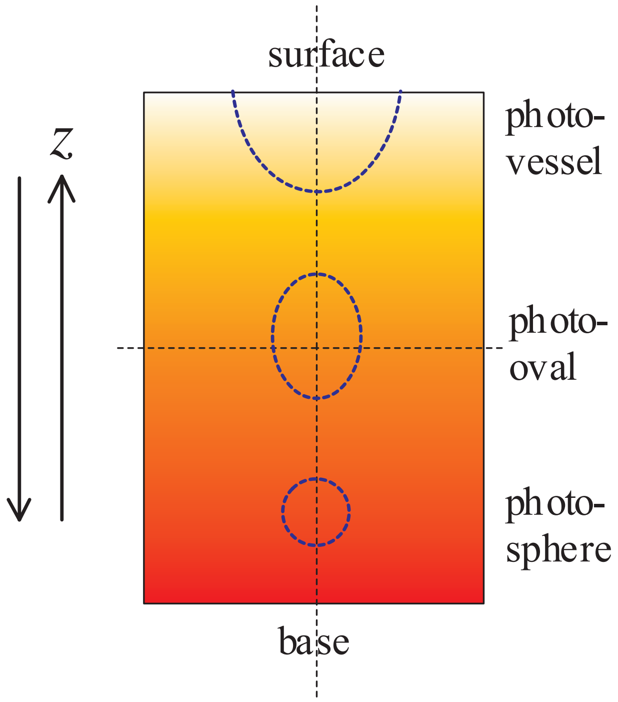


Fig. 1. Schematic picture of a relativistic radiative flow in the vertical direction. The flow is accelerated in the vertical (z) direction, and has a velocity gradient. The dashed curves are one-tau photo-ovals observed by a comoving observer.

In previous papers (Fukue 2008a, b), we have examined the one-tau photo-oval, derived the variable Eddington factor semi-analytically, and proved that the Eddington factor decreases with the velocity gradient. In these studies, however, the treatments were rather restricted, and in the present study we thus examine the behavior of the RVEF for more general cases, and seek an appropriate form of the Eddington factor, $f(\tau, u, du/d\tau)$, where τ is the optical depth and u $(= \gamma\beta)$ is the four velocity, β and γ $(= 1/\sqrt{1-\beta^2})$ being the normalized velocity and the Lorentz factor, respectively.

We first derive the shape of the one-tau photo-oval using the optical depth τ, while in the previous papers we have used the vertical coordinate z.

We define the one-tau photo-oval as the surface where the optical depth measured by the comoving observer is unity. The situation is schematically illustrated in figure 2. We assume that the comoving observer in the vertical flow is located at $z = z_0$ or $\tau = \tau_0$, where the flow four-speed is $u = u_0$. In the s-direction, which forms an angle θ with the downstream direction, the mean free path of photons is l_0. The relation among these quantities is

$$z - z_0 = s \cos\theta. \quad (1)$$

The continuity equation for the stationary, one-dimensional relativistic flow is

$$\rho c u = J \ (= \text{const.}), \quad (2)$$

where ρ is the proper gas density, u the four-velocity, and J the mass-flow rate per unit area. In addition, the optical depth

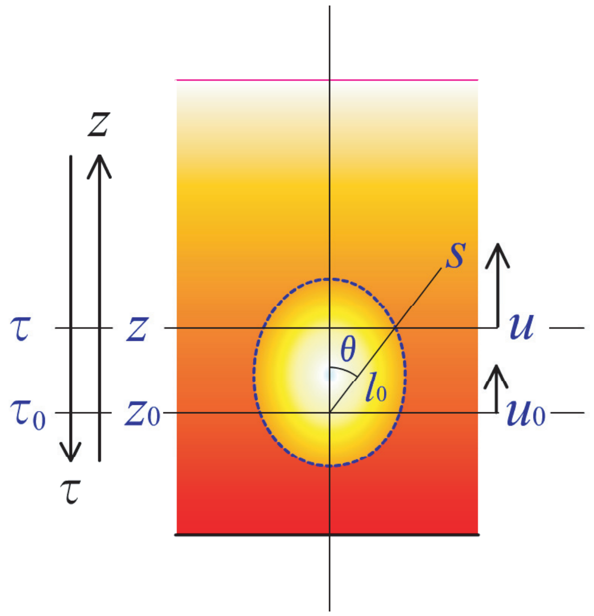


Fig. 2. One-tau photo-oval around a comoving observer in the vertical (z) one-dimensional radiative flow. The comoving observer is located at $\tau = \tau_0$, where the flow four speed is $u = u_0$. In the s-direction, that forms an angle θ with the downstream direction, the mean free path is set to be l_0.

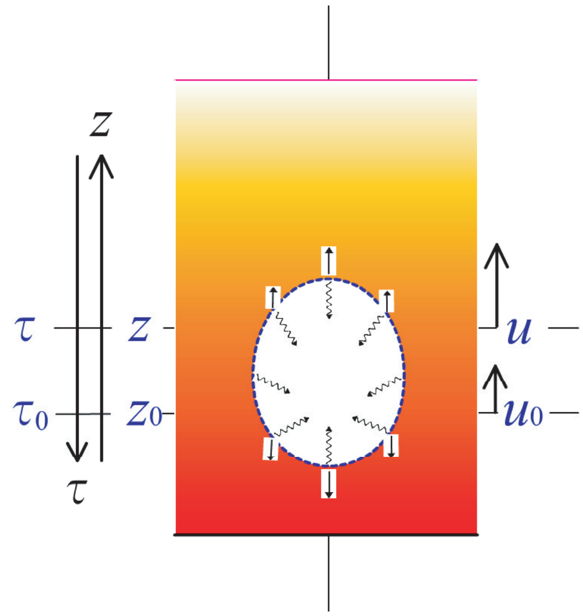


Fig. 3. Radiation field in the one-tau photo-oval around a comoving observer in the vertical one-dimensional radiative flow. The radiation field may become anisotropic, since in general the emitted intensity is not uniform and isotropic, it is redshifted due to the velocity difference, and there is aberration between the comoving and inertial frames.

τ is defined by

$$d\tau \equiv -\kappa\rho dz = -\kappa\rho ds \cos\theta, \tag{3}$$

where κ is the opacity, which is assumed to be constant in the present analysis, and we used equation (1).

In this paper we use a linear approximation for the flow field; that is to say, around the position of the comoving observer the flow four speed is expanded as

$$u = u_0 + \left.\frac{du}{d\tau}\right|_0 (\tau - \tau_0), \tag{4}$$

where $du/d\tau|_0$ is assumed to be constant.

Using equation (3), the optical depth τ_s along the s-direction is expressed as

$$\tau_s = \int_0^{l_0} \kappa\rho ds = -\int_{\tau_0}^{\tau} \frac{d\tau}{\cos\theta} = \frac{\tau_0 - \tau}{\cos\theta}. \tag{5}$$

Thus, the shape of the one-tau photo-oval, where $\tau_s = 1$, is finally determined by the condition

$$\tau = \tau_0 - \cos\theta. \tag{6}$$

Obviously, for $\tau_0 > 1$ the one-tau region is close to be a photo-oval, whereas it is open towards a downstream direction to be a photo-vessel for $\tau_0 < 1$.

3. Comoving Radiation Field and the Relativistic Variable Eddington Factor

We can now compute the radiation field received by the comoving observer, and derive an expression for the Eddington factor in the comoving frame (figure 3).

In a static and optically thick atmosphere, the radiation field is isotropic and uniform. In the present moving atmosphere, on the other hand, there are three reasons for which the radiation field is not uniform and isotropic, as described in the introduction: the optical depth effect, the velocity gradient effect, and the effect of the relativistic speed itself.

First, the radiative intensity, I, in the comoving frame emitted from the one-tau photo-oval walls is generally a function of τ and μ. In general the radiative intensity increases with the optical depth, and therefore, the intensity from the upstream direction is slightly larger than that from the downstream direction in the present one-dimensional vertical flow. This effect of non-uniformity of intensity generally acts as a force to accelerate a comoving observer. This optical depth effect is prominent for small optical depth, similar to the static atmosphere.

Second, although in the usual static atmosphere the μ-dependence is safely ignored for large optical depth, this is not the present relativistic case. Namely, in the relativistic flow there appear the aberration and Doppler effects between the inertial and comoving frames. Hence, except for very special cases, the comoving radiative intensity would have strong anisotropy.

Thirdly, if there is a velocity gradient in the flow, the intensity observed by the comoving observer is redshifted (Doppler shifted) due to the velocity difference between the comoving observer and the one-tau photo-oval walls. In an accelerating flow, where the flow speed increases toward the downstream direction, the relative velocity is generally positive (figure 3), except for some special direction ($\theta = \pi/2$). Hence, the Doppler shift of intensity also causes anisotropy of the radiation field at the position of the comoving observer.

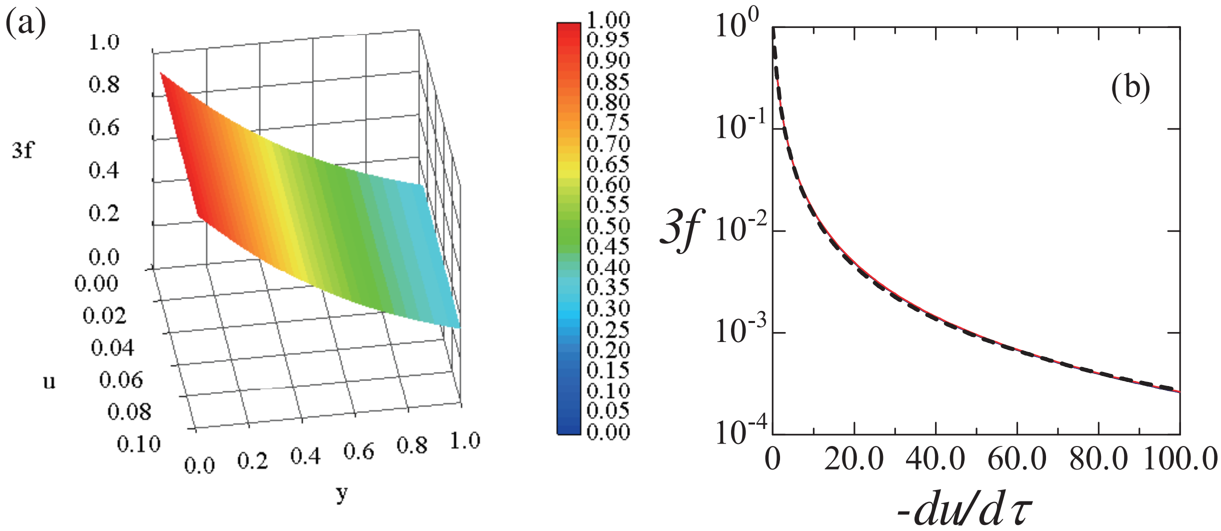


Fig. 4. Behavior of the relativistic variable Eddington factor, $f(u, -du/d\tau)$, multiplied by 3 in the case of small u. (a) $3f$ are plotted as a function of the four velocity u_0 (x axis) and the four-velocity gradient $-du/d\tau|_0$ (y axis). (b) $3f$ are plotted as a function of the four-velocity gradient $-du/d\tau|_0$ for four fixed u_0 (= 0.01, 0.05, 0.1; solid curves), and one fitting curve is also shown (dashed curve).

Neglecting the local emissivity, the frequency-integrated intensity, $I_{\rm co}$, observed in the comoving frame at the position τ_0 of the comoving observer is related to the frequency-integrated intensity, $I(\tau, \mu)$, emitted at the one-tau photo-oval and to the redshift z by

$$I_{\rm co} = \frac{I(\tau, \mu)}{(1+z)^4}. \tag{7}$$

The relative speed $\Delta\beta$ between the comoving observer and the one-tau photo-oval walls is given by the relativistic summation law as

$$\Delta\beta = \frac{\beta - \beta_0}{1 - \beta\beta_0}, \tag{8}$$

where

$$\beta = \frac{u}{\sqrt{1+u^2}}, \quad \text{and} \tag{9}$$

$$\beta_0 = \frac{u_0}{\sqrt{1+u_0^2}}. \tag{10}$$

It should be noted that as long as the velocity gradient is positive (accelerating flow) the relative speed (8) is always positive (or zero), and the one-tau photo-oval is seen to expand (figure 3).

Using the relative speed above, the redshift z is expressed as

$$1 + z = \frac{1 + \Delta\beta\cos\theta}{\sqrt{1 - (\Delta\beta)^2}}. \tag{11}$$

Now, all the quantities are calculated for given τ_0, u_0, $du/d\tau|_0$, and $I(\tau, \mu)$. Once the observed intensity $I_{\rm co}$ is obtained by equation (7), the radiation energy density $E_{\rm co}$, the radiative flux $F_{\rm co}$, and the radiation pressure $P_{\rm co}$ measured by the comoving observer are calculated respectively as

$$cE_{\rm co} \equiv \int I_{\rm co} d\Omega_{\rm co}, \tag{12}$$

$$F_{\rm co} \equiv \int I_{\rm co} \cos\theta \, d\Omega_{\rm co}, \tag{13}$$

$$cP_{\rm co} \equiv \int I_{\rm co} \cos^2\theta \, d\Omega_{\rm co}. \tag{14}$$

Finally, the relativistic variable Eddinton factor in the comoving frame is given by

$$f \equiv \frac{P_{\rm co}}{E_{\rm co}}. \tag{15}$$

In the following subsections, we consider in turn several cases: the velocity gradient effect, the optical depth effect, and the strong anisotropy of the radiative intensity.

3.1. *Uniform Case with Photo-Vessel*

We first consider the velocity gradient effect, similar to the previous studies (Fukue 2008a, b), and also the photo-vessel case, which was not considered in previous studies. To clarify the velocity gradient effect, we here assume that the comoving radiation field is uniform,

$$I(\tau, \mu) = \bar{I} = \text{const.} \tag{16}$$

The numerical results of this uniform case are shown in figure 4 for small u_0 and figure 5 for large u_0. In figures 4a and 5a the relativistic variable Eddington factors $f(u, -du/d\tau)$ multiplied by 3 are plotted in the parameter space, whereas they are plotted for several fixed u_0 with fitting curves in figures 4b and 5b.

The behavior of RVEF for small u_0 is shown in figure 4 (cf. Fukue 2008a). In this case RVEF decreases, as the velocity gradient becomes large, as analytically proved in Fukue (2008a). In figure 4b for fixed u_0 (= 0.01, 0.05, 0.1) the values of $3f$ are plotted as a function of $-du/d\tau|_0$ by solid curves; all curves overlap each other. One fitting curve is also plotted by a thick-dashed one, and the numerical results are well fitted by

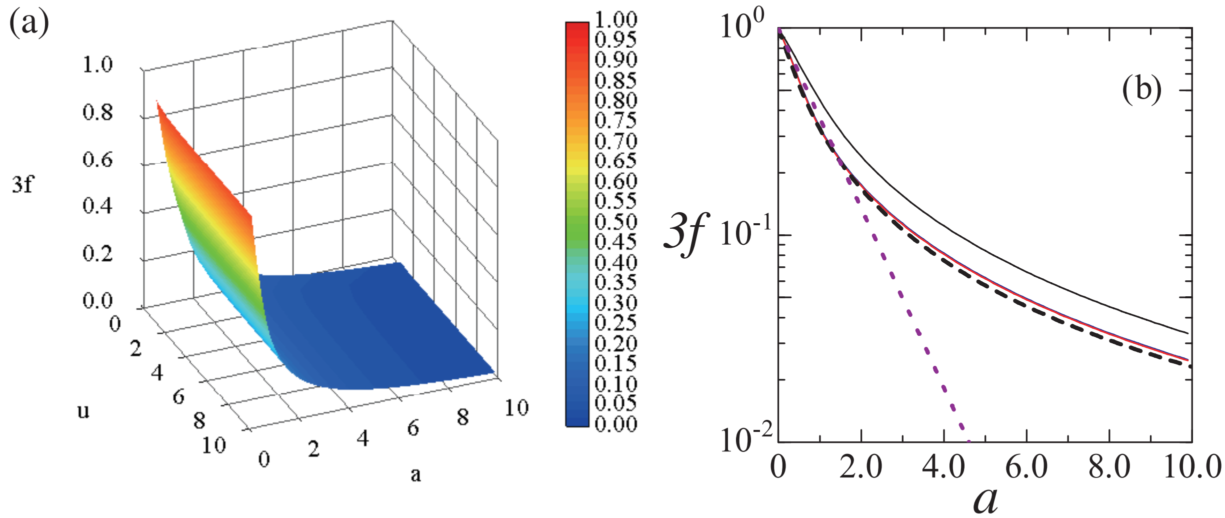


Fig. 5. Behavior of the relativistic variable Eddington factor $f(u, -du/d\tau)$ multiplied by 3 in the case of large u. (a) $3f$ are plotted as a function of the four velocity u_0 (x axis) and the logarithmic four-velocity gradient $-d\ln u/d\tau|_0$ (a axis). (b) $3f$ are plotted as a function of the logarithmic four-velocity gradient $-d\ln u/d\tau|_0$ for three fixed u_0 (= 1, 5, 10; solid curves) from top to bottom, and one fitting curve is also shown (dashed one), with an exponential curve (dotted curve).

$$3f \sim \left(1 + \frac{16}{15}y + 0.9y^{1.8}\right)^{-1}, \tag{17}$$

where

$$y = -\left.\frac{du}{d\tau}\right|_0, \tag{18}$$

within an error of a few percent. It should be noted that we chose the coefficient, 16/15, so that the fitting equation reduces to

$$3f = 1 - \frac{16}{15}y \tag{19}$$

in the linear regime, as analytically proved in Fukue (2008a).

The behavior of RVEF for large u_0 is shown in figure 5 (cf. Fukue 2008b). In this case RVEF decreases, as the *logarithmic* velocity gradient becomes large, as shown in Fukue (2008b). In figure 5b for a fixed u_0 (= 1, 5, 10) the values of $3f$ are plotted as a function of $-d\ln u/d\tau|_0$ by solid curves; except for the marginally case of $u_0 = 1$, other curves overlap each other. One fitting curve is also plotted by a thick dashed one, and the numerical results are well fitted by

$$3f \sim \left(1 + \frac{16}{15}a + a^{1.5}\right)^{-1}, \tag{20}$$

where

$$a = -\left.\frac{d\ln u}{d\tau}\right|_0. \tag{21}$$

Although in Fukue (2008b) we used an exponential fitting curve (the dotted curve in figure 5b), the present power-law type is well fitted at large a.

Mixing the small and large u_0 cases, and considering several other patterns, the relativistic variable Eddington factor in the uniform field is well fitted by

$$3f \sim \left[1 + \frac{16}{15}\left(-\frac{1}{\gamma}\frac{du}{d\tau}\right) + \left(-\frac{1}{\gamma}\frac{du}{d\tau}\right)^{1.6-2}\right]^{-1}. \tag{22}$$

We here assume that the comoving intensity is uniform, and the above result is only due to the redshift effect, similar to the previous studies (Fukue 2008a, b).

Here, we briefly examine the case of small optical depth, a photo-vessel (figure 1). That is, as stated, the one-tau region is open towards a downstream direction to be a photo-vessel for $\tau_0 < 1$ [equation (6)].

The behavior of RVEF for small τ is shown in figure 6. In figure 6a the values of $3f$ are plotted as a function of $-du/d\tau|_0$ for several values of τ_0 by dashed curves ($\tau_0 = 0.1$, 0.2, 0.3, 0.4, 0.5 from bottom to top) and by solid curves ($\tau_0 = 0.6$, 0.7, 0.8, 0.9, 1.0 from bottom to top) for $u_0 = 0.1$. The tendencies are similar, but there exist weak deviations.

In figure 6b the value of $3f$ is plotted as a function of τ_0 for $u_0 = 0.1$ and $-du/d\tau|_0 = 0$. Even if there is no velocity gradient, the value of RVEF changes with the optical depth. This is understood as follows. When the optical depth τ_0 is larger than unity (photo-oval), $f = 1/3$ in the uniform radiation field without the velocity gradient. When the optical depth becomes smaller than unity (photo-vessel), the Eddington factor also becomes smaller than 1/3, since the radiation field is no longer isotropic; the radiation from the downstream direction vanishes. When the optical depth approaches zero, however, the Eddington factor recovers to 1/3, since the Eddington factor above the infinite plane is 1/3. The dependence on the optical depth in figure 6b is well fitted by

$$f \sim \frac{1}{3}\left(\tau^2 - \tau + 1\right), \tag{23}$$

which is shown by a thick dashed curve in figure 6b, without any significant error.

3.2. *Weakly Non-Uniform Case*

Next, we examine the optical depth effect, where the background radiation field is not uniform. We assume that the comoving radiation field is the Milne-Eddington type,

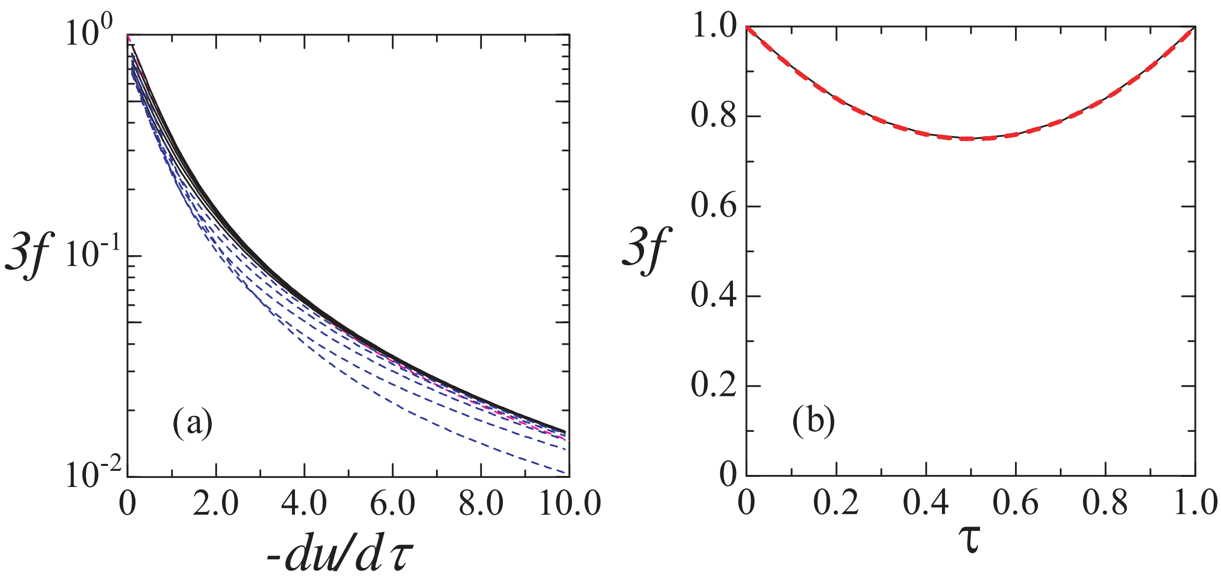


Fig. 6. Behavior of the relativistic variable Eddington factor, $f(u, -du/d\tau)$, multiplied by 3 for small τ. (a) The dependence on the velocity gradient is shown for several values of τ_0 (= 0.1, 0.2, 0.3, 0.4, 0.5 from bottom to top of the dashed curves) (= 0.6, 0.7, 0.8, 0.9, 1.0 from bottom to top of solid curves) for $u_0 = 0.1$. (b) The dependence on the optical depth is shown in the case of $u_0 = 0.1$ and $-du/d\tau|_0 = 0$. One fitting curve is also shown (dashed curve),

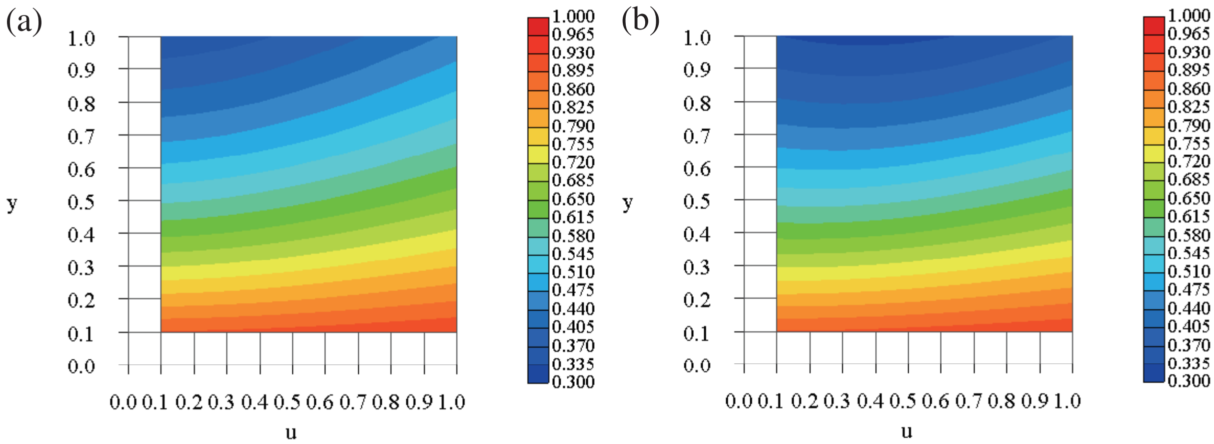


Fig. 7. Behavior of the relativistic variable Eddington factor, $f(u, -du/d\tau)$, multiplied by 3. It is plotted as a function of the four velocity u_0 (x axis) and the four-velocity gradient $-du/d\tau|_0$ (y axis) in (a) the uniform case and in (b) the Milne-Eddington case for $\tau_0 = 1$.

$$I(\tau, \mu) = \frac{3}{4}\bar{I}\left(\tau + \frac{2}{3} + \mu\right). \tag{24}$$

Here, τ is the optical depth, which is related to the observer's quantities by equation (6), and μ is the direction cosine in the comoving frame, which is related to the observer's quantities by

$$\mu = -\frac{\cos\theta + \Delta\beta}{1 + \Delta\beta\cos\theta}, \tag{25}$$

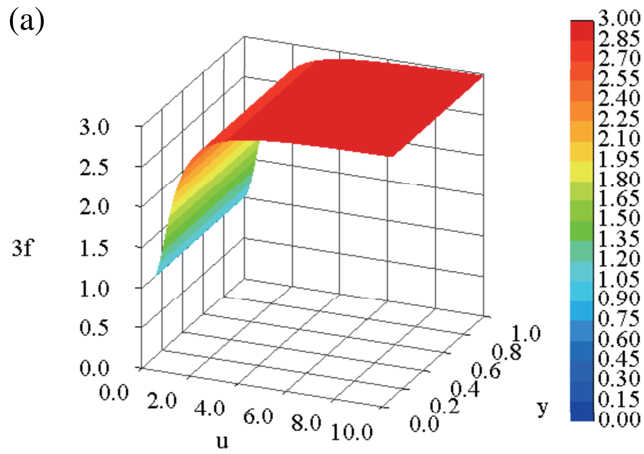
if we consider the aberration effect between the observer and the radiation site on the photo-oval in the comoving frame.

Using this intensity distribution in the comoving frame, we easily computed the radiation field and the Eddington factor. The numerical results are shown in figure 7. In figures 7a and 7b the relativistic variable Eddington factors, $f(u, -du/d\tau)$, multiplied by 3 are plotted in the parameter space for the uniform case (figure 7a) and for the Milne-Eddington case (figure 7b).

As can be seen in figure 7, the distribution of RVEF of the Milne-Eddington case is not so much different from that of the uniform case. For larger u_0 the difference becomes much small. Hence, the weak non-uniformity or weak anisotropy does not significantly affect the values of RVEF.

3.3. Strongly Anisotropic Case

Thirdly, we investigate the effect of the relativistic speed itself. Although there may be various kinds of anisotropy, we consider the simple limiting case; we assume that the radiation field in the inertial frame is uniform. Then, the comoving radiation field is expressed as



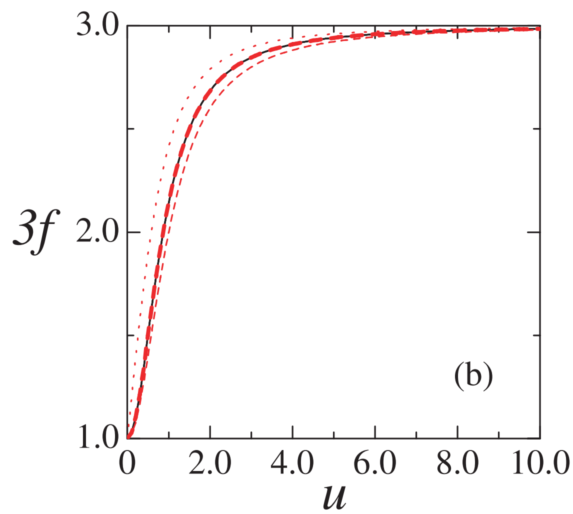


Fig. 8. Behavior of the relativistic variable Eddington factor $f(u, -du/d\tau)$ multiplied by 3 in the strongly anisotropic case. (a) $3f$ are plotted as a function of the four velocity u_0 (x axis) and the four-velocity gradient $-du/d\tau|_0$ (y axis). (b) $3f$ are plotted as a function of the four velocity u_0 for three fixed $du/d\tau|_0$ (= 0, 0.5, 1.0; solid curves), and one fitting curve is also shown (dashed curve), with two comparisons (thin-dashed and dotted curves).

$$I(\tau, \mu, \beta) = \frac{1}{[\gamma(1+\beta\mu)]^4}\bar{I}, \tag{26}$$

where β is the flow speed in units of the speed of light, γ the Lorentz factor, and μ the direction cosine in the comoving frame given by equation (25). Thus, due to the aberration effect the comoving radiation field becomes strongly anistropic. Namely, when the radiative intensity in the comoving frame is uniform, the intensity in the inertial frame becomes anisotropic in the sense that the observer in the inertial frame receives the strong radiation from the upstream direction. When the radiative intensity in the inertial frame is uniform, on the other hand, the intensity in the comoving frame becomes anisotropic in the sense that the comoving observer receives strong radiation from the downstream direction.

The numerical results of this anisotropic case are shown in figure 8. In figure 8a the relativistic variable Eddington factor, $f(u, -du/d\tau)$, multiplied by 3 is plotted in the parameter space, whereas it is plotted as a function of u_0 with fitting curves in figure 8b.

The behavior of RVEF shown in figure 8a is quite impressive, since it seems that f does not depend on the velocity gradient, but depends only on the velocity. Indeed, in figure 8b the values of $3f$ are plotted as a function of u_0 for several fixed $-du/d\tau|_0$ (= 0, 0.5, 1) by solid curves, and all curves overlap each other. One fitting curve is also plotted by a thick dashed one, and the numerical results are well fitted by

$$3f = \frac{3(1+4u^2)}{3+4u^2} = \frac{3(1+3\beta^2)}{3+\beta^2}. \tag{27}$$

As comparisons, RVEF proposed in Fukue (2006) are also plotted by the thin dashed one ($3f = 1 + 2\beta^2$) and the thin dotted one ($3f = 1 + 2\beta$).

As can be seen in figure 8b, equation (27) reproduces the numerical results very well. This is interpreted as follows. When the radiation intensity in the inertial frame is uniform, the radiation energy density $E_{\rm lab}$, the radiative flux $F_{\rm lab}$, and the radiation pressure $P_{\rm lab}$ in the inertial (laboratory) frame become, respectively:

$$cE_{\rm lab} = 4\pi\bar{I}, \tag{28}$$

$$F_{\rm lab} = 0, \tag{29}$$

$$cP_{\rm lab} = \frac{4}{3}\pi\bar{I}, \tag{30}$$

for $\tau_0 > 1$. Hence, from the Lorentz transformation the radiation energy density $E_{\rm co}$, the radiative flux $F_{\rm co}$, and the radiation pressure $P_{\rm co}$ in the comoving frame become, respectively:

$$cE_{\rm co} = \gamma^2\left(cE_{\rm lab} - 2\beta F_{\rm lab} + \beta^2 cP_{\rm lab}\right) = 4\pi\bar{I}\gamma^2\left(1 + \frac{1}{3}\beta^2\right), \tag{31}$$

$$F_{\rm co} = \gamma^2\left[(1+\beta^2)F_{\rm lab} - \beta(cE_{\rm lab} + cP_{\rm lab})\right] = -4\pi\bar{I}\frac{4}{3}\gamma^2\beta, \tag{32}$$

$$cP_{\rm co} = \gamma^2\left(\beta^2 cE_{\rm lab} - 2\beta F_{\rm lab} + cP_{\rm lab}\right) = 4\pi\bar{I}\gamma^2\left(\beta^2 + \frac{1}{3}\right). \tag{33}$$

Thus, the RVEF analytically becomes

$$f \equiv \frac{P_{\rm co}}{E_{\rm co}} = \frac{1+3\beta^2}{3+\beta^2}. \tag{34}$$

4. Concluding Remarks

In this paper, we have derived the relativistic variable Eddington factor (RVEF) for a plane-parallel flow that is accelerating in the vertical direction; we have introduced the one-tau photo-oval observed by the comoving observer, and then calculated the comoving radiation field and the Eddington factor for several situations, and found that the Eddington factor in the relativistic regime generally depends on the optical depth, the flow velocity, and the velocity gradient.

Of these, the dependence of RVEF on the optical depth is weak, similar to the non-relativistic plane-parallel atmosphere. As for the dependence on the velocity gradient, on the other hand, RVEF generally decreases as the velocity gradient increases. When the velocity gradient is small, RVEF linearly decreases with the velocity gradient, as proved in Fukue (2008a). When the velocity gradient becomes large, RVEF decreases in the power-law manner of the logarithmic velocity gradient (cf. Fukue 2008b). Finally, as for the dependence on the relativistic speed, RVEF can increase to be on the order of unity, if the comoving intensity is sufficiently anisotropic.

We have considered the case of plane-parallel vertical flow. In spherical flow, on the other hand, there exists a geometrical dilution effect, and the situation can be somewhat different, since r^2 terms appear (e.g., Peraiah 2002). The present approach, however, can be extended to also treat the spherical case, and we will examine it in future work.

The treatment of the relativistic Eddington factor presented in this paper may turn out to be applicable in various aspects of relativistic astrophysics with radiation transfer; i.e., black-hole accretion flows with supercritical accretion rates, relativistic jets and winds driven by luminous central objects, relativistic explosions including gamma-ray bursts, neutrino transfers in supernova explosions, and various events occured in the proto universe (cf. Fukue 2008b). The treatment of the relativistic Eddington factor presented in this paper will help to investigate these types of astrophysical problems that involve the solution of the fully relativistic radiation hydrodynamical equations.

The author would like to express his sincere thanks to M. Umemura, T. Koizumi, and C. Akizuki for their enlightening and stimulated discussions. This work has been supported in part by a Grant-in-Aid for Scientific Research (No. 18540240) of the Ministry of Education, Culture, Sports, Science and Technology.

References

Akizuki, C., & Fukue, J. 2008, PASJ, 60, 337
Baschek, B., Efimov, G.V., von Waldenfels, W., & Wehrse, R. 1997, A&A, 317, 630
Baschek, B., Hafner, M., & Wehrse, R. 1995, A&A, 301, 511
Castor, J. I. 2004, Radiation Hydrodynamics (Cambridge: Cambridge University Press)
Chandrasekhar, S. 1934, MNRAS, 94, 444
Chandrasekhar, S. 1960, Radiative Transfer (New York: Dover Publishing, Inc.)
Eddington, A. S. 1926, The Internal Consitution of the Stars (Cambridge: Cambridge University Press)
Flammang, R. A. 1982, MNRAS, 199, 833
Flammang, R. A. 1984, MNRAS, 206, 589
Fukue, J. 2006, PASJ, 58, 461
Fukue, J. 2008a, PASJ, 60, 377
Fukue, J. 2008b, PASJ, 60, 1209
Fukue, J., & Akizuki, C. 2006, PASJ, 58, 1073
Fukue, J., & Akizuki, C. 2007, PASJ, 59, 1027
Fukue, J., Kato, S., & Matsumoto, R. 1985, PASJ, 37, 383
Hsieh, S.-H., Spiegel, E. A. 1976, ApJ, 207, 244
Hummer, D. G., & Rybicki, G. B. 1971, MNRAS, 152, 1
Kato, S., Fukue, J., & Mineshige, S. 1998, Black-Hole Accretion Disks (Kyoto: Kyoto University Press)
Kato, S., Fukue, J., & Mineshige, S. 2008, Black-Hole Accretion Disks – Towards a New Paradigm (Kyoto: Kyoto University Press)
Koizumi, T., & Umemura, M. 2008, submitted to MNRAS
Kosirev, N. A. 1934, MNRAS, 94, 430
Masaki, I., & Unno, W. 1978, PASJ, 30, 547
Mihalas, D. 1970, Stellar Atmospheres (San Francisco: W. H. Freeman and Co.)
Mihalas, D., & Auer, L. H. 2001, J. Quant. Spectrosc. Radiat. Transfer, 71, 61
Mihalas, D., & Mihalas, B. W. 1984, Foundations of Radiation Hydrodynamics (Oxford: Oxford University Press)
Mihalas, D., Kunasz, P. B., & Hummer, D. G. 1975, ApJ, 202, 465
Mihalas, D., Kunasz, P. B., & Hummer, D. G. 1976a, ApJ, 203, 647
Mihalas, D., Kunasz, P. B., & Hummer, D. G. 1976b, ApJ, 206, 515
Milne, E. A. 1921, MNRAS, 81, 361
Nobili, L., Turolla, R., & Zampieri, L. 1991, ApJ, 383, 250
Nobili, L., Turolla, R., & Zampieri, L. 1993, ApJ, 404, 686
Park, M.-G. 2001, J. Korean Astron. Soc., 34, S305
Park, M.-G. 2006, MNRAS, 367, 1739
Peraiah, A. 2002, An Introduction to Radiative Transfer: Methods and applications in astrophysics (Cambridge: Cambridge University Press)
Rybicki, G. B., & Lightman, A. P. 1979, Radiative Processes in Astrophysics (New York: John Wiley & Sons)
Sen, K. K., & Wilson, S. J. 1993; Ap&SS, 203, 227
Shu, F. H. 1991, The Physics of Astrophysics Vol. 1: Radiation (Mill Valley: University Science Books)
Takahashi, R. 2007, MNRAS, 382, 1041
Tamazawa, S., Toyama, K., Kaneko, N., & Ôno, Y. 1975, Ap&SS, 32, 403
Thorne, K. S. 1981, MNRAS, 194, 439
Thorne, K. S., Flammang, R. A., & Żytkow, A. N. 1981, MNRAS, 194, 475
Unno, W., & Kondo, M. 1976, PASJ, 28, 347
Wilson, S. J., Tung, C. T., & Sen, K. K. 1972, MNRAS, 160, 349