ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 62, nº 2 (1995), p. 103-144
<http://www.numdam.org/item?id=AIHPA_1995__62_2_103_0>

Ann. Inst. Henri Poincaré,

Vol. 62, n° 1, 1995, p. 103-144. *Physique théorique*

Relativity, spherical functions and the hypergeometric equation

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ABSTRACT. – Besides its relevance to physics, special relativity contains in germ a wealth of new mathematics. Starting from the Klein-Gordon calculus, a relativistic substitute for the Weyl calculus, we show here how to deform the theory of spherical functions on rank-one symmetric spaces or, what amounts roughly to the same, that of hypergeometric functions.

RÉSUMÉ. – Outre sa position centrale en physique, la relativité restreinte ouvre la voie à des faits mathématiques nouveaux. Partant du calcul de Klein-Gordon, un analogue relativiste du calcul de Weyl, on déforme ici la théorie des fonctions sphériques sur un espace symétrique de rang un ou, si l'on veut, celle des fonctions hypergéométriques.

0. INTRODUCTION

We wish here to convey the feeling that, once you have been trapped into a genuinely relativistic mathematical area, there may not exist any turning back, at least not before you have coped with a much broader task than what you had in mind to start with: namely, in order that a given mathematical theory retain its internal coherence after the relativistic deformation process has taken place, it may be necessary to get involved simultaneously with several branches of mathematics, like e.g. quantization theory, harmonic analysis and special function theory.

Start from such a commonplace appliance as the one-dimensional harmonic oscillator $L = \pi (x^2 + D^2)$, with $D = (2 i \pi)^{-1} \frac{d}{dx}$. This is a self-adjoint operator on $L^2 (\mathbb{R})$, the Hilbert space of initial data for the free Schrödinger equation

$$(0.1) \qquad 4 i \pi \frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}.$$

The harmonic oscillator is intimately linked with the Heisenberg representation (of which the operators x and D are the infinitesimal generators) and with the Fourier transformation $\mathcal{F}$ characterized by

$$(0.2) \qquad (\mathcal{F} u)(x) = \hat{u}(x) = \int u(y) e^{-2i\pi xy} dy,$$

since L and $\mathcal{F}$ commute. Also, there is a strong link between L and the Weyl calculus of operators. Recall that this is the well-known rule $f \mapsto 0p(f)$ that assigns an operator (generally unbounded) on $L^2 (\mathbb{R})$ to each function $f = f(x, \mathbf{p})$ on the phase space $\mathbb{R}^2$ (where $\mathbf{p}$ stands for the momentum variable): f is also called the symbol of $0p(f)$. As is well-known too, functions of L in the spectral-theoretic sense are just the operators $0p(f)$, where f is any function of $x^2 + \mathbf{p}^2$. In particular, denoting as

$$(0.3) \qquad \varphi(x) = 2^{1/4} e^{-\pi x^2}$$

the ground state of L, the Wigner function $W(\varphi, \varphi)$, by which is meant the symbol of the projection operator $u \mapsto (u, \varphi) \varphi$, is the function $\Phi(x^2 + \mathbf{p}^2)$, with $\Phi(r) = 2 e^{-2\pi r}$. One may wish to link Φ to φ by the formula

$$(0.4) \qquad \Phi(r) = \left(-\frac{1}{\pi} \frac{d}{dr} \right)^{1/2} [(\mathcal{F} \varphi)(r^{1/2})]^2,$$

where the Fourier transformation $\mathcal{F}$ can of course be dispensed with, and where the non-local operator on the right-hand side accounts for the missing factor $2^{1/2}$.

What can be left alive of the preceding facts after L has been replaced by a suitable relativistic generalization? *Everything*! Only the whole structure, not just L, must bear the relativistic mark; also, things are considerably harder to prove. To proceed towards the relativistic deformation, first replace (0.1) by its relativistic counterpart

$$(2\,i\,\pi)^{-1}\,\frac{\partial\psi}{\partial t} = c^2\left(1+\frac{D^2}{c^2}\right)^{1/2}\psi, \tag{0.5}$$

which is the positive-frequency part of the Klein-Gordon equation for a free particle of mass 1. As is well-known from the theory of the one-particle space of free quantum field theory (*cf.* e.g. [1], vol. 2, p. 402), the correct space of initial data for (0.5) is no longer $L^2(\mathbb{R})$, but $H_c^{1/2}(\mathbb{R})$ characterized as the space of $u \in L^2(\mathbb{R})$ satisfying

$$\int \langle \mathbf{p}\rangle\,|\hat{u}(\mathbf{p})|^2\,d\mathbf{p} < \infty, \tag{0.6}$$

where we have set $\langle \mathbf{p}\rangle = (1+c^{-2}\,\mathbf{p}^2)^{1/2}$; we also set $\langle D\rangle = (1+c^{-2}\,D^2)^{1/2}$. The Heisenberg representation is unsuitable for a description of relativistic particles and must be replaced by the Bargmann-Wigner representation of the orthochronous Poincaré group in the Hilbert space $H_c^{1/2}(\mathbb{R})$. Now the infinitesimal generators of the B. W. representation are $\langle D\rangle$, D and $B = x\,\langle D\rangle$, corresponding to time or space translations and to boosts respectively. This yields a definition of the relativistic oscillator L:

$$L = \pi\,(B^2 + D^2). \tag{0.7}$$

There is a relativistic version $\mathcal{G}$ of the Fourier transformation: it is defined as

$$(\mathcal{G}\,u)\,(x) = \langle x\rangle\,\hat{u}\,(x). \tag{0.8}$$

and does commute with L. This transformation, on the other hand, retains it fundamental role as a connecting link between the position and momentum space realizations since, looking at the positive-frequency sheet $\mathcal{M}$ of the mass-hyperboloid of equation $E^2 = c^2\,\mathbf{p}^2 + c^4$ and setting (with a slight abuse of language) $(\mathcal{G}\,u)\,(E,\,\mathbf{p}) = (\mathcal{G}\,u)\,(\mathbf{p})$ for $p = (E,\,\mathbf{p}) \in \mathcal{M}$, one gets the standard isometry from $H_c^{1/2}(\mathbb{R})$ onto $L^2(\mathcal{M}) = L^2(\mathcal{M};\,\langle \mathbf{p}\rangle^{-1}\,d\mathbf{p})$. Even though this is not immediately obvious from (0.7), the relativistic oscillator is a differential operator: under the disguise induced by the isometry $\mathcal{G}$ just described, one may also write

$$\mathcal{G}\,(4\,\pi\,L)\,\mathcal{G}^{-1} = \Delta_{\mathcal{M}} + 4\,\pi^2\,\mathbf{p}^2 \tag{0.9}$$

with

$$\Delta_{\mathcal{M}} = -\frac{d^2}{d\mathbf{p}^2} - c^{-2}\left(\mathbf{p}\,\frac{d}{d\mathbf{p}}\right)^2, \tag{0.10}$$

the Laplacian on $\mathcal{M}$ with respect to the Lorentz-invariant metric. All this is described, in n variables rather than just one, in [17].

Now the Weyl calculus is inoperant in this setting, since it is covariant with respect to the Heisenberg, not the Bargmann-Wigner, representation. It has to be replaced here by the Klein-Gordon analysis, a symbolic calculus of operators on $H_c^{1/2}(\mathbb{R})$ developed in full in our monograph [18]. Let us just recall here from ([17], (3.5)) that in the one-dimensional case which we have decided to consider in this introduction, the definition of the operator $0p_{KG}(f)$ with Klein-Gordon symbol $f = f(x,\, \mathbf{p})$ can be written as

$$\begin{aligned}(\mathcal{G}\,0p_{KG}(f)\,u)\,(c\,\mathrm{sh}\,\xi) &\\ = c\int e^{-2i\pi c x\,(\mathrm{sh}\,\xi - \mathrm{sh}\,\eta)}\, f\left(x,\, c\,\mathrm{sh}\,\frac{\xi+\eta}{2}\right)&\\ \times\,(\mathcal{G}\,u)\,(c\,\mathrm{sh}\,\eta)\,\mathrm{ch}\,\frac{\xi+\eta}{2}\,dx\,d\eta.&\end{aligned} \tag{0.11}$$

Then the symbol of L itself is the function

$$(x,\, \mathbf{p}) \mapsto \pi\, r - (16\,\pi\, c^2)^{-1} \tag{0.12}$$

with

$$r = x^2 + \mathbf{p}^2 + c^{-2}\, x^2\, \mathbf{p}^2. \tag{0.13}$$

Observe at once that r is much more complicated than $x^2 + \mathbf{p}^2$, its non-relativistic analogue (and limit), since its level curves are curves of genus one, not circles; also, the eigenfunctions φ_k of L (associated in what follows to the eigenvalues arranged in increasing order, starting from $k = 0$) are no more elementary functions, but are related to Mathieu functions, as will be recalled at the end of section 1. No explicit representation of φ_k by means of series or integrals is known, but the functions φ_k do satisfy some nice identities ([17], corollary 3.5) which it is one of the purposes of the present paper to generalize.

Finally, here is the relativistic generalization of (0.4) (which does not depend on how we normalize φ_k). Again (but here, contrary to the non-relativistic case, this is far from trivial), functions of r, as defined in (0.13), are just the K.G. symbols of functions of L. In particular, the symbol of the rank-one operator $u \mapsto (u,\ \varphi_k)\,\varphi_k$ (of course, the scalar product is taken in $H_c^{1/2}(\mathbb{R})$) is some function $\Phi_k\,(x^2 + \mathbf{p}^2 + c^{-2}\,x^2\mathbf{p}^2)$. As a consequence of proposition 3.6 and of (3.14) in [17], one may write

$$\Phi_k\,(r) = (-1)^k \left(-\frac{1}{\pi}\,\frac{d}{dr}\right)^{1/2} [(\mathcal{G}\,\varphi_k)\,(r^{1/2})]^2, \tag{0.14}$$

where the square-root on the right-hand side makes sense since it applies to a function which is the Laplace transform of some function supported in $\mathbb{R}^+$: some details will be provided at the end of section 3.

When $c = \infty$, all results concerning the relativistic oscillator or the Klein-Gordon calculus reduce to facts relative to the harmonic oscillator which have been reported here and there (e.g. [19], where the Wigner function of any two eigenstates of any two, possibly distinctly "squeezed", harmonic oscillators is made explicit), with the possible exception of (0.14): indeed it reduces to an identity equivalent, after some easy work, to the formula

$$\frac{(-2)^{-k}}{k\,!}\,[H_k\,(\sqrt{2\,\pi\,r})]^2 = [L_k\left(-2\,\frac{d}{dt} + 4\,\pi\,r\right)\cdot t^{-1/2}]\,(t=1) \tag{0.15}$$

connecting Hermite and Laguerre polynomials; we have not found it in the special function literature, but it is easy to derive it in an elementary way from the identities ([10], p. 252 and p. 242)

$$(1-z^2)^{-1/2}\,\exp\frac{2\,x^2\,z}{1+z} = \sum \frac{2^{-k}}{k\,!}\,(H_k\,(x))^2\,z^k$$

and

$$(1+z)^{-1}\,\exp\frac{\xi\,z}{1+z} = \sum L_k\,(\xi)\,(-z)^k.$$

Much of the story we have just told extends (in a sometimes non trivial way) to the n-dimensional case [17]: starting from the n-dimensional analogue of (0.9), one can view the relativistic oscillator L as a deformation (one extra term) of the Laplace-Beltrami operator on the mass hyperboloid. Since L commutes with the group of (spatial) rotations, one may consider its radial part, which in some appropriate coordinate $s \in\,]0,\ \infty[$ is just an ordinary differential operator $L\,(\lambda,\ \nu)$ with $\lambda = \dfrac{n-2}{2}$ and $\nu = \pi\,c^2$.

The present work is devoted to studying and establishing various exact formulas relative to the eigenfunctions of $L(\lambda, \nu)$ or, rather, of some larger three-parameter family $L(\lambda, \mu, \nu)$: the preceding case would then be that with $\mu = -\frac{1}{2}$. The full family of equations $L(\lambda, \mu, \nu)\, u = \rho u$ is a generalization of the hypergeometric equation, which is the $\nu = 0$ case. We suggest to call it the *chronogeometric* equation, as it is related to the Klein-Gordon equation, as explained above, when $\mu = -\frac{1}{2}$; it may be poor physics, but it is sound mathematics, to allow for an m-dimensional *time* too, from which general values of μ come into the picture: section 4 will explain how solving the chronogeometric equation permits to solve some initial-value problem related to the *bi-radial* part of some (m, n)-dimensional Klein-Gordon equation.

As the mass hyperboloid $\mathcal{M}$ is a (rank one) symmetric space, the study of the radial eigenfunctions of its Laplace-Beltrami operator $\Delta_{\mathcal{M}}$ is a chapter of the theory of *spherical functions*: in a surprising way, part of it extends to the case when $\Delta_{\mathcal{M}}$ is replaced by the complete relativistic oscillator (*cf.* (1.20)). As an example, consider the well-known identity

$$P(\operatorname{ch} x)\, P(\operatorname{ch} y) = \frac{1}{\pi} \int_0^{\pi} P(\operatorname{ch} x \operatorname{ch} y + \operatorname{sh} x \operatorname{sh} y \sin\theta)\, d\theta$$

valid for Legendre functions: this is properly a result in the theory of spherical functions on the two-dimensional mass hyperboloid. More generally, one gets a related identity if one starts form Gegenbauer functions, which are a subfamily (one parameter has to be kept frozen) of that of *hypergeometric* functions. Section 5 gives a generalization of this Gegenbauer function identity to the "relativistic" $\nu \neq 0$ case $\left(\mu = -\frac{1}{2}\right.$ is the frozen parameter$\left.\right)$.

In this kind of identities, the eigenfunctions under consideration are normalized by their value at $s = 0$. In the hypergeometric case, nothing exotic will happen if you substitute for this normalization some normalization at infinity, since the two can be related. In the ("relativistic") chronogeometric case, things are different, since $L(\lambda, \mu, \nu)$ has a regular (Fuchs type) singularity at $s = 0$ or -1 but an irregular one at ∞. It turns out that, if one normalizes eigenfunctions ψ of $L(\lambda, \mu, \nu)$ by some relevant condition at ∞, one gets a brand-new identity for $\psi \otimes \psi$ which, this time, is valid without our having to freeze any of the parameters

λ, μ, ν: it is described in section 3. Another topic discussed in this section is the expression of the function introduced right before (0.14) as a chronogeometric function, and a rephrasing of the identity (0.14) in terms of chronogeometric functions. Replacing the mass hyperboloid by a sphere of the same dimension amounts to changing the domain of $L(\lambda, \mu, \nu)$, in particular replacing $]0, \infty[$ by $]-1, 0[$: this is discussed in section 6.

In all that precedes, the familiar case $\nu = 0$ (or $c = 0$ since $\nu = \pi c^2$) was in some sense the *ultrarelativistic* case. It is necessary, would it be only in order to set up, in section 1, various spectral problems relative to the operator $L(\lambda, \mu, \nu)$, to consider also its *non-relativistic* limit as c goes to ∞: as it turns out, this is when $\mu = -\frac{1}{2}$ and $\lambda = \frac{n-2}{2}$ the radial part of an n-dimensional harmonic oscillator or, more generally, the infinitesimal generator e_0 corresponding to the subgroup $SO(2)$ of some representation in the discrete series of $SL(2, \mathbb{R})$. Mostly for the technical reasons just alluded to, we include, in section 2, a precise discussion of the differential operator e_0, and link $L(\lambda, \mu, \nu)$ to the representation under consideration.

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1. THE CHRONOGEOMETRIC EQUATION

Consider the differential equation

$$s(1+s)\frac{d^2 u}{ds^2} + [\lambda + 1 + (\lambda + \mu + 2) s]\frac{du}{ds} + (\rho - \nu^2 s) u = 0. \tag{1.1}$$

When $\nu = 0$, it reduces to the hypergeometric equation for ${}_2F_1(\alpha, \beta, \gamma; -s)$ provided we set $\lambda = \gamma - 1$, $\mu = \alpha + \beta - \gamma$ and $\rho = \alpha\beta$. Besides regular singular points (*i.e.* points of Fuchs type) at 0 and -1, with pairs of exponents $(0, -\lambda)$ and $(0, -\mu)$, it has when $\nu \neq 0$ an irregular singularity at ∞, and does not have in general any known solution in terms of an explicit series or integral any more. An exception occurs when $\lambda = -1$ *and* $\rho = 0$, or $\mu = -1$ *and* $\rho = -\nu^2$, since one can then divide the whole equation by s or $1+s$: in the first case, for instance, a solution of (1.1) is the function $k_{\mu,\nu}((1+s)^{1/2})$, where the (elementary) function $k_{\mu,\nu}$ is defined in (3.6).

We suggest to call (1.1) the chronogeometric equation to recall its relativistic origin (the reader may find this word even more appropriate in view of section 4 in this paper), at the same time recalling its link to the hypergeometric equation. We shall treat ρ, in (1.1), as a spectral parameter, and shall be interested, actually, in the operator

$$(1.2)\quad L(\lambda,\,\mu,\,\nu) = -s\,(1+s)\,\frac{d^2}{ds^2} - [\lambda+1+(\lambda+\mu+2)\,s]\,\frac{d}{ds} + \nu^2\,s.$$

Assuming λ, μ, ν real, $\nu \neq 0$ (say $\nu > 0$) and $\lambda > -1$, $\mu > -1$, we shall consider, presently, the spectral problem for $L(\lambda,\,\mu,\,\nu)$ on $(0,\,\infty)$: observe, however, that changing s to $-1-s$ and exchanging λ and μ would reduce to the preceding case the spectral problem on $(-\infty,\,-1)$ in the case when $\nu \in i\,\mathbb{R}$.

From the theory of the relativistic oscillator as well as from the "Klein-Gordon-Bessel equation" that will arise in section 4, it will be useful to set, if $\lambda+\mu+1 \neq 0$ or $\mu = -\frac{1}{2}$,

$$(1.3)\qquad h = \frac{\lambda+1/2}{\lambda+\mu+1},\qquad \nu = \frac{\pi\,c^2}{h}$$

with the understanding that when $\mu = -\frac{1}{2}$, one has $h = 1$ even when $\lambda = -\frac{1}{2}$ too. The parameters λ and μ should be thought of as related to the dimensions of space and time respectively $\left(\text{for } \mu \neq -\frac{1}{2}, \text{ this is rather fancy physics!}\right)$, c is indeed the velocity of light, and we shall not argue with the reader whether h should be regarded as some kind of Planck's constant or not: it certainly plays a role akin to such in equations (1.18), (2.5), (4.2).

Then, the hypergeometric equation will appear as the ultrarelativistic (*i.e.* $c \to 0$) limit of the chronogeometric equation. To get an interesting non-relativistic limit ($c \to \infty$), one must perform first the change of variable

$s = \dfrac{r}{c^2}$ that reduces $L(\lambda, \mu, \nu)$ to the operator characterized by

$$\begin{aligned} (1.4) \qquad & c^{-2} L(\lambda, \mu, \nu) \\ & = -\left[r \frac{d^2}{dr^2} + (\lambda + 1) \frac{d}{dr} - \frac{\pi^2}{h^2} r \right. \\ & \qquad \left. + c^{-2} \left(r^2 \frac{d^2}{dr^2} + (\lambda + \mu + 2) r \frac{d}{dr} \right) \right]. \end{aligned}$$

The limit as $c \to \infty$ of the right-hand side is the Laguerre operator

$$-r \frac{d^2}{dr^2} - (\lambda + 1) \frac{d}{dr} + \frac{\pi^2}{h^2} r,$$

a particular self-adjoint realization of which has a complete set of eigenstates

$$u_n(r) = \exp\left(-\frac{\pi}{h} r\right) L_n^{(\lambda)}\left(\frac{2\pi r}{h}\right),$$

corresponding to the eigenvalues $\dfrac{2\pi}{h}\left(n + \dfrac{\lambda + 1}{2}\right)$.

THEOREM 1.1. – $L(\lambda, \mu, \nu)$ *is formally self-adjoint on the Hilbert space* $H^{\lambda, \mu}$ *that consists of all functions* u *on* $(0, \infty)$ *satisfying*

$$\|u\|_{\lambda, \mu}^2 = \int_0^\infty |u(s)|^2 s^\lambda (1 + s)^\mu \, ds < \infty.$$

It is essentially self-adjoint if and only if $\lambda \geq 1$. *When* $-1 < \lambda < 1$, *it has two particular self-adjoint extensions, the domains of which are the spaces of all* $u \in H^{\lambda, \mu}$ *with* $L(\lambda, \mu, \nu) u \in H^{\lambda, \mu}$ *in the distribution sense on* $]0, \infty[$, *satisfying the additional property that* $s^{\lambda+1} u'(s)$ *vanishes at zero for the first extension, and that* $su'(s) + \lambda u(s)$ *vanishes at zero for the second one.*

Proof. – Set $s = \operatorname{sh}^2 \dfrac{\xi}{2}$ with $\xi > 0$ and

$$(1.5) \qquad w(\xi) = \left(\operatorname{sh} \frac{\xi}{2}\right)^{\lambda + \frac{1}{2}} \left(\operatorname{ch} \frac{\xi}{2}\right)^{\mu + \frac{1}{2}} u\left(\operatorname{sh}^2 \frac{\xi}{2}\right)$$

so that

$$(1.6) \qquad \|u\|_{\lambda, \mu}^2 = \int_0^\infty |u(s)|^2 s^\lambda (1 + s)^\mu \, ds = \int_0^\infty |w(\xi)|^2 \, d\xi.$$

The transformation $u \mapsto w$ is called a Liouville transformation (*cf.* Dieudonné [2] or Nourrigat [13] or Gangolli-Varadarajan [4], p. 134). It takes the operator $L(\lambda,\ \mu,\ \nu)$ to the operator (of Schrödinger type)

$$\tag{1.7} \begin{aligned} &-\frac{d^2}{d\xi^2}+\frac{1}{2}\left(\lambda^2-\frac{1}{4}\right)(\operatorname{ch}\xi-1)^{-1} \\ &\quad -\frac{1}{2}\left(\mu^2-\frac{1}{4}\right)(\operatorname{ch}\xi+1)^{-1}+\frac{1}{4}(\lambda+\mu+1)^2+\frac{\nu^2}{2}(\operatorname{ch}\xi-1) \end{aligned}$$

as a rather tedious calculation, which we shall not reproduce here, shows: as a hint, it is easier to check this if one starts from the operator in (1.7), noting first that $\frac{d}{d\xi}$ transfers to the operator

$$\tag{1.8} \begin{aligned} &(s\,(1+s))^{1/2}\,\frac{d}{ds}+\frac{1}{2}\left(\lambda+\frac{1}{2}\right)s^{-1/2}\,(1+s)^{1/2} \\ &\qquad +\frac{1}{2}\left(\mu+\frac{1}{2}\right)s^{1/2}\,(1+s)^{-1/2} \end{aligned}$$

whose square is not that bad to compute. This shows in particular that $L(\lambda,\ \mu,\ \nu)$ is formally self-adjoint on $H^{\lambda,\,\mu}$.

At ∞, the operator $L(\lambda,\ \mu,\ \nu)$ has an irregular singularity. The WKB method (Dieudonné [2], p. 423 or Nourrigat [13], proposition 12.3) applies more directly to the equation satisfied $v=v(t)$ with

$$\tag{1.9} w(\xi)=e^{-\xi/4}\,v\,(\nu\,e^{\xi/2}):$$

indeed, the equation for v reduces to $v''-V\,v=0$ where, as t goes to ∞, $V(t)-1$ admits an asymptotic expansion as a sum $\sum c_k\,t^{-2k}$ with $k\geq 1$; thus, any eigenfunction of the operator in (1.7) is a linear combination of two functions equivalent, as $\xi\to\infty$, to $\exp\left(-\frac{\xi}{4}\pm\nu\,e^{\xi/2}\right)$. Only one of them is in L^2 near infinity: thus (*cf.* Reed-Simon ([15], vol. 2, p. 152), one always is in the limit-point case at infinity. For future reference, note that the eigenfunctions of $L(\lambda,\ \mu,\ \nu)$ which lie in $H^{\lambda,\,\mu}$ near ∞ are equivalent to a constant times

$$\tag{1.10} s^{-\frac{1}{2}\left(\lambda+\mu+\frac{3}{2}\right)}\,e^{-2\nu s^{1/2}}$$

as shown by (1.5).

At zero, the operator in (1.7) has a regular singularity with the pair of exponents $\frac{1}{2} \pm \lambda$ since it reduces to

$$E_\lambda = -\frac{d^2}{d\xi^2} + \left(\lambda^2 - \frac{1}{4}\right) \xi^{-2} \tag{1.11}$$

up to a zero-order term bounded near 0. Only one of the functions $\xi^{\frac{1}{2} \pm \lambda}$ lies in L^2 near 0 if and only if $\lambda \geq 1$ (since $\lambda > -1$). According to Weyl's limit point-limit circle theorem (Reed-Simon, *loc. cit.*), this is the exact condition that will make $L(\lambda, \mu, \nu)$, initially defined on $C_0^\infty(]0, [\infty[)$, essentially self-adjoint on $H^{\lambda, \mu}$.

We postpone to next section the proof that, when $-1 < \lambda < 1$, each of the two boundary conditions indicated in theorem 1.1 yields a self-adjoint extension of $L(\lambda, \mu, \nu)$. Meanwhile, let us note that

$$s^\lambda L(\lambda, \mu, \nu) s^{-\lambda} = L(-\lambda, \mu, \nu) + \lambda(1 + \mu) \tag{1.12}$$

and that $u = s^{-\lambda} v$ implies $s^{\lambda+1} u' = sv' - \lambda v$: thus the isometry $v \mapsto u$ from $H^{-\lambda, \mu}$ onto $H^{\lambda, \mu}$ transforms the second boundary condition relative to $L(-\lambda, \mu, \nu)$ into the first one relative to $L(\lambda, \mu, \nu)$. The two cases in theorem 1.1 can thus be reduced to the first one.

Inverting (1.5) as

$$u(s) = s^{-\frac{1}{2}\left(\lambda+\frac{1}{2}\right)} (1+s)^{-\frac{1}{2}\left(\mu+\frac{1}{2}\right)} w(\ln(2s+1+2\sqrt{s(s+1)})), \tag{1.13}$$

one may write

$$s^{\lambda+1} u'(s) = \left(\frac{\operatorname{sh} \xi/2}{\xi}\right)^{\lambda+\frac{1}{2}} \left(\operatorname{ch} \frac{\xi}{2}\right)^{-\mu-\frac{3}{2}} f(\xi) \tag{1.14}$$

with

$$f(\xi) = \xi^{\lambda+\frac{1}{2}} w'(\xi) - \left(\lambda + \frac{1}{2}\right) \xi^{\lambda-\frac{1}{2}} (1 + \xi \varepsilon(\xi)) w(\xi) \tag{1.15}$$

where ε denotes some function continuous on $[0, \infty[$, satisfying $\varepsilon(\xi) = O(\xi)$ as $\xi \to 0$. Choosing any indefinite integral b of ε, one may solve

(1.15) as

$$\text{(1.16)}\quad \xi^{\lambda+\frac{3}{2}}\, w(\xi) = \xi^{2\lambda+2} \exp\left(\left(\lambda+\frac{1}{2}\right) b(\xi)\right) \times \left[C + \int_1^{\xi} s^{-2\lambda-1} \exp\left(-\left(\lambda+\frac{1}{2}\right) b(s)\right) f(s)\, ds\right]$$

where C is some constant, which shows that the vanishing at zero of $f(\xi)$ as defined in (1.15) does not depend on which function ε appears on the right-hand side. Finally, to finish the proof of theorem 1.1, we just have to show that E_λ, as defined on $(0, \infty)$ in (1.11), has some self-adjoint extension for which the boundary condition at zero is

$$\text{(1.17)}\qquad \xi^{\lambda+\frac{1}{2}}\, w'(\xi) - \left(\lambda+\frac{1}{2}\right) \xi^{\lambda-\frac{1}{2}}\, w(\xi) = 0.$$

We shall wait until next section to do that. □

Consider now, on $\mathbb{R}^n$, the differential operator L^n defined by

$$\text{(1.18)}\quad -4\pi L^n = h^2 \sum \frac{\partial^2}{\partial x_j^2} - 4\pi^2 |x|^2 + c^{-2}\left[h^2\left(\sum x_j \frac{\partial}{\partial x_j}\right)^2 + (n-1)\, h \sum x_j \frac{\partial}{\partial x_j}\right].$$

When $h = 1$, this is just the relativistic oscillator in n variables introduced in ([17], (2.2)), another definition of which, as a sum of squares of infinitesimal generators of the Bargmann-Wigner representation (*loc. cit*, definition 2.1), is also possible, in a way that reduces to (0.7) when $n = 1$: in that case, one may dispense with h through an obvious rescaling (changing c at the same time); when $n \neq 1$, the operators L^n are genuinely distinct for distinct values of h. Coming back to the case when $h = 1$, one may consider L^n as *a deformation of the Laplace-Beltrami operator on a rank-one symmetric space*. Indeed, denoting (as in the introduction, this time in n variables) as $\mathcal{M}$ the sheet of hyperboloid in $\mathbb{R}^{n+1}$ of equation $E = (c^2\,|\mathrm{p}^2| + c^4)^{1/2}$ and identifying $H_c^{1/2}(\mathbb{R}^n)$ [the obvious generalization of (0.6)] with $L^2(\mathcal{M}) = L^2(\mathcal{M}, \langle \mathbf{p} \rangle^{-1} d\mathbf{p})$ through the isometry $\mathcal{G}$ defined by

$$(\mathcal{G}\, u)(E,\, \mathbf{p}) = \langle \mathbf{p} \rangle\, \hat{u}(\mathbf{p}), \tag{1.19}$$

one may write ([17], (2.3))

$$\mathcal{G}(4\pi L^n)\mathcal{G}^{-1} = \Delta_{\mathcal{M}} + 4\pi |\mathbf{p}|^2 \tag{1.20}$$

where

$$\Delta_{\mathcal{M}} = -\sum \frac{\partial^2}{\partial \mathbf{p}_j^2} - c^{-2}\left[\left(\sum \mathbf{p}_j \frac{\partial}{\partial \mathbf{p}_j}\right)^2 + (n-1)\sum \mathbf{p}_j \frac{\partial}{\partial \mathbf{p}_j}\right] \tag{1.21}$$

is the Laplace-Beltrami operator on $\mathcal{M}$ associated with the ds^2 that is the negative of $c^{-2}\, dE^2 - |d\mathbf{p}|^2$. Taking the radial part of L^n, we shall thus get a "relativistic" deformation of the hypergeometric operator that corresponds to the radial part of $\Delta_{\mathcal{M}}$.

There is no need any more to assume that $h = 1$. Letting L^n, as defined in (1.18), act on a radial function

$$f(x) = u\left(\frac{|x|^2}{c^2}\right) = u(s), \tag{1.21}$$

we find

$$\begin{aligned} -4\pi L^n_{\mathrm{rad}} &= h^2\left[\frac{4s}{c^2}\frac{d^2}{ds^2} + \frac{2n}{c^2}\frac{d}{ds}\right] - 4\pi^2 c^2 s \\ &\quad + c^{-2}\left[h^2\left(2s\frac{d}{ds}\right)^2 + 2h(n-1)s\frac{d}{ds}\right] \\ &= -\frac{4h^2}{c^2} L\left(\frac{n-2}{2},\, \frac{n-1}{2h} - \frac{n}{2},\, \frac{\pi c^2}{h}\right), \end{aligned} \tag{1.22}$$

thus

$$\frac{\pi c^2}{h^2} L^n_{\mathrm{rad}} = L(\lambda,\, \mu,\, \nu) \tag{1.23}$$

with $(\lambda,\, \mu,\, \nu)$ as defined in (1.3).

In the case when $n = 1$, there is no loss of generality in assuming that $h = 1$: then L^1 is essentially self-adjoint both on the space $H_c^{1/2}(\mathbb{R})$ and the space $L^2\left(\mathbb{R},\ \left(1+\frac{x^2}{c^2}\right)^{-1/2}\frac{dx}{c}\right)$. For the radial, or even, part, (1.24) applies and yields

$$\pi\, c^2\, L^1_{\text{even}} = L\left(-\frac{1}{2},\ -\frac{1}{2},\ \pi\, c^2\right). \tag{1.24}$$

One can also, in this case, transfer the odd part under the map $f(x) = \frac{x}{c}\, u\left(\frac{x^2}{c^2}\right)$, which then yields

$$\pi\, c^2\, L^1_{\text{odd}} = L\left(\frac{1}{2},\ -\frac{1}{2},\ \pi\, c^2\right) - \frac{1}{4}. \tag{1.25}$$

Under these transfers, the even (odd) part of $L^2\left(\mathbb{R},\ \left(1+\frac{x^2}{c^2}\right)^{-1/2}\frac{dx}{c}\right)$ becomes $H^{-1/2,\,-1/2}$ (resp. $H^{1/2,\,-1/2}$) with the notation of theorem 1.1. Finally, it is the first self-adjoint extension of $L(\lambda,\ \mu,\ \nu)$ (as described in theorem 1.1 by the condition that $s^{\lambda+1}\,u'(s)$ should vanish at 0) which we are interested in, since in terms of $f = f(x)$ it expresses that $f'(x)$ (resp. $x f'(x) - f(x)$) vanishes at zero.

Observe that, in the coordinate ξ such that $x = c \operatorname{sh} \xi$, one has

$$-4\,\pi\, c^2\, L^1 = \frac{d^2}{d\xi^2} - 4\,\pi^2\, c^4 \operatorname{sh}^2 \xi.$$

The search for eigenfunctions of the operator on the right-hand side is called the modified Mathieu equation. Replacing the function sh by sin leads to the Mathieu equation proper [*cf.* (6.11)]: the usual spectral problem associated with the Mathieu equation is the search for its $2\,\pi$-periodic eigenfunctions; it will occur naturally in section 6. Both the Mathieu equation and the modified Mathieu equation were introduced on the occasion of Mathieu's discussion [11] of the Dirichlet problem in an elliptic-shaped plane domain, as the result of an appropriate separation of variables. No explicit series or integral representation of Mathieu functions is known yet (to our knowledge), despite the fact that several books, among which [21] and [22], (and many papers) have been devoted in full or in part to this subject: *see* Jager [8], however, for an interpretation of the Fourier coefficients of a Mathieu function in terms of polynomials linked to the Klein-Gordon calculus.

That the Mathieu equation transfers, under some appropriate change of variable, to some special case of the "chronogeometric" equation, was already known to and used by Lindemann, as reported in ([22], p. 417).

2. THE DISCRETE SERIES OF $SL(2, \mathbb{R})$

It will be helpful in our understanding of $L(\lambda, \mu, \nu)$. Only it is the full projective series $\{\pi_\lambda;\ \lambda > -1\}$, as considered in connection with the universal covering group of $SL(2, \mathbb{R})$ (Pukanszky [14]) that we need here: when λ is a non-negative integer, π_λ will be a genuine representation, namely the one denoted as $\mathcal{D}^+_{\lambda+1}$ in Knapp ([9], p. 35). We use the real type realization of the holomorphic discrete series of $SL(2, \mathbb{R})$ which, for various classes of groups, has been described by Gross-Kunze ([5], [6]) in terms of Bessel kernels.

Recall, for instance from ([20], p. 89-90) that there exists a unique map π_λ from $SL(2, \mathbb{R})$ to the unitary group of $H^{\lambda, 0}$ satisfying the following properties:

(i) let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$: if $\gamma = 0$ and $\alpha > 0$, then

$$(\pi_\lambda(g)\,u)(s) = \alpha^{1+\lambda} \exp(i\,\nu\alpha\beta s)\,u(\alpha^2 s);$$

(ii) if $\gamma > 0$,

$$(\pi_\lambda(g)\,u)(s) = \frac{\nu}{\gamma}\,e^{-i\pi(\lambda+1)/2} \int_0^\infty u(t)\left(\frac{t}{s}\right)^{\lambda/2} \times \exp\left(\frac{i\,\nu}{\gamma}(\alpha\,s + \delta t)\right) J_\lambda\left(\frac{2\,\nu}{\gamma}\sqrt{st}\right) dt;$$

(iii) if $\gamma < 0$, $\pi_\lambda(g) = e^{i\pi(\lambda+1)}\,\pi_\lambda(-g)$; if $\gamma = 0$ and $\alpha < 0$, $\pi_\lambda(g) = e^{-i\pi(\lambda+1)}\,\pi_\lambda(-g)$.

Then π_λ is a projective representation: more precisely, for all g, $g_1 \in SL(2, \mathbb{R})$, one has

$$\text{(iv)} \qquad \pi_\lambda(g)\,\pi_\lambda(g_1) = \omega\pi_\lambda(g\,g_1)$$

for some number $\omega \in \exp(2\,i\pi\lambda\,\mathbb{Z})$ depending on the pair (g, g_1). One can go to *loc. cit.* for a proof valid for $\lambda > 0$, then use analytic continuation. A more self-contained method is along the following lines. Consider the Hankel transformation $\mathcal{H}$ defined (Magnus-Oberhettinger-Soni [10], p. 397) on $(0, \infty)$ by

$$(\mathcal{H}\,u)(\sigma) = \int_0^\infty (\sigma\tau)^{1/2}\,u(\tau)\,J_\lambda(\sigma\tau)\,d\tau. \tag{2.1}$$

Setting $s = \frac{\gamma\sigma^2}{2\,\nu}$, $t = \frac{\gamma\tau^2}{2\,\nu}$ and using $\mathcal{H}^2 = I$, one immediately sees that the maps $\pi_\lambda(g)$ are isometries of $H^{\lambda, 0}$. On the other hand, the set of functions $\psi_z\,(z \in \mathbb{C},\ \mathrm{Im}\, z > 0)$, with $\psi_z\,(s) = e^{i\nu z s}$, is total in $H^{\lambda,\,0}$. As a consequence of the formula for the Laplace transform of $t^{\lambda/2}\, J_\lambda\,(\mathrm{a}t^{1/2})$ ([10], p. 446), one finds after an elementary calculation that, when $\gamma > 0$ (this was case (ii) on our list), one has

$$\pi_\lambda\,(g)\,\psi_z = (\gamma\, z + \delta)^{-\lambda-1}\,\psi_{\frac{\alpha z+\beta}{\gamma z+\delta}} \tag{2.2}$$

with the argument of $\gamma\, z + \delta$ in $]0,\ \pi[$. The property (iv) easily follows.

A particularly interesting unitary transformation of $H^{\lambda,\,0}$ is the involutive transformation

$$\mathcal{F}_{\lambda,\,\nu} = e^{i\pi\,(\lambda+1)/2}\,\pi_\lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.3}$$

i.e.

$$(\mathcal{F}_{\lambda,\,\nu}\, u) = \nu \int_0^\infty u\,(t) \left(\frac{t}{s}\right)^{\lambda/2} J_\lambda\,(2\,\nu\,\sqrt{st})\, dt. \tag{2.4}$$

Assume $\lambda = \dfrac{n-2}{2}$ for some integer $n \geq 1$. Under the identification $f\,(x) = u\left(\dfrac{|x|^2}{c^2}\right)$, introduced in (1.22), of $H^{\lambda,\,0}$ with a space of radial functions on $\mathbb{R}^n$, and with $\nu = \dfrac{\pi\, c^2}{h}$, $\mathcal{F}_{\lambda,\,\nu}$ just corresponds to the radial part of the Fourier transformation on $\mathbb{R}^n$, defined as

$$(\mathcal{F}\, u)\,(x) = h^{-\frac{n}{2}} \int \exp\left(\frac{2\,i\,\pi}{h}\,\langle x,\ y\rangle\right) u\,(y)\ dy. \tag{2.5}$$

This is a consequence of the classical formula (*cf.* e.g. Schwartz [16], vol. 2, p. 115) for the Fourier transform of radial functions.

Choose

$$\varepsilon_0 = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \qquad \varepsilon_1 = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix}, \qquad \varepsilon_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \tag{2.6}$$

as a basis of the Lie algebra of $SL\,(2,\ \mathbb{R})$: then

$$e^{-\theta\varepsilon_0} = \begin{pmatrix} \cos\dfrac{\theta}{2} & -\sin\dfrac{\theta}{2} \\ \sin\dfrac{\theta}{2} & \cos\dfrac{\theta}{2} \end{pmatrix} \tag{2.7}$$

so that, as a consequence of (2.2) (the calculation is detailed in ([20], proposition 2.1), but some interwining operator must take place since the space denoted H^{λ} there would be denoted as $H^{-\lambda,\,0}$ here), the infinitesimal operator

$$e_0 = -\frac{1}{2\,i\,\pi}\,\frac{d}{d\theta}\,(\pi_\lambda\,(e^{-\theta\varepsilon_0}))\quad (\theta = 0) \tag{2.8}$$

is given formally, on $H^{\lambda,\,0}$, as

$$e_0 = -\frac{1}{4\,\pi\nu}\left[s\,\frac{d^2}{ds^2} + (\lambda+1)\,\frac{d}{ds} - \nu^2\,s\right]. \tag{2.9}$$

The other infinitesimal generators of the projective representation π_λ are quite simple to compute: with obvious notations, they are given formally as

$$e_0 - e_1 = \frac{\nu\,s}{2\,\pi},\qquad e_2 = \frac{1}{2\,i\,\pi}\left(s\,\frac{d}{ds} + \frac{1+\lambda}{2}\right). \tag{2.10}$$

At least when $-1 < \lambda < 1$, we now want a characterization of the domain, in the sense of Stone's theorem, of the self-adjoint extension of e_0 that is the infinitesimal generator of the one-parameter unitary group which coincides with $\theta \mapsto \pi_\lambda\,(e^{-\theta\varepsilon_0})$ for $0 < \theta < 2\,\pi$. More precisely, we are interested in the boundary condition at zero which characterizes this domain. Now a complete orthonormal basis $(\varphi_n)_{n\geq 0}$ of $H^{\lambda,\,0}$ consisting of eigenstates of the extension of e_0 which we are interested in is given by

$$\varphi_n\,(s) = (2\,\nu)^{\frac{\lambda+1}{2}}\left(\frac{n!}{\Gamma\,(\lambda+n+1)}\right)^{1/2} e^{-\nu s}\,L_n^{(\lambda)}\,(2\,\nu\,s) \tag{2.11}$$

with the corresponding eigenvalues

$$a_n = \frac{1}{2\,\pi}\left(n + \frac{\lambda+1}{2}\right). \tag{2.12}$$

Indeed, the orthogonality and normalization come from [10], p. 241. With $\psi_z\,(s) = e^{i\nu z s}$, $\operatorname{Im} z > 0$, it follows from ([10], p. 244) that

$$\begin{aligned}(\varphi_n,\,\psi_z) &= \int_0^\infty \varphi_n\,(s)\,\bar{\psi}_z\,(s)\,s^\lambda\,ds \\ &= \left(\frac{2}{\nu}\right)^{\frac{\lambda+1}{2}}\left(\frac{\Gamma\,(\lambda+n+1)}{n!}\right)^{1/2} \\ &\quad\times (1+i\,\bar{z})^{-\lambda-n-1}\,(-1+i\,\bar{z})^n.\end{aligned} \tag{2.13}$$

Thus, using (2.2), one gets for $0 < \theta < 2\pi$ that

$$(2.14) \qquad (\varphi_n, \pi_\lambda(e^{-\theta\varepsilon_0})\psi_z) = \exp i\theta\left(n + \frac{\lambda+1}{2}\right) \cdot (\varphi_n, \psi_z),$$

from which the equation $e_0\,\varphi_n = a_n\varphi_n$ follows in a true, not just formal, sense. The completeness is standard ([10], p. 239).

As we have found no reference for this certainly known fact, we now prove that the boundary condition at zero that characterizes the domain of e_0 is $s^{\lambda+1}\,u'(s) = 0$. Assume that $u = \sum c_n\,\varphi_n$ lies in this domain, so that

$$(2.15) \qquad \sum \left(n + \frac{\lambda+1}{2}\right)^2 |c_n|^2 < \infty.$$

Then, calling α_n the numerical coefficient on the right-hand side of (2.11), and using the fact that the derivative of $L_n^{(\lambda)}$ is $-L_{n+1}^{(\lambda+1)}$, one may write, for $s > 0$,

$$(2.16) \qquad s^{\lambda+1}\,u'(s) = -\,\nu \sum c_n\,\alpha_n\,s^{\lambda+1}\,e^{-\nu s} \times [2\,L_{n-1}^{(\lambda+1)}(2\,\nu\,s) + L_n^{(\lambda)}(2\,\nu\,s)].$$

Now, from ([10], p. 242), one has for $0 \leq x < 1$ the identity

$$(2.17) \qquad \frac{1}{2\,\nu}\sum \alpha_n^2\,e^{-2\nu s}\,s^\lambda\,(L_n^{(\lambda)}(2\nu s))^2\,x^n = \frac{x^{-\lambda/2}}{1-x}\exp\left(-2\nu s\,\frac{1+x}{1-x}\right) I_\lambda\left(\frac{4\,\nu\,s x^{1/2}}{1-x}\right).$$

Using

$$(2.18) \qquad \left(n + \frac{\lambda+1}{2}\right)^{-2} = -\int_0^1 x^{n+\frac{\lambda-1}{2}}\,(\ln x)\,dx$$

and the Cauchy-Schwarz inequality to estimate each of the two terms on the right-hand side of (2.16), one is left with proving that, with $\delta = 0$ or 1, the expression

$$s^{\lambda+2-\delta}\int_0^1 |\ln x|^{1-\delta}\,(1-x)^{-1} \times \exp\left(-2\,\nu\,s\,\frac{1+x}{1-x}\right) I_{\lambda+\delta}\left(\frac{4\,\nu\,s x^{1/2}}{1-x}\right) dx$$

goes to zero as $s \to 0$, an elementary task for $-1 < \lambda < 1$.

In the reverse direction, we still have to show that if $u \in H^{\lambda,0}$ vanishes near ∞ and satisfies $s^{\lambda+1} u'(s) = 0$ at 0, then, defining $e_0 u$ on $]0, \infty[$ in the distribution sense and assuming that $e_0 u \in H^{\lambda,0}$, one may write

$$\int_0^\infty (e_0 u)(s)\, \bar{\varphi}_n(s)\, s^\lambda\, ds = \int_0^\infty u(s)\, (e_0\, \bar{\varphi}_n)(s)\, s^\lambda\, ds. \tag{2.19}$$

As

$$-4\pi\nu\, s^\lambda\, e_0 = \frac{d}{ds}\, s^{\lambda+1}\, \frac{d}{ds} - \nu^2\, s^{\lambda+1}, \tag{2.20}$$

this requires two integrations by parts, where the vanishing of the boundary terms at zero just requires the vanishing there of $s^{\lambda+1} u(s)$ and $s^{\lambda+1} u'(s)$. Finally, it is elementary that the second of these two conditions implies the first one when $\lambda > -1$, which concludes our characterization near zero of the domain of e_0.

End of proof of theorem 1.1. – Set

$$u(s) = s^{-\frac{1}{2}\left(\lambda+\frac{1}{2}\right)}\, w(2\, s^{1/2}), \tag{2.21}$$

so that $\|u\|^2_{\lambda,0} = \|w\|^2_{L^2}$. With $\xi = 2\, s^{1/2}$, e_0, as defined in (2.9), transfers to

$$\tilde{e}_0 = -\frac{1}{4\pi\nu}\left[\frac{d^2}{d\xi^2} - \left(\lambda^2 - \frac{1}{4}\right)\xi^{-2} - \frac{\nu^2}{4}\,\xi^2\right] \tag{2.22}$$

which differs from $(4\pi\nu)^{-1} E_\lambda$, as defined in (1.11), only by a term of order 0, bounded near zero. Moreover,

$$2^{\lambda+\frac{1}{2}}\, s^{\lambda+1}\, u'(s) = \xi^{\lambda+\frac{1}{2}}\, w'(\xi) - \left(\lambda + \frac{1}{2}\right)\xi^{\lambda-\frac{1}{2}}\, w(\xi) \tag{2.23}$$

so that, looking back at what was said just before (1.17), we are done. □

From now on, we shall always look at $L(\lambda, \mu, \nu)$ as realized as a self-adjoint operator on $H^{\lambda,\mu}$, under the boundary condition $s^{\lambda+1} u'(s) = 0$ at zero in the case $-1 < \lambda < 1$. One may observe that the difference between the operator in (1.7) and $4\pi\nu\, \tilde{e}_0$ (recall that these two operators are self-adjoint on $L^2((0, \infty); d\xi)$: they are the transfers, under appropriate isometries, of $L(\lambda, \mu, \nu)$ on $H^{\lambda,\mu}$ and $4\pi\nu\, e_0$ on $H^{\lambda,0}$) is the multiplication by some smooth function on $[0, \infty[$, bounded from below. This, together with the WKB analysis at the beginning of the proof of theorem 1.1, shows that the spectrum of $L(\lambda, \mu, \nu)$ is discrete, simple and can be arranged as an increasing sequence (ρ_k) going to infinity. From (1.10) we can give the following definition.

DEFINITION 2.1. – *Let (ρ_k) be the increasing sequence of eigenvalues of $L(\lambda, \mu, \nu)$. We shall denote as $\varphi_k(\lambda, \mu, \nu; s)$ the eigenfunction corresponding to the eigenvalue ρ_k, normalized by the condition that*

$$\varphi_k(\lambda, \mu, \nu; s) \sim e^{-2\nu s^{1/2}} s^{-\frac{1}{2}\left(\lambda+\mu+\frac{3}{2}\right)}, \qquad s \to \infty.$$

The operator $L(\lambda, \mu, \nu)$ lies in the enveloping algebra of the representation π_λ:

THEOREM 2.2. – *With e_0, e_1, e_2 defined in* (2.9) *and* (2.10), *one has the identity*

$$(4\pi\nu)^{-1} L(\lambda, \mu, \nu) = e_0 + \frac{\pi}{\nu}\left(e_0 + \frac{\mu}{4i\pi}\right)^2 + \frac{(\lambda+\mu+1)^2}{16\pi\nu}$$

where the two sides are considered as operators on $C_0^\infty(]0, \infty[)$. When extended as an operator on the space of C^∞ vectors of the representation π_λ, $L(\lambda, \mu, \nu)$ commutes with

$$\mathcal{G}(\lambda, \mu, \nu) = (1+s)^{-\mu} \mathcal{F}_{\lambda, \nu}.$$

Proof. – The first identity is just a trivial computation, based on (2.9) and (2.10); of course, the right-hand side extends as an endomorphism of the space of C^∞ vectors of π_λ. On that space, one may write

$$\mathcal{F}_{\lambda,\nu}\, e_0\, \mathcal{F}_{\lambda,\nu}^{-1} = e_0, \qquad \mathcal{F}_{\lambda,\nu}\, e_j\, \mathcal{F}_{\lambda,\nu}^{-1} = -e_j, \qquad (j = 1, 2), \tag{2.24}$$

where $\mathcal{F}_{\lambda,\nu}$ was defined in (2.3). The easiest way to prove this is to observe that, for $j = 1, 2$, one has

$$\pi_\lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \pi_\lambda(e^{t\varepsilon_j}) \left(\pi_\lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)^{-1} = (\pi_\lambda(e^{t\varepsilon_j}))^{-1}$$

up to some phase factor in the group $\exp(2i\pi\lambda\mathbb{Z})$ which has to be 1 for small t, and to take the derivative at $t = 0$.

From this identity one immediately gets

$$(2.25)\quad \mathcal{F}_{\lambda,\,\nu}\, L(\lambda,\, \mu,\, \nu)\, \mathcal{F}_{\lambda,\,\nu}^{-1} = L(\lambda,\, \mu,\, \nu) + 4\, i\, \pi \mu\, e_2$$
$$= L(\lambda,\, \mu,\, \nu) + 2\, \mu s\, \frac{d}{ds} + \mu\, (\lambda + 1),$$

and it is a routine matter to check that this is just the same as $(1 + s)^{\mu}\, L(\lambda,\, \mu,\, \nu)\, (1 + s)^{-\mu}$. □

3. A FEW EXACT FORMULAS

In this section, we exploit the fact that, on a symmetric space of rank one, the differential equation that characterizes spherical functions is a particular case of the hypergeometric equation: looking at the chronogeometric equation as a relativistic deformation of the former one, we shall prove for the function $\varphi_k\, (\lambda,\, \mu,\, \nu;\, s)$ introduced in definition 2.1 an integral identity similar to the one which characterizes spherical functions. This identity is valid for the functions φ_k normalized at infinity in the way indicated. In the case when $\mu = -\frac{1}{2}$, we shall give in section 5 another identity, valid for eigenfunctions normalized at zero. The latter one, as will be seen there, is a striking generalization of the one familiar for Legendre functions. As a reference for spherical functions, we use Helgason [7], Faraut [3] and Gangolli-Varadarajan [4].

Consider a rank-one symmetric space $G/_K$ of the non-compact type and set $p = m_\alpha$, $q = m_{2\alpha}$: here, α is a simple root and m_α (resp. $m_{2\alpha}$) is the dimension of the root space corresponding to α (resp. $m_{2\alpha}$). Denote as x the coordinate $\log a$ (it is usually denoted as t in harmonic analysis) arising from the Iwasawa decomposition nak of an element of G. In terms of x, the differential equation for spherical functions expresses itself ([7], p. 317; [3], p. 80; [4], p. 135) as the search of an eigenfunction for the operator

$$(3.1)\qquad \frac{d^2}{dx^2} + [(p + q) \coth x + q \operatorname{th} x]\, \frac{d}{dx}.$$

To get the hypergeometric equation for ${}_2F_1\, (\alpha,\, \beta,\, \gamma;\, -s)$ one must set $s = \operatorname{sh}^2 x$: note that α, as an argument in ${}_2F_1$, has nothing to do with a root!

Let us do this the other way round, transforming the operator $L(\lambda, \mu, \nu)$ under this change of variable: we get the operator $\tilde{L}(\lambda, \mu, \nu)$ with

$$(3.2) \qquad -4\tilde{L}(\lambda, \mu, \nu) = \frac{d^2}{dx^2} + [(2\lambda+1)\coth x + (2\mu+1)\operatorname{th} x]\frac{d}{dx} - 4\nu^2 \operatorname{sh}^2 x.$$

This is the same as (3.1), under the sole assumptions $p+q > -1$, $q > -1$, except for the additional "relativistic" term $-4\nu^2 \operatorname{sh}^2 x$. Two special cases should be singled out. When $\mu = -\frac{1}{2}$ and $\lambda = \frac{n-2}{2}$ with $n \in \mathbb{N}^*$, one may identify (3.1) with a truncation of (3.2) if $p = n-1$ and $q = 0$, which are precisely the multiplicities m_α and $m_{2\alpha}$ that correspond to the symmetric space $SO_0(1, n)/SO(n)$. Now this is not surprising since, as shown in (1.23), $L(\lambda, \mu, \nu)$ corresponds in that case to the radial part of the relativistic oscillator L^n on $\mathbb{R}^n$ with $h = 1$, and (1.20) actually shows that $4\pi L^n$ itself, not just its radial part, is a deformation of the Laplace-Beltrami operator on $G/_K$. The case when $\lambda = \mu = 0$ is of interest too (*see* the end of this section): then $p = 0$ and $q = 1$, which is the situation that arises from $G = SU(1, 1)$. Since the symmetric space then obtained is the same as the one arising from $G = SO_0(1, 2)$, one gets essentially the same equation (3.1) whether one chooses $(p, q) = (0, 1)$ or $(1, 0)$: one gets the second one from the first one by setting $x = \frac{y}{2}$. However, it is not indifferent whether one has an extra term $-4\nu^2 \operatorname{sh}^2 x$ or $-\nu^2 \operatorname{sh}^2 \frac{x}{2}$ in the complete chronogeometric operator (3.2): actually, the two cases correspond to $h = \frac{1}{2}$ of $h = 1$ respectively, and the operators are intrinsically different.

Recall the identity

$$(3.3) \qquad \varphi(g_1)\varphi(g_2) = \int_K \varphi(g_1 k g_2)\, dk,$$

a characterization of spherical functions ([7], p. 400 or [3], p. 4). In terms of the coordinate $x = \log a$, this can be written as

$$(3.4) \qquad \varphi(x)\varphi(y) = \int_0^\infty f(x, y, z)\varphi(z)\, dz$$

for some kernel $f(x, y, z)$ that need not be made explicit at this point but does not depend on which spherical function φ we have in mind. We now proceed to build such a kernel in connection with the eigenfunctions of $L(\lambda, \mu, \nu)$.

First note that under the change of variable $s = \operatorname{sh}^2 x$, the measure $s^\lambda (1+s)^\mu \, ds$ which defines the spaces $H^{\lambda, \mu}$ (on which $L(\lambda, \mu, \nu)$ is self-adjoint) transfers to

$$dm(x) = 2 (\operatorname{sh} x)^{2\lambda+1} (\operatorname{ch} x)^{2\mu+1} \, dx. \tag{3.5}$$

We must now use the Bessel functions I_λ, J_λ, K_λ as defined in the book [10] by Magnus, Oberhettinger and Soni; also, as a shorthand, set

$$\begin{aligned} i_{\lambda,\nu}(t) &= t^{-\lambda} I_\lambda(2\nu t), \\ k_{\lambda,\nu}(t) &= t^{-\lambda} K_\lambda(2\nu t), \\ j_{\lambda,\nu}(t) &= t^{-\lambda} J_\lambda(2\nu t) \end{aligned} \tag{3.6}$$

for $t > 0$. It is then a routine matter to check that the differential equations

$$\begin{aligned} i''_{\lambda,\nu} &= 4\nu^2 i_{\lambda,\nu} - \frac{2\lambda+1}{t} i'_{\lambda,\nu}, \\ k''_{\lambda,\nu} &= 4\nu^2 k_{\lambda,\nu} - \frac{2\lambda+1}{t} k'_{\lambda,\nu}, \\ j''_{\lambda,\nu} &= -4\nu^2 j_{\lambda,\nu} - \frac{2\lambda+1}{t} j'_{\lambda,\nu} \end{aligned} \tag{3.7}$$

are valid. Observe that $i_{\lambda,\nu}$ and $j_{\lambda,\nu}$ are the restrictions to $\mathbb{R}_*^+$ of smooth even functions on $\mathbb{R}$.

DEFINITION 3.1. – *Given $z \geq 0$, define F_z as the operator on $H^{\lambda,\mu}$ such that*

$$(F_z u)(\operatorname{sh}^2 x) = \int_0^\infty f(z, x, y)\, u(\operatorname{sh}^2 y)\, dm(y)$$

with

$$f(z, x, y) = 2\left(\frac{\nu^3}{\pi}\right)^{1/2} i_{\lambda,\nu}(\operatorname{sh} z \operatorname{sh} x \operatorname{sh} y)\, k_{\mu,\nu}(\operatorname{ch} z \operatorname{ch} x \operatorname{ch} y)$$

and dm as defined in (3.5).

THEOREM 3.2. – *For every $z \geq 0$, F_z is a bounded operator from $H^{\lambda,\mu}$ to itself, and sends $H^{\lambda,\mu}$ into the domain of $L(\lambda, \mu, \nu)$. The operator F_z commutes with $L(\lambda, \mu, \nu)$. As z goes to ∞, the operator*

$$e^{2\nu \operatorname{sh} z} (\operatorname{sh} z)^{\lambda+\mu+\frac{3}{2}} F_z$$

converges strongly to the identity operator.

Proof. – First observe that the kernel $f(z, x, y)$ is C^∞ on the closed set $\{z \geq 0, x \geq 0, y \geq 0\}$. The first point amounts to proving that, on $L^2(0, \infty; dy)$, the operator with kernel (with respect to dy)

$$\tilde{f}(z, x, y) = 2(\operatorname{sh} x \operatorname{sh} y)^{\lambda+\frac{1}{2}} (\operatorname{ch} x \operatorname{ch} y)^{\mu+\frac{1}{2}} f(z, x, y) \tag{3.8}$$

is bounded. Now this is a consequence of the estimates ([10], p. 139)

$$\begin{aligned} i_{\lambda,\nu}(t) &\sim (4\pi\nu)^{-\frac{1}{2}} t^{-\lambda-\frac{1}{2}} e^{2\nu t}, \\ k_{\mu,\nu}(t) &\sim \left(\frac{\pi}{4\nu}\right)^{\frac{1}{2}} t^{-\mu-\frac{1}{2}} e^{-2\nu t}, \qquad t \to +\infty \end{aligned} \tag{3.9}$$

together with

$$e^{2\nu(\operatorname{sh} z \operatorname{sh} x \operatorname{sh} y - \operatorname{ch} z \operatorname{ch} x \operatorname{ch} y)} \leq e^{-2\nu \operatorname{ch} z \operatorname{ch}(x-y)}. \tag{3.10}$$

The estimates (3.9) also show that, given $x > 0$ and $y > 0$, one has

$$\tilde{f}(z, x, y) \sim \left(\frac{\nu}{\pi}\right)^{1/2} (\operatorname{sh} z)^{-\lambda-\mu-1} \exp(-2\nu \operatorname{sh} z \operatorname{ch}(x-y)) \tag{3.11}$$

as $z \to +\infty$, from which the last assertion of theorem 3.2 follows.

It is clear from the definition of $f(z, x, y)$ that, for every $u \in H^{\lambda,\mu}$, the function $x \mapsto (F_z u)(\operatorname{sh}^2 x)$ extends in a natural way as a C^∞ *even* function on $\mathbb{R}$: thus, with $s = \operatorname{sh}^2 x$,

$$s^{\lambda+1} \frac{d}{ds}(F_z u)(s) = \frac{(\operatorname{sh} x)^{2\lambda+1}}{2 \operatorname{ch} x} \frac{d}{dx}(F_z u)(\operatorname{sh}^2 x) \tag{3.12}$$

vanishes at $s = x = 0$ since $\lambda > -1$.

To complete the proof of theorem 3.2, it thus suffices to show that the kernel $f(z, x, y)$ satifies the identity

$$\tilde{L}(\lambda, \mu, \nu; x) f(z, x, y) = \tilde{L}(\lambda, \mu, \nu; y) f(z, x, y) \tag{3.13}$$

where the differential operator $\tilde{L}(\lambda, \mu, \nu)$, self-adjoint on $L^2(0, \infty; dm)$ (whose precise domain is the image of that of $L(\lambda, \mu, \nu)$ under the change of variable $s = \operatorname{sh}^2 x$), was made explicit in (3.2), and where the extra letter x or y is to indicate that this differential operator is supposed to act on $f(z, x, y)$ considered as a function of x or y. To get rid of useless burden, set

$$f_0 = i_{\lambda,\nu} k_{\mu,\nu} \tag{3.14}$$

with the convention that $i_{\lambda,\,\nu}$ and its derivatives shall always be evaluated at $(\operatorname{sh} z \operatorname{sh} x \operatorname{sh} y)$, $k_{\mu,\,\nu}$ and its derivatives at $(\operatorname{ch} z \operatorname{ch} x \operatorname{ch} y)$. Then

$$\frac{\partial f_0}{\partial x} = \operatorname{sh} z \operatorname{ch} x \operatorname{sh} y \, i'_{\lambda,\,\nu} \, k_{\mu,\,\nu} + \operatorname{ch} z \operatorname{sh} x \operatorname{ch} y \, i_{\lambda,\,\nu} \, k'_{\mu,\,\nu} \tag{3.15}$$

and, taking (3.7) into account,

$$\begin{aligned} \frac{\partial^2 f_0}{\partial x^2} &= 2 \operatorname{sh} z \operatorname{ch} z \operatorname{sh} x \operatorname{ch} x \operatorname{sh} y \operatorname{ch} y \, i'_{\lambda,\,\nu} \, k'_{\mu,\,\nu} \\ &\quad + \left[\operatorname{ch} z \operatorname{ch} x \operatorname{ch} y - (2\,\mu + 1) \operatorname{ch} z \, \frac{\operatorname{sh}^2 x}{\operatorname{ch} x} \operatorname{ch} y \right] i_{\lambda,\,\nu} \, k'_{\mu,\,\nu} \\ &\quad + \left[\operatorname{sh} z \operatorname{sh} x \operatorname{sh} y - (2\,\lambda + 1) \operatorname{sh} z \, \frac{\operatorname{ch}^2 x}{\operatorname{sh} x} \operatorname{sh} y \right] i'_{\lambda,\,\nu} \, k_{\mu,\,\nu} \\ &\quad + 4\,\nu^2 [\operatorname{sh}^2 z \operatorname{ch}^2 x \operatorname{sh}^2 y + \operatorname{ch}^2 z \operatorname{sh}^2 x \operatorname{ch}^2 y] \, i_{\lambda,\,\nu} \, k_{\mu,\,\nu}. \end{aligned} \tag{3.16}$$

Thus, from (3.2),

$$\begin{aligned} & - 4 \tilde{L} (\lambda,\ \mu,\ \nu;\ x) \, f_0 \\ &= \frac{1}{4} (\operatorname{sh} 2\,z) (\operatorname{sh} 2\,x) (\operatorname{sh} 2\,z) \, i'_{\lambda,\,\nu} \, k'_{\mu,\,\nu} \\ &\quad + (2\,\lambda + 2) \operatorname{ch} z \operatorname{ch} x \operatorname{ch} y \, i_{\lambda,\,\nu} \, k'_{\mu,\,\nu} \\ &\quad + (2\,\mu + 2) \operatorname{sh} z \operatorname{sh} x \operatorname{sh} y \, i'_{\lambda,\,\nu} \, k_{\mu,\,\nu} \\ &\quad + 4\,\nu^2 \, [2 \operatorname{sh}^2 z \operatorname{sh}^2 x \operatorname{sh}^2 y + \operatorname{sh}^2 z \operatorname{sh}^2 x \\ &\qquad\qquad + \operatorname{sh}^2 x \ \operatorname{sh}^2 y + \operatorname{sh}^2 y \operatorname{sh}^2 z] \, i_{\lambda,\,\nu} \, k_{\mu,\,\nu}. \end{aligned} \tag{3.17}$$

This, as well as f_0, is a symmetric expression in $(x,\ y)$, which concludes the proof of theorem 3.2. □

Remark. – In ([17], section 4), we defined in a rather natural way a family of operators (F_s) in connection with the n-dimensional relativistic oscillator; then, taking the radial part and transferring the result under (1.22), we got the present family (F_z) in the case when $\mu = -\frac{1}{2}$ and $\lambda = \frac{n-2}{2}$: it was guessed in general from an extrapolation of this case.

On the other hand, that Mathieu functions do satisfy certain integral equations analogous to special cases of part (ii) in the theorem which follows (*cf.* the very last remark of the present paper) was reported by Whittaker (Whittaker-Watson [22], p. 407) as early as 1904; related integral equations hold in the theory of Lamé functions ([22], p. 565). We think that the present work puts this kind of integral identities into a more general perspective, at the same time linking them to the theory of spherical functions.

THEOREM 3.3. – *Set, for short,*

$$\varphi_k(s) = \varphi_k(\lambda,\ \mu,\ \nu;\ s)$$

to denote the function introduced in definition 2.1. *Then:*

(i) *for all* $z \geq 0$, *and all* $k \in \mathbb{N}$,

$$F_z\,\varphi_k = \varphi_k(\mathrm{sh}^2 z)\,\varphi_k;$$

(ii) *for all* $x,\ y \geq 0$, *and* $k \in \mathbb{N}$,

$$\varphi_k(\mathrm{sh}^2 x)\,\varphi_k(\mathrm{sh}^2 y) = \int_0^\infty f(x,\ y,\ z)\,\varphi_k(\mathrm{sh}^2 z)\,dm(z);$$

(iii) *one has the expansion*

$$f(x,\ y,\ z) = \sum_{k\geq 0} \|\varphi_k\|_{\lambda,\mu}^{-2}\,\varphi_k(\mathrm{sh}^2 x)\,\varphi_k(\mathrm{sh}^2 y)\,\varphi_k(\mathrm{sh}^2 z).$$

Proof. – From theorem 3.2, $F_z\,\varphi_k$ is an eigenfunction of $L(\lambda,\ \mu,\ \nu)$ relative to the same eigenvalue ρ_k as φ_k: thus $F_z\,\varphi_k = g(z,\ k)\,\varphi_k$ for some $g(z,\ k)$. As a function of z for fixed x, $(F_z\,\varphi_k)(\mathrm{sh}^2 x)$ lies in the domain of $\tilde{L}(\lambda,\ \mu,\ \nu)$, of which it is an eigenfunction relative to the eigenvalue ρ_k: this is a consequence of theorem 3.2, of (3.17) and of the symmetry with respect to the pair $(z,\ x)$ of $f(z,\ x,\ y)$ as well as of the right-hand side of (3.17). This proves (i), up to some constant factor depending only on k, which is taken care of by means of the last assertion in theorem 3.2, in view of the normalization of φ_k chosen in definition 2.1. Part (iii) is an immediate consequence of (i) (Mercer's theorem) and (ii) follows from (iii): it has been singled out in view of its similarity with the identity for spherical functions. □

Remark. – All this is equivalent to the operator identity

$$F_x\,F_y = \int_0^\infty f(x,\ y,\ z)\,F_z\,dm(z). \tag{3.18}$$

Going back to the relativistic oscillator on $\mathbb{R}^n$ with $h = 1$ $\left(\text{this corresponds to } \mu = -\frac{1}{2}\right)$ as described in (1.18)-(1.21), one should not believe that this kind of equation only holds for *radial* eigenfunctions of L^n. Indeed, assume $n = 2$ and, with $\mathcal{G}$ as in (1.19) and setting $p = (E,\ \mathbf{p}) \in \mathcal{M},\ \ldots$, define F_p by the formula

$$(\mathcal{G}\,F_p\,u)\,(p') = c^2 \int e^{-2\pi\,\{p,p',p''\}}\,(\mathcal{G}\,u)\,(p'')\,\frac{d\mathbf{p}''}{E''} \tag{3.19}$$

with

$$\begin{aligned}\{p,\ p',\ p''\} =\,& c^{-4}\,E\,E'\,E'' + c^{-1}\,p_1\,(p_1'\,p_1'' + p_2'\,p_2'') \\ &+ c^{-1}\,p_2\,(p_1'\,p_2'' - p_2'\,p_1'').\end{aligned} \tag{3.20}$$

This time, we have a *two*-parameter family of operators that commute with the given relativistic oscillator, and one can still show that

$$F_p\,F_{p'} = c^2 \int_{\mathcal{M}} e^{-2\pi\,\{p,p',p''\}}\,F_{p''}\,\frac{d\mathbf{p}''}{E''}. \tag{3.21}$$

This is obtained by an expansion of the integral kernel of the conjugation under $\mathcal{G}$ of each of the two sides, using the formula

$$\int_{\mathcal{M}} e^{-2\pi\,\langle q,p\rangle}\,\frac{d\mathbf{p}}{E} = c\,(c^2\,q_0^2 - |\mathbf{q}|^2)^{-1/2}\,e^{-2\pi c\,(c^2\,q_0^2 - |\mathbf{q}|^2)^{1/2}} \tag{3.22}$$

proved in ([18], proposition 4.3), and valid for every $q = (q_0,\ \mathbf{q}) \in \mathbb{R}^3$ with $c^2\,q_0^2 - |\mathbf{q}|^2 > 0,\ q_0 > 0$.

It is our belief that the theory of the chronogeometric equation is full of formulas yet to be discovered, of various depth. Here is a pair, based on the concept of Wigner function as recalled in the introductory section.

THEOREM 3.4. – *Denote as* $\rho_k\,(\lambda,\ \mu,\ \nu)$ *the* k-th *eigenvalue* $(k \geq 0)$ *of* $L\,(\lambda,\ \mu,\ \nu)$, *denoted as* ρ_k *in definition* 2.1. *Then, for every* $k \in \mathbb{N}$:

(i) $$\rho_{2k}\,(0,\ 0,\ 2\,\nu) = 4\,\rho_k\left(-\frac{1}{2},\ -\frac{1}{2},\ \nu\right) + \frac{1}{4}$$

and

$$\rho_{2k+1}\,(0,\ 0,\ 2\,\nu) = 4\,\rho_k\left(\frac{1}{2},\ -\frac{1}{2},\ \nu\right) - \frac{3}{4};$$

(ii) $$\varphi_{2k}\,(0,\ 0,\ 2\,\nu;\ s) = \left(-\frac{1}{2\,\nu}\,\frac{d}{ds}\right)^{1/2}\left[\varphi_k\left(-\frac{1}{2},\ -\frac{1}{2},\ \nu;\ s\right)\right]^2$$

and

$$\varphi_{2k+1}(0,\,0,\,2\nu;\,s)=\left(-\frac{1}{2\nu}\frac{d}{ds}\right)^{1/2}\left[s^{1/2}\varphi_k\left(\frac{1}{2},\,-\frac{1}{2},\,\nu;\,s\right)\right]^2,$$

where the non-local operator that appears on the right-hand sides makes sense as it applies to a function which is the Laplace transform of a measure supported in $\mathbb{R}^+$.

Proof. – Let $(\sigma_n)_{n\geq 0}$ denote the spectrum of the one dimensional relativistic oscillator (0.7), and let (φ_n) be the corresponding sequence of eigenfunctions, normalized by the condition ([17], theorem 3.4) that

$$\varphi_n(-x)\sim\left(\frac{x}{c}\right)^{-1/2}e^{-2\pi cx},\qquad x\mapsto+\infty \tag{3.23}$$

($\varphi_n(x)$ is associated with the relatistic oscillator, $\varphi_k(\lambda,\,\mu,\,\nu;\,s)$ with $L(\lambda,\,\mu,\,\nu)$: no confusion should arise). In ([18], (16.32)) we introduced the "Mathieu-Laguerre" operator

$$\text{M.L.}=-\frac{1}{4\pi}\left[r\left(1+\frac{r}{c^2}\right)\frac{d^2}{dr^2}+\left(1+\frac{2r}{c^2}\right)\frac{d}{dr}-4\pi^2r\right] \tag{3.24}$$

on $]0,\,\infty[$: setting $\nu=\pi c^2$ and switching to the new variables $s=\frac{r}{c^2}$, we get [from (1.2)]

$$\text{M.L.}=(4\nu)^{-1}L(0,\,0,\,2\nu). \tag{3.25}$$

The self-adjoint extension of M.L., on $L^2(\mathbb{R}^+,\,dr)$, we were interested in was defined ([18], proposition 15.7) so that the eigenfunctions of M.L. should be the restrictions to $\mathbb{R}^+$ of C^∞ even functions on $\mathbb{R}$: thus (3.25) holds in a genuine, not only formal sense, and the result of ([18], theorem 15.11 and statement just before (16.32)) is that the n-th eigenvalue of M.L. is $\sigma_n+\frac{1}{16\nu}$. As proved by Jager [8], the function φ_n is even or odd according to whether n is: from (1.25) and (1.26), which link the even and odd parts of the relativistic oscillator to $\nu^{-1}L\left(\pm\frac{1}{2},\,-\frac{1}{2},\,\nu\right)$, and from (3.25), we get the part (i) in theorem 3.4.

From ([18], theorem 15.11), an eigenfunction of M.L. corresponding to the n-th eigenvalue is $\Phi_n(r)$ [with r as in (0.12)], the Wigner function (in the Klein-Gordon calculus) associated with φ_n. According to ([17], proposition 3.6 and corollary 3.5), one has

$$(3.26)\quad \begin{aligned} \Phi_n(r) &= c\,K \int_{\mathbf{R}} (2\,e^t)^{1/2}\, e^{-2\pi c^2 \operatorname{ch} t}\, e^{-2\pi r e^t}\, \varphi_n(c \operatorname{sh} t)\, dt \\ &= K \left(-\frac{1}{\pi}\frac{d}{dr}\right)^{1/2} c \int_{\mathbf{R}} e^{-2\pi c^2 \operatorname{ch} t}\, e^{-2\pi r e^t}\, \varphi_n(c \operatorname{sh} t)\, dt \\ &= K \left(-\frac{1}{\pi}\frac{d}{dr}\right)^{1/2} (\varphi_n(r^{1/2}))^2 \end{aligned}$$

where K is some constant (the value of which, namely $(-1)^n\,(\varphi_n(ic))^{-2}$, led to (0.14)). What matters here is that, as a consequence of (3.23) and of the stationary phase method applied to the integral (3.26) one has the estimate [8]

$$(3.27)\qquad \Phi_n(r) \sim K.2^{1/2}\, c^{3/2}\, r^{-3/4}\, e^{-4\pi c r^{1/2}}, \qquad r \mapsto \infty.$$

Then, under the transform $f \mapsto u$, with

$$f(x) = u\left(\frac{x^2}{c^2}\right) \left[\text{resp.}\ f(x) = \frac{x}{c}\, u\left(\frac{x^2}{c^2}\right)\right],$$

that led to (1.25) [resp. (1.26)], $\varphi_{2k}(x)$ [resp. $\varphi_{2k+1}(x)$] transfers to $\varphi_k\left(-\frac{1}{2}, -\frac{1}{2}, \nu;\ s\right)$ $\left[\text{resp.}\ -\varphi_k\left(\frac{1}{2}, -\frac{1}{2}, \nu;\ s\right)\right]$, as follows from a comparison between (3.23) and the normalization introduced in definition 2.1. Finally, as shown by (3.27), $\Phi_n(r)$ transfers to $2^{1/2}\, K\, \varphi_n(0,\ 0,\ 2\,\nu;\ s)$ under the change of variable $r = c^2\, s$, which proves part (ii) of theorem 3.4. □

4. THE KLEIN-GORDON-BESSEL EQUATION

In his very first paper on the functions which bear his name, Mathieu [11] introduced the change of variables defined by $x = \operatorname{ch}\xi \operatorname{ch}\eta$, $y = \operatorname{sh}\xi \operatorname{sh}\eta$ in order to separate the variables in the Helmholtz problem $\Delta u + k^2\, u = 0$ in such a way as to make a solution of the associated Dirichlet problem in an elliptic-shaped plane domain possible; a related change of variables (Miller [12]) permits to separate variables in the 2-dimensional Klein-Gordon equation (*cf.* also [17], end of section 3, for the solution of the associated initial-value problem). In connection with chronogeometric functions, we here do the same for the Klein-Gordon-Bessel equation

$$\left[t\,\frac{\partial^2}{\partial t^2}+(\mu+1)\,\frac{\partial}{\partial t}-s\,\frac{\partial^2}{\partial s^2}-(\lambda+1)\,\frac{\partial}{\partial s}+\nu^2\right]\psi=0. \tag{4.1}$$

With $t=\tau^2$, $s=\sigma^2$, and setting $\nu=\frac{\pi\,c^2}{h}$, we can put the K.G.B. equation in the form

$$\frac{h^2}{4\,\pi^2}\left[\frac{\partial^2\,\psi}{\partial\tau^2}+\frac{2\,\mu+1}{\tau}\,\frac{\partial\psi}{\partial\tau}-\frac{\partial^2\,\psi}{\partial\sigma^2}-\frac{2\,\lambda+1}{\sigma}\,\frac{\partial\psi}{\partial\sigma}\right]+c^4\,\psi=0. \tag{4.2}$$

Now, on $\mathbb{R}^d$, $\frac{d^2}{d\sigma^2}+\frac{d-1}{\sigma}\,\frac{d}{d\sigma}$ represents the radial part of the Laplacian: thus, when $2\,\lambda$ and $2\,\mu$ are integers, (4.2) may be thought of as the bi-radial part of a fancy "Klein-Gordon"-like equation in any number of time, or space, variables: this may be another justification for the name "chronogeometric" which we have adopted. The ultrahyperbolic *wave* equation was already considered by Asgeirsson (*cf.* Helgason [7], p. 318).

Here is, briefly stated, a solution of the initial-value problem for the K.G.B. equation. Still assume $\lambda>-1$, $\mu>-1$ and, in a way similar to definition 3.1, consider for $z\geq 0$ the operator E_z defined by

$$(E_z\,u)\,(\mathrm{sh}^2\,x)=\int_0^\infty e\,(z,\,x,\,y)\,u\,(\mathrm{sh}^2\,y)\,dm\,(y) \tag{4.3}$$

with

$$e\,(z,x,y)=\Gamma\,(\mu{+}1)\,\nu^{1-\mu}\,j_{\lambda,\,\nu}\,(\mathrm{ch}\,z\,\mathrm{sh}\,x\,\mathrm{sh}\,y)\,j_{\mu,\,\nu}\,(\mathrm{sh}\,z\,\mathrm{ch}\,x\,\mathrm{ch}\,y). \tag{4.4}$$

In terms of the s-coordinate, one may write

$$\begin{aligned}(E_z\,u)\,(s)=&\,\Gamma\,(\mu+1)\,\nu^{1-\mu}\,(\mathrm{ch}\,z)^{-\lambda}\,(\mathrm{sh}\,z)^{-\mu}\\ &\times\int_0^\infty\left(\frac{t}{s}\right)^{\lambda/2}\left(\frac{1+t}{1+s}\right)^{\mu/2}J_\lambda\,(2\,\nu\,(\mathrm{ch}\,z)\,\sqrt{st})\\ &\times J_\mu\,(2\,\nu\,(\mathrm{sh}\,z)\,\sqrt{(1+s)\,(1+t)})\,u\,(t)\,dt.\end{aligned} \tag{4.5}$$

One should compare (4.5) to (2.4) and get in particular the relation

$$E_0=[\mathcal{G}\,(\lambda,\,\mu,\,\nu)]^{-1} \tag{4.6}$$

with $\mathcal{G}(\lambda, \mu, \nu)$ as defined in theorem 2.2. With the same proof as that of theorem 3.2, one can show that when defined, say, on the space $C_0^\infty(]0, \infty[)$, E_z commutes with $L(\lambda, \mu, \nu)$.

Finally, the standard method of separation of variables shows the following. Given $u \in H^{\lambda, 0}$ actually in $D(e_0 + e_1)$, the domain of the infinitesimal operator of π_λ formally given by

$$e_0 + e_1 = -\frac{1}{2\pi\nu}\left[s\frac{d^2}{ds^2} + (\lambda + 1)\frac{d}{ds}\right], \tag{4.7}$$

the K.G.B. equation (4.1) admits exactly one solution satisfying the following two conditions: (i) the map $t \mapsto \psi(t, \cdot)$ is a C^1 function from $[0, \infty[$ to $H^{\lambda, 0}$; (ii) $\psi(0, s) = u(s)$. Moreover, the solution is given by

$$\psi(t, s) = (E_z\,\mathcal{G}(\lambda, \mu, \nu)\,u)(\operatorname{sh}^2 x) \tag{4.8}$$

with

$$t^{1/2} \pm s^{1/2} = \operatorname{sh}(z \pm x). \tag{4.9}$$

5. AN EXTENSION OF THE IDENTITY OF SPHERICAL FUNCTIONS

Theorem 3.3 cannot have an interesting limit as $\nu\left(= \frac{\pi c^2}{h}\right)$ goes to zero, since it depends on a normalization at ∞ of the eigenfunctions of $L(\lambda, \mu, \nu)$ which is meaningless when $\nu = 0$. In the present section we introduce, in the case when $\mu = -\frac{1}{2}$ (*i.e.* $h = 1$) and $\lambda > -\frac{1}{2}$, another identity, fully satisfactory in this respect: it depends on a normalization at zero, and applies to generalized eigenfunctions of $L\left(\lambda, -\frac{1}{2}, \nu\right)$, still satisfying the condition $s^{\lambda+1}\varphi'(s)$ at $s = 0$ but without any restriction at infinity. As the roots of the indicial equation at zero are 0 and $-\lambda$, the boundary condition just amounts, for an eigenfunction φ of $L\left(\lambda, -\frac{1}{2}, \nu\right)$, to its being analytic (or C^1) at zero.

Set $\psi(\operatorname{ch} x) = \varphi(\operatorname{sh}^2 x)$, *i.e.* $\psi(s) = \varphi(s^2-1)$, so that (1.1) transforms to

$$(s^2-1)\frac{d^2\psi}{ds^2} + 2(\lambda+1)\,s\,\frac{d\psi}{ds} + 4\nu^2(1-s^2)\,\psi = -4\rho\psi. \tag{5.1}$$

THEOREM 5.1. – *Assume that* $\lambda > -\frac{1}{2}$. *Let* ψ *be an eigenfunction of the operator on the left-hand side of* (5.1), *analytic on* $[1, \infty[$ *and such that* $\psi(1) = 1$. *For all* $x \geq 0$, $z \geq 0$, *one has*

$$\psi(\operatorname{ch} x)\,\psi(\operatorname{ch} z) = \frac{\Gamma(\lambda+1)}{\pi^{1/2}\,\Gamma\left(\lambda+\frac{1}{2}\right)} \times \int_0^\pi \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\nu^{\lambda-\frac{1}{2}}}\, j_{\lambda-\frac{1}{2},\nu}(\operatorname{sh} x\operatorname{sh} z\,\sin\theta) \times \psi(\operatorname{ch} x\operatorname{ch} z + \operatorname{sh} x\operatorname{sh} z\,\cos\theta)\,\sin^{2\lambda}\theta\,d\theta.$$

Remark. – When $\nu = 0$ or $t = 0$, one has

$$\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\nu^{\lambda-\frac{1}{2}}}\, j_{\lambda-\frac{1}{2},\nu}(t) = 1, \qquad (\nu t = 0) \tag{5.2}$$

as a consequence of definition 3.6. Thus, when $\nu = 0$ and $\lambda = \frac{n-2}{2}$ with $n \geq 0$, theorem 5.1 reduces (*cf.* [3]) to the identity for spherical functions on $SO_0(1, n)/SO(n)$. Also, for any value of ν, the identity is trivially true when x or z is zero.

Proof of theorem 5.1. – Since $j_{\lambda-\frac{1}{2},\nu}$ is a C^∞ even function on $\mathbb{R}$, the right-hand side $R(x, z)$ of the claimed identity is a smooth function on $[0, \infty[\times[0, \infty[$. With the same proof as that of theorem 3.3 (i), it thus suffices to show that if ψ satisfies (5.1), then so does $R(x, z)$, for any given $z > 0$, as a function of $s = \operatorname{ch} x$: also, it is sufficient to prove this on the open set where $x > 0$, $x \neq z$.

To this effect, introduce the function

$$\chi(x) = (\operatorname{sh} x)^{2\lambda}\,\psi(\operatorname{ch} x) = (\operatorname{sh} x)^{2\lambda}\,\varphi(\operatorname{sh}^2 x)\ : \tag{5.3}$$

then the equation (5.1) transforms to $\bar{L}\chi = \left(\rho - \frac{\lambda}{2}\right)\chi$ with

$$-4\,\bar{L} = \frac{d^2}{dx^2} + (1 - 2\,\lambda)\,\frac{\operatorname{ch} x}{\operatorname{sh} x}\,\frac{d}{dx} - 4\,\nu^2 \operatorname{sh}^2 x, \tag{5.4}$$

as is readily seen from (3.2); the operator $\bar{L}$ is formally self-adjoint on $(0,\ \infty)$, with respect to the measure $(\operatorname{sh} x)^{1-2\lambda}\,dx$, as seen from (3.5).

Besides the change of function from ψ to χ, let us perform in the integral on the right-hand side of the claimed identity the change of variable defined by $y > 0$,

$$\operatorname{ch} y = \operatorname{ch} x \operatorname{ch} z + \operatorname{sh} x \operatorname{sh} z\, \cos\,\theta. \tag{5.5}$$

Then y describes the interval $(|x - z|,\ x + z)$; also

$$\left|\frac{d\theta}{dy}\right| = \frac{\operatorname{sh} y}{\operatorname{sh} x \operatorname{sh} z\, \sin\,\theta} \tag{5.6}$$

and

$$(\operatorname{sh} x \operatorname{sh} z\, \sin\,\theta)^2 = \operatorname{sh}^2 x \operatorname{sh}^2 z - (\operatorname{ch} y - \operatorname{ch} x \operatorname{ch} z)^2, \tag{5.7}$$

a function fully symmetric in $(x,\ y,\ z)$ as can be seen at once.

Denoting as $g_\lambda\,(\xi)$ the (locally summable) function on $\mathbb{R}$ that vanishes for $\xi \leq 0$, such that

$$g_\lambda\,(\xi) = \frac{\Gamma\,(\lambda + 1)}{\pi^{1/2}\,\nu^{\lambda - \frac{1}{2}}}\,\xi^{\lambda - \frac{1}{2}}\,j_{\lambda - \frac{1}{2},\ \nu}\,(\xi^{1/2}) \tag{5.8}$$

for $\xi > 0$, one finally sees that, in terms of χ, theorem 5.1 would express itself as

$$\begin{aligned} \chi\,(x)\,\chi\,(z) = \int_0^\infty & g_\lambda\,(\operatorname{sh}^2 x \operatorname{sh}^2 z - (\operatorname{ch} y - \operatorname{ch} x \operatorname{ch} z)^2) \\ & \times\ \chi\,(y)\,(\operatorname{sh} y)^{1-2\lambda}\,dy. \end{aligned} \tag{5.9}$$

For any fixed z, set

$$\begin{aligned} h\,(x,\ y) &= \operatorname{sh}^2 x \operatorname{sh}^2 z - (\operatorname{ch} y - \operatorname{ch} x \operatorname{ch} z)^2 \\ &= (\operatorname{ch}\,(y + z) - \operatorname{ch} x)\,(\operatorname{ch} x - \operatorname{ch}\,(y - z)). \end{aligned} \tag{5.10}$$

On the open set defined by $x > 0,\ x \neq z,\ y > 0,\ y \neq z$, the function $w_\lambda(x,\ y) = g_\lambda(h(x,\ y))$ is locally integrable since, as seen from (5.10), h is a submersion at each point $(x,\ y) \in \Sigma = \{(x,\ y) : h(x,\ y) = 0\}$; also, the two projections from Σ, the singular support of w_λ, are proper maps. Thus, in order to finish the proof of theorem 5.1, all that remains to be done is to show that, with

$$
\begin{aligned}
(5.11)\quad P &= \bar{L}_x - \bar{L}_y \\
&= -\frac{1}{4}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) + \left(\frac{\lambda}{2} - \frac{1}{4}\right) \times \left[\frac{\operatorname{ch} x}{\operatorname{sh} x}\frac{\partial}{\partial x} - \frac{\operatorname{ch} y}{\operatorname{sh} y}\frac{\partial}{\partial y}\right] \\
&\quad + \nu^2(\operatorname{sh}^2 x - \operatorname{sh}^2 y),
\end{aligned}
$$

w_λ satisfies the equation $P\,w_\lambda = 0$, in the distribution sense, in the open set $\{(x,\ y) \in \mathbb{R} : x > 0,\ y > 0,\ x \neq z,\ y \neq z\}$.

Now w_λ still makes sense, as a locally integrable function, for any complex λ with $\operatorname{Re}\lambda > -\frac{1}{2}$, and the map $\lambda \mapsto w_\lambda$ is a holomorphic family of distributions: so is then the map $\lambda \mapsto P\,w_\lambda$, and it suffices to prove that $P\,w_\lambda = 0$ for real $\lambda > \frac{5}{2}$, in which case g_λ is a C^2 function on $\mathbb{R}$; formal computations valid in the open set where $h(x,\ y) > 0$ will do the work in that case. With $\xi = h(x,\ y)$, one has

$$
(5.12)\qquad \frac{\partial}{\partial x} w_\lambda = -2\operatorname{sh} x\,(\operatorname{ch} x - \operatorname{ch} y \operatorname{ch} z)\, g'_\lambda(\xi),
$$

$$
\begin{aligned}
(5.13)\qquad \frac{\partial^2}{\partial x^2} w_\lambda &= 4\operatorname{sh}^2 x\,(\operatorname{ch} x - \operatorname{ch} y \operatorname{ch} z)^2\, g''_\lambda(\xi) \\
&\quad + 2(\operatorname{ch} x \operatorname{ch} y \operatorname{ch} z - \operatorname{ch}^2 x - \operatorname{sh}^2 x)\, g'_\lambda(\xi).
\end{aligned}
$$

To compute $P\,w_\lambda$, we note that

$$
\begin{aligned}
(5.14)\quad & \operatorname{sh}^2 x\,(\operatorname{ch} x - \operatorname{ch} y \operatorname{ch} z)^2 - \operatorname{sh}^2 y\,(\operatorname{ch} y - \operatorname{ch} x \operatorname{ch} z)^2 \\
&= \frac{1}{4}[(\operatorname{sh} 2x)^2 - (\operatorname{sh} 2y)^2] \\
&\quad + \operatorname{ch} x \operatorname{ch} y \operatorname{ch} z\,(\operatorname{ch} 2y - \operatorname{ch} 2x) \\
&\quad + \operatorname{ch}^2 z\,(\operatorname{sh}^2 x \operatorname{ch}^2 y - \operatorname{sh}^2 y \operatorname{ch}^2 x)
\end{aligned}
$$

$$
\begin{aligned}
&= \operatorname{sh}(x-y)\operatorname{ch}(x+y)\operatorname{sh}(x+y)\operatorname{ch}(x-y)\\
&\quad - [\operatorname{ch}(x+y)+\operatorname{ch}(x-y)]\operatorname{ch} z\operatorname{sh}(x+y)\operatorname{sh}(x-y)\\
&\quad + \operatorname{ch}^2 z\operatorname{sh}(x+y)\operatorname{sh}(x-y)\\
&= -\xi\operatorname{sh}(x-y)\operatorname{sh}(x+y)
\end{aligned}
$$

according to (5.10). Thus

$$
\begin{aligned}
(5.15)\quad (P\,w_\lambda)(x,\,y) &= \operatorname{sh}(x-y)\operatorname{sh}(x+y)\,\xi\,g_\lambda''(\xi)\\
&\quad + \left(\frac{3}{2}-\lambda\right)(\operatorname{ch}^2 x - \operatorname{ch}^2 y)\,g_\lambda'(\xi)\\
&\quad + \nu^2(\operatorname{sh}^2 x - \operatorname{sh}^2 y)\,g_\lambda(\xi)\\
&= \operatorname{sh}(x-y)\operatorname{sh}(x+y)\\
&\quad \times \left[\xi\,g_\lambda''(\xi) + \left(\frac{3}{2}-\lambda\right)g_\lambda'(\xi) + \nu^2 g_\lambda(\xi)\right].
\end{aligned}
$$

Finally, as a consequence of (3.7), g_λ satisfies when $\lambda > \frac{5}{2}$ the differential equation

$$
(5.16)\qquad \xi\,g_\lambda''(\xi) + \left(\frac{3}{2}-\lambda\right)g_\lambda'(\xi) + \nu^2 g_\lambda(\xi) = 0,
$$

which concludes the proof of theorem 5.1. □

Here is a rephrasing of theorem 5.1, meant only to harmonize notations with the existing ($\nu = 0$) literature.

COROLLARY 5.2. – *Given $\alpha \in \mathbb{C}$ with* $\operatorname{Re}\alpha > 0$, and $\lambda > -\frac{1}{2}$, *denote as*

$$
\frac{\Gamma(2\lambda+1)\,\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda+1)}\,C_\alpha^{\lambda+\frac{1}{2}}(\nu;\,s)
$$

the solution ψ of the equation

$$
\begin{aligned}
&(s^2-1)\,\psi''(s) + (2\lambda+2)\,s\,\psi'(s) + 4\nu^2(1-s^2)\,\psi(s)\\
&\quad = \alpha(\alpha+2\lambda+1)\,\psi(s)
\end{aligned}
$$

which is analytic on $[1, \infty[$ *and such that* $\psi(1) = 1$. *Then, for all* $x \geq 0$, $y \geq 0$, *one has*

$$\frac{2^{2\lambda}\Gamma(\alpha+1)\left(\Gamma\left(\lambda+\frac{1}{2}\right)\right)^2}{\Gamma(\alpha+2\lambda+1)} C_\alpha^{\lambda+\frac{1}{2}}(\nu;\, \operatorname{ch} x)\, C_\alpha^{\lambda+\frac{1}{2}}(\nu;\, \operatorname{ch} y)$$

$$= \int_0^\pi \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\nu^{\lambda-\frac{1}{2}}} j_{\lambda-\frac{1}{2},\nu}(\operatorname{sh} x \operatorname{sh} y \sin\theta)$$

$$\times C_\alpha^{\lambda+\frac{1}{2}}(\nu;\, \operatorname{ch} x \operatorname{ch} y + \operatorname{sh} x \operatorname{sh} y \cos\theta) \sin^{2\lambda}\theta\, d\theta.$$

Proof and remark. – No proof is needed except for the fact that

$$\frac{2^{2\lambda}\Gamma(\alpha+1)\left(\Gamma\left(\lambda+\frac{1}{2}\right)\right)^2}{\Gamma(\alpha+2\lambda+1)} = \frac{\Gamma(2\lambda+1)\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda+1)} \times \frac{\pi^{1/2}\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda+1)},$$

a consequence of the duplication formula for the gamma function. □

The function just introduced generalizes Gegenbauer's function since, in accordance with the standard notation ([10], p. 199), one has

$$C_\alpha^{\lambda+\frac{1}{2}}(0;\, s) = C_\alpha^{\lambda+\frac{1}{2}}(s). \tag{5.17}$$

In the case when $\nu = 0$ and α is an integer, the coefficient in front of the left-hand side of the statement in corollary 5.2 may be checked against ([21], p. 290).

Remark. – In contrast with the (standard) $\nu = 0$ case, there exist when $\nu > 0$ two genuinely distinct integral formulas (theorems 3.3 and 5.1) for $\varphi_k \otimes \varphi_k$. Actually there is some link between them, and our first derivation of theorem 5.1 was based on theorem 3.3: however, such a (slightly simpler) proof would work only for eigenfunctions of $L\left(\lambda, -\frac{1}{2}, \nu\right)$ decaying at infinity, besides satisfying the relevant boundary condition at zero.

6. THE OPERATOR $L(\lambda, \mu, \nu)$ ON $]-1, 0[$

Consider now the equation

$$L(\lambda, \mu, \nu)\, u = -\rho\, u \tag{6.1}$$

(in contrast with (1.1), (1.2), note the minus sign) for $u \in L^2((-1, 0); (-s)^\lambda (1+s)^\mu\, ds)$.

It is now preferable to use $-s$ as a variable, so we set $M(\lambda, \mu, \nu) = -L(\lambda, \mu, \nu)$ as expressed in the $t = -s$ coordinate: thus

$$M(\lambda, \mu, \nu) = -t(1-t)\frac{d^2}{dt^2} - [\lambda + 1 - (\lambda+\mu+2)\, t]\frac{d}{dt} + \nu^2 t \tag{6.2}$$

and we are interested in solving

$$M(\lambda, \mu, \nu)\, u = \rho\, u \tag{6.3}$$

with

$$u \in H_{(0,1)}^{\lambda, \mu} = L^2((0, 1); t^\lambda (1-t)^\mu\, dt). \tag{6.4}$$

It is a trivial computation, using only

$$t^\lambda (1-t)^\mu \frac{d}{dt} t^{-\lambda} (1-t)^{-\mu} = \frac{d}{dt} - \frac{\lambda}{t} + \frac{\mu}{1-t}, \tag{6.5}$$

to check that $M(\lambda, \mu, \nu)$ is formally self-adjoint on $H_{(0,1)}^{\lambda, \mu}$, whether ν is real or pure imaginary; also note that changing t to $1-t$ changes $L(\lambda, \mu, \nu)$ to $L(\lambda, \mu, i\nu) + \nu^2$.

We still assume $\lambda > -1$, $\mu > -1$ (the latter condition was not needed in the analysis on $(1, \infty)$). From the proof of theorem 1.1 the condition $t^{\lambda+1} u'(t) = 0$ (resp. $(1-t)^{\mu+1} u'(t) = 0$) is a valid boundary condition at zero (resp. 1); if, moreover, $\lambda < 1$ (resp. $\mu < 1$), so is the boundary condition $t\, u'(t) - \lambda\, u(t) = 0$ at 0 (resp. $(1-t)\, u'(t) - \mu\, u(t) = 0$ at 1). In this way, we get four, two or one possible (particular except in the essentially self-adjoint case, *i.e.* when $\lambda \geq 1$ and $\mu \geq 1$) self-adjoint extensions of $M(\lambda, \mu, \nu)$. However, as in (1.12), we have

$$t^\lambda M(\lambda, \mu, \nu)\, t^{-\lambda} = M(-\lambda, \mu, \nu) - \lambda(1+\mu) \tag{6.6}$$

and $u = t^{-\lambda} v$ implies $t^{\lambda+1} u' = t v' - \lambda v$: thus the isometry $v \mapsto u$ from $H^{-\lambda, \mu}_{(0,1)}$ onto $H^{\lambda, \mu}_{(0,1)}$ transforms the second boundary condition at zero relative to $M(-\lambda, \mu, \nu)$, in the case when $-1 < \lambda < 1$, into the first one relative to $M(\lambda, \mu, \nu)$; at the same time, it preserves any of the two boundary conditions at $t = 1$ since the conditions $v \in C^1(]0, 1[)$, $v \in L^2((1-t)^{\mu}\, dt)$ near $t = 1$ and $(1-t)^{\mu+1} v'(t) = 0$ at $t = 1$ imply $(1-t)^{\mu+1} v(t) = 0$ at $t = 1$.

In this way, the four self-adjoint boundary problems mentioned above can be reduced to just one: from now on, $M(\lambda, \mu, \nu)$ *shall denote the self-adjoint extension of the formal operator* $M(\lambda, \mu, \nu)$ *on* $H^{\lambda, \mu}_{(0,1)}$ *characterized by the boundary conditions*

$$(6.7) \qquad t^{\lambda+1} u'(t) = 0 \quad \text{at } t = 0, \qquad (1-t)^{\mu+1} u'(t) = 0 \quad \text{at } t = 1.$$

Since the roots of the indicial equation at 0 (resp. 1) are $(0, -\lambda)$ (resp. $(0, -\mu)$) and since $\lambda > -1$, $\mu > -1$, an eigenfunction of $M(\lambda, \mu, \nu)$ can be characterized as a function u analytic on $[0, 1]$, satisfying (6.3).

Then case when $\nu = 0$ is not excluded from the analysis. Up to the multiplication by some constant, the only solution of $M(\lambda, \mu, 0)\, u = \rho\, u$ analytic on $[0, 1[$ is $u(t) = {}_2F_1(\alpha, \beta, \gamma, t)$ with $\gamma = \lambda + 1$, $\alpha + \beta = \lambda + \mu + 1$ and $\alpha\beta = -\rho$. Now this function is analytic at $t = 1$ if and only if α or β is a non-positive integer: this is a consequence of the last formula in ([10], p. 47) in the case when $\mu \neq 0, 1, \ldots$; of the second formula on p. 49 in the remaining cases. This yields the spectrum of $M(\lambda, \mu, 0)$, which consists of the sequence $m(\lambda + \mu + 1 + m)$, where m is a non-negative integer.

This time, the additional term $\nu^2 t$ is a bounded perturbation, and the spectrum of $M(\lambda, \mu, \nu)$ is still discrete, simple, and can be arranged as a sequence going to infinity.

In Cartan's duality of symmetric spaces, the dual of a mass hyperboloid $SO_0(1, n)/(SO(n)$ is a sphere $SO(n+1)/SO(n)$. Going from (1.18) (or, rather, its momentum-space realization, which just asks for replacing x_j by p_j in view of the fact that L^n commutes with $\mathcal{G}$) to its analogue on the sphere demands no more than a few sign changes. However, c has no significance any longer, so rather than the sphere $E^2 + c^2\, |\mathbf{p}|^2 = c^4$, we use the standard unit sphere in $\mathbb{R}^{n+1} = \{(q_0, \mathbf{q}) : q_0 \in \mathbb{R}, \mathbf{q} \in \mathbb{R}^n\}$ of equation $q_0^2 + |\mathbf{q}|^2 = 1$, by setting $E = c^2 q_0$, $\mathbf{p} = c\,\mathbf{q}$. Then the analogue of (1.18) is defined in the $\mathbf{q}$-coordinates as

$$(6.8)\qquad -4\pi L^n = \frac{h^2}{c^2}\sum \frac{\partial^2}{\partial q_j^2} - 4\pi^2 c^2 |\mathbf{q}|^2 - c^{-2}\left[h^2\left(\sum q_j \frac{\partial}{\partial q_j}\right)^2 + (n-1)\,h \sum q_j \frac{\partial}{\partial q_j}\right]$$

or, still setting $\nu = \dfrac{\pi c^2}{h}$,

(6.9)
$$\frac{\nu}{h} L^n = -\frac{1}{4}\left[\sum \frac{\partial^2}{\partial q_j^2} - \left(\sum q_j \frac{\partial}{\partial q_j}\right)^2 + \frac{n-1}{h}\sum q_j \frac{\partial}{\partial q_j}\right] + \nu^2 |\mathbf{q}|^2.$$

Zonal functions on the sphere ("radial", though technically correct from the point of view of harmonic analysis, would be somewhat misleading) are those functions that depend only on q_0. On each of the two hemispheres $q_0 > 0$, $q_0 < 0$, such a function can be written as $f(\mathbf{q}) = u(|\mathbf{q}|^2)$ for some $u = u(t)$ living on $(0, 1)$. As in (1.24), the zonal part of L^n can then, in terms of u, be expressed as

$$(6.10)\qquad \frac{\nu}{h} L^n_{\text{zonal}} = M(\lambda, \mu, \nu)$$

with (λ, μ, ν) as defined in (1.3). To describe the boundary conditions on u that will characterize the zonal eigenfunctions of L^n, one must split the space of functions on the sphere into its even and odd parts and associate $u(|\mathbf{q}|^2) = f(\mathbf{q})$ with, say, the restriction of the given function to the upper hemisphere $q_0 > 0$. It is clear that we get a function on the sphere smooth near the poles if and only if the function $t \mapsto u(t^2)$ is C^∞ up to $t = 0$ in both cases; we get a function smooth near the equator if and only if the function u (resp. $t \mapsto (1-t)^{-1/2} u(t)$) is smooth up to $t = 1$ in the even (resp. odd) case.

Let us assume from now on, until further warning, that $\mu = -\dfrac{1}{2}$ (*i.e.* $h = 1$): then the operator within brackets on the right-hand side of (6.9) is just the Laplacian on the sphere. Also, the search for zonal eigenfunctions of L^n on the sphere is fully equivalent with the search for eigenfunctions of $M\left(\lambda, -\dfrac{1}{2}, \nu\right)$ on $(0, 1)$, together with the boundary condition of the first type $(t^{\lambda+1} u'(t) = 0)$ at 0, and the boundary condition of the first type $(1-t)^{1/2} u'(t) = 0$ $\left(\text{resp. of the second type } (1-t)\, u'(t) + \frac{1}{2} u(t) = 0\right)$ at 1 in the even (resp. odd) case.

The case when $n = 1$ needs further splitting. With $t = \sin^2 x$ $\left(0 < x < \frac{\pi}{2}\right)$, equation (6.3) becomes

$$\frac{d^2 u}{dx^2} - 4\nu^2 (\sin^2 x)\, u = -4\rho u, \tag{6.11}$$

an ordinary Mathieu equation ([22], p. 405). The original Mathieu problem calls for looking towards 2π-periodic solutions of (6.11): as is well-known, solutions are classified into four types, according to whether u is even or odd, π-periodic or π-antiperiodic (this is the origin of the standard Mathieu functions ce_{2n}, se_{2n}, ce_{2n+1}, se_{2n+1}). An equivalent classification is according to whether u is even or odd, and whether it is even or odd around $\frac{\pi}{2}$ [*i.e.* $u(\pi - t) = \pm u(t)$]: now this fits perfectly well with our four types of boundary problems for $M\left(-\frac{1}{2}, -\frac{1}{2}, \nu\right)$ on $(0, 1)$, the boundary condition at 0 (resp. 1) being of the first or second type there according to whether u is even or odd around 0 $\left(\text{resp. } \frac{\pi}{2}\right)$.

Finally, the spherical functions identities: there is no need to rephrase theorem 5.1, which extends analytically from $(0, \infty)$ to $(-1, \infty)$ (in the s-variable) since it only depends on a condition at 0. For the generalization to come of theorem 3.3, *we do not assume that* $\mu = -\frac{1}{2}$ *any longer.* However, let us assume that ν is real > 0 (if it were pure imaginary, it would suffice to change t to $1 - t$). As in the beginning of section 3, we transfer, by means of the change of variable $t = \sin^2 x$, the operator $M(\lambda, \mu, \nu)$ to $\tilde{M}(\lambda, \mu, \nu)$ with

$$\begin{aligned} -4\tilde{M}(\lambda, \mu, \nu) &\\ = \frac{d^2}{dx} &+ [(2\lambda + 1)\operatorname{cotan} x - (2\mu + 1)\tan x]\frac{d}{dx} \\ &- 4\nu^2 \sin^2 x \end{aligned} \tag{6.12}$$

and the measure that defines the space $H^{\lambda, \mu}_{(0, 1)}$ to

$$dm(x) = 2(\sin x)^{2\lambda+1} (\cos x)^{2\mu+1}\, dx. \tag{6.13}$$

Also, we note that for an eigenfunction of $\tilde{M}(\lambda, \mu, \nu)$ on $\left(0, \frac{\pi}{2}\right)$, the boundary conditions that are the transfer, under the given change of variable, of (6.7), just mean that this eigenfunction is C^∞ on $\left[0, \frac{\pi}{2}\right]$.

Any non-zero eigenfunction φ_k of $M(\lambda, \mu, \nu)$ [under the boundary conditions (6.7)] satisfies $\varphi_k(0) \neq 0$ and $\varphi_k(1) \neq 1$. Also,

$$\int_0^1 \varphi_k(t)\, t^\lambda (1-t)^\mu \, dt \neq 0 \tag{6.14}$$

since otherwise φ_k would be orthogonal, in the space $H^{\lambda, \mu}_{(0, 1)}$, to all functions of the kind $[M(\lambda, \mu, \nu)]^n 1$ with $n \in \mathbb{N}$, that is, as a consequence of (6.2), to all polynomials. This gives a meaning to the following theorem, whose proof (except for the normalization, checked by sheer inspection) is identical to that of theorems 3.2 and 3.3.

THEOREM 6.1. – *Consider, on* $\left[0, \frac{\pi}{2}\right]^3$, *the kernel*

$$\begin{aligned} g(x, y, z) = & \frac{\Gamma(\lambda+1)\Gamma(\mu+1)}{\nu^{\lambda+\mu}} \\ & \times j_{\lambda, \nu}(\sin x \sin y \sin z)\, i_{\mu, \nu}(\cos x \cos y \cos z). \end{aligned}$$

Given any eigenfunction φ_k *of* $M(\lambda, \mu, \nu)$ [*under the boundary conditions* (6.7)] *satisfying the normalization condition*

$$\varphi_k(0)\, \varphi_k(1) = \int_0^{\frac{\pi}{2}} \varphi_k(\sin^2 z)\, dm(z),$$

one has the identity

$$\varphi_k(\sin^2 x)\, \varphi_k(\sin^2 y) = \int_0^{\frac{\pi}{2}} g(x, y, z)\, \varphi_k(\sin^2 z)\, dm(z).$$

Remark. – When $\lambda = \pm\frac{1}{2}$ and $\mu = \pm\frac{1}{2}$, we are actually dealing, as explained in the present section, with a disguised version of some Mathieu function: if, moreover, x or $y = 0$, the identity above then reduces to the one in ([22], p. 409).

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(Manuscript received February 22, 1994;
revised version received September 22, 1994.)