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ABSTRACT

We propose a relaxation framework for general fluid models which can be understood as a natural extension of the Suliciu approach in the Euler setting. In particular, the relaxation system may be totally degenerate. Several stability properties are proved. The relaxation procedure is shown to be efficient in the numerical approximation of the entropy weak solutions of the original PDEs. The numerical method is particularly simple in the case of a fully degenerate relaxation system for which the solution of the Riemann problem is explicit. Indeed, the Godunov solver for the homogeneous relaxation system results in an HLLC-type solver for the equilibirum model. Discrete entropy inequalities are established under a natural Gibbs principle.

Contents

1 Introduction

We intend to approximate the solutions of rather general *fluid systems* by those of simpler relaxation models, with an hyperbolic structure, and endowed with an entropy. These fluid systems share some common algebraic structure which has allowed us to treat their coupling from a general point of view in [1]. We now make the most of this structure to extend the relaxation approximation initiated in [19], and for the coupling in [2] [3], to more general systems in (Eulerian or) Lagrangian coordinates; this corresponds to the relaxation of the pressure p when restricting to Euler system, an approach first introduced in the work of Suliciu [35]. We thus introduce a relaxation system with only linearly degenerate fields; such a linear degeneracy of acoustic fields also appears for a Chaplygin gas [34]. Linked to the total linear degeneracy of the homogeneous relaxation system, the associated numerical procedure leads to a rather simple HLL type numerical scheme for the original fluid system.

While some relaxation approaches have more physical backgrounds, see [27] for instance, our approach is mainly motivated by numerical purposes, and may be seen as some generalization of the Jin and Xin relaxation scheme [26] for fluid systems. However we are also interested in proving some convergence results: we consider a sequence of smooth solutions of the Cauchy problem for the relaxation extension (associated to a smooth initial data) when the relaxation coefficient tends to zero and aim at using Yong's results in [39] or [40], in order to prove the convergence to a solution of the corresponding fluid system. The results are a priori local in time and may be global if we make the assumption that the initial condition is close to a constant equilibrium state as in [40]. Similar investigations are done in [11] for the Born-Infeld equations or closer to our approach in [17] for Euler system. Note that the subject of relaxation approximation has gained interest in the last few years, and many interesting papers have been recently published following the pioneering work [18], we refer to the survey [31], and for the most recent ones, see for instance [6], [25], [13], [5], [17] and the references therein.

The outline of the paper is as follows. In the spirit of [21], we first recall the physical assumptions characterizing our fluid models. This is followed by some technical computations which are required in order to define the particular set of variables used in our Lagrangian approach and to prove the equivalence with the system in Eulerian coordinates; two classical examples are detailed. We introduce the relaxation system in section 2 and describe different aspects concerning the stability of the relaxation process which all rely on the same kind of assumption on the relaxation energy (involving a a Whitham-like

condition). We check some structural properties in section 3, in order to obtain existence and approximation results. Then, in section 4, we detail the global approximation procedure for the fluid system, which results in a numerical scheme involving explicit solutions of the Riemann problem for the relaxation system; we state the good stability properties inherited from the underlying continuous relaxation system. The approach and results are detailed in the much simpler Lagrangian framework and the results are then directly stated in the Eulerian framework. We illustrate on some examples that the numerical procedure is indeed simple and well adapted.

Some part of the present work was announced in [20].

1.1 The structure of general Lagrangian systems

We consider *fluid* systems modelled by PDE in the general form of n conservation laws

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{0}, \tag{1}$$

which meet some common properties in their Lagrangian description (here x stands for a mass variable, and t is the time variable):

- they are endowed with a strictly convex entropy $s = s(\mathbf{U})$, with null associated entropy flux, so that for smooth solutions $\mathbf{U}$
$$\partial_t s(\mathbf{U}) = 0;$$
- we can decompose $n = r + d + 1$ so that $\mathbf{U} = (U_1, ...U_n)^T$ is made of $r \equiv n - d - 1$ *state* variables $\mathbf{v}$, taking their value in some open convex set $\Omega_{\mathbf{v}} \subset \mathbb{R}^r$, and d *velocity* variables $\mathbf{u} \in \mathbb{R}^d$ (d is a space dimension). The last component of $\mathbf{U}$ is the total energy which we will denote by e; it can be decomposed as
$$U_n \equiv e = \frac{1}{2}|\mathbf{u}|^2 + \epsilon \tag{2}$$
where the internal energy $\epsilon = \epsilon(\mathbf{v}, s)$ is a *state* variable, then s is also a *state* variable. We will assume that $s(\mathbf{U})$, satisfies $\frac{\partial s}{\partial e}(\mathbf{U}) \equiv s_e(\mathbf{U}) < 0$. The model is then called a *fluid model*. Setting $\mathbf{V} = (\mathbf{v}, \mathbf{u}, s)$, we observe that $\mathbf{V} \mapsto e(\mathbf{V})$ is convex;
- they satisfy Galilean invariance and
- reversibility in time for smooth solutions.

We refer to [21] for more details. Observe that $e_{\mathbf{v},\mathbf{v}}$ is non negaitive, hereafter we shall assume it is (strictly) positive $\forall \mathbf{v} \in \Omega_{\mathbf{v}}$. The above properties result in some reduction formalism of system (1) wrt. state and velocity variables and some specific eigenstructure of the Jacobian. First, the flux $\mathbf{F}(\mathbf{U})$ of the system can be written in a canonical form in $\mathbb{R}^{n-1} \times \mathbb{R}$: there exists $\Psi : \mathbf{U} \to \Psi(\mathbf{U}) \in \mathbb{R}^{n-1}$ and $B \in \mathbb{R}^{(n-1)\times(n-1)}$ a square *constant* matrix such that

$$\mathbf{F}(\mathbf{U}) = (B\Psi(\mathbf{U}), -\frac{1}{2}\Psi(\mathbf{U})^T B\Psi(\mathbf{U})), \tag{3}$$

moreover B is a symmetric matrix and can be written in block form

$$B = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}, \tag{4}$$

where N is a rectangular $r \times d$ constant matrix. We assume, without loss of generality in practice, that the first line of the matrix N is $(1, 0...)$. For what concerns the function $\Psi(\mathbf{u})$, it is derived from the entropy variables that symmetrize the system (see [24]). Indeed, defining these entropy variables

$$\mathbf{U}^* \equiv s'(\mathbf{U})^T = (s_{U_1}, \cdots, s_{U_{n-1}}, s_e)^T$$

then $\Psi(\mathbf{u})$ in (3) is given by

$$\Psi(\mathbf{U}) = (\frac{s_{U_1}}{s_e}, \cdots, \frac{s_{U_{n-1}}}{s_e})^T. \tag{5}$$

As put forward in [1], it turns convenient to address a closely related vector of variables together with its polar form, setting

$$\mathbf{V} = (U_1, U_2, \cdots, U_{n-1}, s)^T = (\mathbf{v}, \mathbf{u}, s)^T$$

where $\mathbf{u} \in \mathbb{R}^d$, is the velocity vector as defined above, $\mathbf{v} \in \mathbb{R}^r$, represents the state variable vector, if we define the conjugate function or polar variables by

$$\mathbf{V}^* = e'(\mathbf{V})^T = (e_{V_1}, \cdots, e_{V_{n-1}}, e_s)^T \equiv (\mathbf{V}^*_{n-1}, s^*)^T, \tag{6}$$

we also have

$$\mathbf{V}^*_{n-1} = -\Psi(\mathbf{U}(\mathbf{V})),$$

with the benefit that the force field Ψ is now expressed very simply in terms of the first $n-1$ components of the polar variable $\mathbf{V}^*$. Hence, in view of (2) (3) (4), the equations (1) in Lagrangian coordinates can also be written equivalently (for smooth solutions) in a first quasilinear simple form

$$\begin{cases} \partial_t \mathbf{v} - N\partial_x \mathbf{u} = \mathbf{0} \\ \partial_t \mathbf{u} - N^T \partial_x \phi(\mathbf{v}, s) = \mathbf{0} \\ \partial_t s = 0 \end{cases} \tag{7}$$

where here $\phi(\mathbf{v}, s) = e_{\mathbf{v}}(\mathbf{U}) = \epsilon_{\mathbf{v}}$, note that $e_{\mathbf{u}}(\mathbf{U}) = \mathbf{u}$. Throughout the present work we use the notation $e_{\mathbf{v}} \equiv \nabla_{\mathbf{v}} e$ for the vector of partial derivatives $(e_{v_1}, e_{v_2}, ..., e_{v_{n-d-1}})^T$, and $e_{v_i} = \frac{\partial e}{\partial v_i}$. From the second equation we deduce

$$\partial_t \mathbf{u} - N^T e_{\mathbf{v},\mathbf{v}} \partial_x \mathbf{v} - N^T e_{\mathbf{v},s} \partial_x s = \mathbf{0}$$

(again, $e_{\mathbf{v},\mathbf{v}}$ denotes the Hessian matrix with coefficients $e_{v_i,v_j} = \frac{\partial^2 e}{\partial v_i \partial v_j}$). Now, the eigenvalues of (7) (system equivalent to (1)) are 0 and the (opposite of the) eigenvalues of the extracted matrix

$$\overline{B} = \begin{pmatrix} 0 & N \\ N^T e_{\mathbf{v},\mathbf{v}} & 0 \end{pmatrix}. \tag{8}$$

If μ is an eigenvalue of $\overline{B}$, there exists an eigenvector of the form $(\mathbf{y}, \mathbf{z})^T \neq (\mathbf{0}, \mathbf{0}) \in \mathbb{R}^r \times \mathbb{R}^d$ such that $N\mathbf{z} = \mu\mathbf{y}$, $N^T e_{\mathbf{v},\mathbf{v}} \mathbf{y} = \mu\mathbf{z}$, and if $\mu \neq 0$, it implies $N^T e_{\mathbf{v},\mathbf{v}} N\mathbf{z} = \mu^2 \mathbf{z}$ and $\mathbf{z} \neq 0$. Thus, if $\mu \neq 0$, denotes a non zero eigenvalue, then $-\mu$ is an eigenvalue too and μ^2 is an eigenvalue of the matrix

$$\mathcal{E} = N^T e_{\mathbf{v},\mathbf{v}} N \tag{9}$$

which, beeing a $d \times d$ symmetric matrix, has d real eigenvalues. We know that $e_{\mathbf{v},\mathbf{v}}$ is positive so that the eigenvalues of $\mathcal{E}$ are positive. Hence $\overline{B}$ has at most $2d$ non null eigenvalues. Finally the spectrum of $\mathbf{F}'(\mathbf{U})$ is real valued and symmetric and there are at least $n - 2d = r - d + 1$ null eigenvalues, and $n - 2d \geq 1$ (there is at least one which is associated to the conservation of entropy).

We may try to reduce even further the system in order to distinguish between the null eigenvalues (say $k + 1 \geq n - 2d, k \geq 0$, with $n - k - 1$ an even integer) and non null eigenvalues $\pm\mu_i$, $i = 1, ..., m$, ($2m = n - k - 1 \leq 2d$). In [1], we have indeed exhibited a change of variables such that

$$\begin{cases} \partial_t \mathbf{w}_k = \mathbf{0}, \\ \partial_t \mathbf{w}_{2m} - \Lambda \partial_x \mathbf{w}^*_{2m} = \mathbf{0}, \\ \partial_t s = 0 \end{cases} \tag{10}$$

where Λ is a $2m \times 2m$ diagonal, invertible matrix with entries the non null symmetric eigenvalues. We now assume for simplicity that

(**A**) the d eigenvalues of $\mathcal{E}(\mathbf{v})$ in (9) are all non null over $\Omega_{\mathbf{v}}$,

so that the eigenvalue 0 of $\mathbf{F}'(\mathbf{U})$ has multiplicity exactly $r-d+1(=n-2d)$, and this multiplicity is thus $r-d$ in the isentropic case. We refer the reader to section 1.3 for significant examples. In section 3.1.2, we will moreover assume that $d=r$ and that N is a square $r \times r$ invertible matrix, and the eigenvalue 0 is then only associated to the equation of entropy conservation.

From now on, we assume these results and we derive when necessary some additional properties needed in the computations.

1.2 From Lagrangian to Eulerian coordinates

The equivalence of the Cauchy problem in Eulerian and Lagrangian coordinates for the equations of gas dynamics is now well known as regards the existence and uniqueness of entropy solutions in $\mathbb{L}^\infty(\mathbb{R}\times\mathbb{R}+)$ (see [37] and also [32]). Recall that in the Eulerian to Lagrangian transformation, the first component u_1 of the conservative Eulerian variable U plays a special role and is assumed to be positive and bounded ($0 < m \leq u_1(x,t) \leq M$). If $U=(u_1,u_2,...,u_n)^T$ solves the Eulerian system

$$\partial_t U + \partial_x f(U) = 0, \tag{11}$$

there is a change of independent variables $(x,t) \to (y,t)$ such that, setting $\tilde{U}(y,t) = U(x,t)$, one has

$$\begin{cases} \partial_t \frac{1}{\tilde{u}_1} - \partial_y \frac{f_1(\tilde{U})}{\tilde{u}_1} = 0, \\ \partial_t \frac{\tilde{u}_i}{\tilde{u}_1} + \partial_y \big(f_i(\tilde{U}) - \frac{f_1(\tilde{U})\tilde{u}_i}{\tilde{u}_1} \big) = 0, \quad i=2,...,n. \end{cases} \tag{12}$$

Indeed from the first equation $\partial_t u_1(x,t) + \partial_x f_1(U(x,t)) = 0$, one can solve $y = Y(x,t)$ with $dy = u_1 dx - f_1(U)dt$, i.e. $\partial_x Y(x,t) = u_1(x,t), \partial_t Y(x,t) = -f_1(U(x,t))$ and check (12). Later, we will rather note u_1 by ϱ in order to identify it in a simpler way, and note by u the corresponding normal velocity term, or 'velocity' for short, $u \equiv \frac{f_1(U)}{u_1}$. Moreover if (η, q), η convex is an entropy-entropy flux pair for system (11), $(\frac{\eta}{u_1}, q - f_1 \frac{\eta}{u_1})$ is an entropy-entropy flux pair for system (12).

How can we express in Eulerian coordinates the general structure developed in the Lagrangian frame? Using, for this section only, the notations in [37], [32], the mass variable in the above equations (1) or (7) will be noted y and the conservative variable in Eulerian frame by $U(x,t)$. We look for a relation between U and $V=(\mathbf{v},\mathbf{u},s)$ and for a flux $f(U)$.

Let us detail the (less usual) Lagrangian to Eulerian transformation. We will thus assume that the first component v_1 of the vector of state variables $\mathbf{v}$ is positive (and bounded). The mapping $(t,y) \to (t,x)$ which makes the equivalence possible starts from the first equation in Lagrangian coordinates, say $\partial_t v_1 + \partial_y g_1(V) = 0$. We mimic the Eulerian to Lagrangian transformation

$$dx = v_1 dy - g_1(V)dt \Rightarrow dy = \frac{1}{v_1}dx + \frac{g_1(V)}{v_1}dt.$$

Thus let $\check{V}(x,t) = V(y,t)$ (we emphasize when necessary the different mathematical functions, the usual convention is rather to employ the same symbol) and let $y = Y(x,t)$ be the unique solution of

$$dy = \frac{1}{\check{v}_1}dx + \frac{g_1(\check{V})}{\check{v}_1}dt. \tag{13}$$

This implies

$$\partial_t \frac{1}{\check{v}_1} = \partial_x \big(\frac{g_1(\check{V})}{\check{v}_1} \big),$$

which reads, setting $u_1 = \frac{1}{\check{v}_1}$, $f_1 = -\frac{g_1(\check{V})}{\check{v}_1} = -u_1 g_1(\check{V})$,

$$\partial_t u_1 + \partial_x f_1 = 0.$$

If we return to more classical notations, set $v_1 \equiv \tau$. We will note u the first component of the vector of velocity variables $\mathbf{u}$ and also since we have assumed that the first line of the matrix N is $(1, 0...)$, the first equation in (7) then writes

$$\partial_t \tau - \partial_y u = 0.$$

We now formalize the result of the above transformation on a system which has the general form (7); the solutions are supposed to be in $\mathbb{L}^\infty(\mathbb{R} \times \mathbb{R}+)$ and we make the assumption that $v_1 \equiv \tau$ is bounded above and below. Following [37], [32], we can prove

Proposition 1.1. *Let $(\mathbf{v}, \mathbf{u}, s)$ where $\mathbf{v} = (\tau, v_2, \dots, v_r)^T$, $\mathbf{u} = (u, u_2, \dots, u_d)^T$ be a solution of*

$$\begin{cases} \partial_t \mathbf{v} - N \partial_y \mathbf{u} = \mathbf{0} \\ \partial_t \mathbf{u} - N^T \partial_y \phi(\mathbf{v}, s) = \mathbf{0} \\ \partial_t s = 0 \end{cases} \tag{14}$$

where N is a rectangular $r \times d$ constant matrix with first line equal to $(1, 0, \dots, 0)$ and $\phi : \mathbb{R}^{r+1} \to \mathbb{R}^d$ is a smooth mapping. Assume that $\tau > 0$. Defining $(\overline{U}, \underline{U}) \in \mathbb{R}^r \times \mathbb{R}^d$ by

$$\overline{U}_1 = \varrho \equiv \frac{1}{\tau},\ \overline{U}_i = \varrho v_i, i = 2, \dots, r,\ \underline{U}_1 = \varrho u,\ \underline{U}_i = \varrho u_i, i = 2, \dots, d \tag{15}$$

setting $U = (\overline{U}, \underline{U}, \varrho s) \in \mathbb{R}^n$; defining $(\overline{f}(U), \underline{f}(U)) \in \mathbb{R}^r \times \mathbb{R}^d$ by

$$\overline{f}_1(U) = \varrho u,\ \overline{f}_i(U) = -(N \frac{\underline{U}}{\varrho})_i + u \overline{U}_i, i = 2, \dots, r,\ \underline{f}_i(U) = -(N^T \check{\phi}(\overline{U}, s))_i + u \underline{U}_i, i = 1, \dots, d \tag{16}$$

where $\check{\phi}(\overline{U}, s) = \phi(\mathbf{v}, s)$ and setting $f(U) = (\overline{f}(U), \underline{f}(U), \varrho s u)$, then U is solution of the system

$$\partial_t U + \partial_x f(U) = 0,$$

which writes equivalently

$$\begin{cases} \partial_t \varrho + \partial_x \varrho u = 0 \\ \partial_t \overline{U}_i + \partial_x (-N \frac{\underline{U}}{\varrho})_i + u \overline{U}_i) = 0, \quad i = 2, \dots, r, \\ \partial_t \varrho u + \partial_x (\varrho u^2 - (N^T \check{\phi})_1) = 0 \\ \partial_t \underline{U}_i + \partial_x (-(N^T \phi)_i + u \underline{U}_i) = 0, \quad i = 2, \dots, d \\ \partial_t \varrho s + \partial_x \varrho s u = 0, \end{cases} \tag{17}$$

Proof. The proof relies on identifying U and f from (12). First with $\varrho = \frac{1}{\tau}$, we get $u_1 = \varrho$, $f_1 = \varrho u$ and as expected

$$\partial_t \varrho + \partial_x \varrho u = 0.$$

Similarly, the entropy conservation equation in (7) becomes thanks to (12)

$$\partial_t \varrho s + \partial_x \varrho u s = 0.$$

We now focus on the remaining equations for identifying U and f from (12). We will decompose U and $f(U)$ according to the decomposition $n = r+d+1$, and set $U = (\overline{U}, \underline{U}, \varrho s)$, $f(U) = (\overline{f}(U), \underline{f}(U), \varrho u s)$

with $\overline{U}, \overline{f}(U) \in \mathbb{R}^r$, $\underline{U}, \underline{f}(U) \in \mathbb{R}^d$. Then, writing $\mathbf{v} = (\tau, v_2, ..., v_r)^T, \mathbf{u} = (u, u_2, \dots, u_d)^T$, we get (15)

$$\overline{U}_1 = \varrho,\ \overline{U}_i = \varrho v_i, i = 2, ...r,\ \underline{U}_1 = \varrho u,\ \underline{U}_i = \varrho u_i, i = 2, ..., d$$

and $\overline{U}$ involves only state variables, while $\underline{U} = \varrho \mathbf{u}$ is the momentum vector. For the flux, we have

$$\overline{f}_1(U) = \varrho u,\ \overline{f}_i(U) = -(N\frac{\underline{U}}{\varrho})_i + u\overline{U}_i, i = 2, \dots, r,\ \underline{f}_i(U) = -(N^T\check{\phi})_i + u\underline{U}_i, i = 1, \dots, d$$

where ϕ, which was defined as a function of $(\mathbf{v}, s)$, is to be considered here as a function of $(\overline{U}, s)$. This gives the system (17). In the applications, $-(N^T\phi)_1 = p_*$ is the total pressure term. □

If we are interested in the isentropic case $s = s_0$ (with s_0 a given constant value), then we set $\epsilon(\mathbf{v}) \equiv \epsilon(\mathbf{v}, s_0)$, $e_{\mathbf{v}} = \epsilon'$, thus (7) reads

$$\left\{ \begin{array}{l} \partial_t \mathbf{v} - N\partial_x \mathbf{u} = \mathbf{0}, \\ \partial_t \mathbf{u} - N^T \partial_x \epsilon'(\mathbf{v}) = \mathbf{0}. \end{array} \right. \tag{18}$$

Then, the energy $e = \frac{1}{2}|\mathbf{u}|^2 + \epsilon(\mathbf{v})$, plays the role of an entropy for system (18). Its Hessian matrix, e'' for short,

$$e'' = \begin{pmatrix} \epsilon'' & 0 \\ 0 & I_r \end{pmatrix} \tag{19}$$

is positive iff $e_{\mathbf{v},\mathbf{v}} = \epsilon''$ is (we have noted I_r the $r \times r$ identity matrix). In Eulerian coordinates, the energy(-entropy) becomes $\varrho e = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho\epsilon$ and the energy flux is $q^E = q^{\mathcal{L}} + \varrho u e$. From (3), the Lagrangian energy flux is $q^{\mathcal{L}} = -\frac{1}{2}\Psi(\mathbf{U})^T B \Psi(\mathbf{U})$ and by the definition (2) and (4), (5), (6),

$$q^{\mathcal{L}} = -\frac{1}{2}(\mathbf{u}^T N^T \epsilon_{\mathbf{v}} + \epsilon_{\mathbf{v}}^T N \mathbf{u}) = -\mathbf{u}^T N^T \epsilon_{\mathbf{v}}$$

so that with short notations

$$q^E = \varrho u e - \mathbf{u}^T N^T \epsilon_{\mathbf{v}}$$

and the usual abuse of notation (we should write for instance $\check{\epsilon}$ instead of ϵ in the last equation where $\check{\epsilon}(\overline{U}) = \epsilon(\mathbf{v})$). Moreover, the entropy(-energy) inequality is transformed accordingly (we refer again to [37], [32]).

We will now work in the Lagrangian frame to derive the method of numerical approximation. The above lines will be used for the relaxation system which has a general structure similar to (14). Thus a method in Lagrangian frame will immediately provide a method in Eulerian frame.

Let us first check the notations on two classical examples.

1.3 Examples

1.3.1 Euler system

Let us explicit the above computations for the Euler system,

$$\left\{ \begin{array}{l} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \\ \partial_t e + \partial_x p u = 0, \end{array} \right. \tag{20}$$

$\mathbf{U} = (\tau, u, e)^T$, $n = 3, d = 1, r = 1$, $e = u^2/2 + \varepsilon$, ε is the internal energy, $\tau > 0$ is the specific volume and p is a given function of (τ, ε) and $\varepsilon = \tilde{\varepsilon}(\tau, s)$ verifies $\tilde{\varepsilon}_\tau = -p$. Then we have $s_e = -\frac{1}{T}$, where T is the temperature, $\mathbf{U}^* = \frac{1}{T}(-p, u, -1)^T$, $\Psi = (p, -u)^T$, $\mathbf{v} = \tau$, $\Omega_{\mathbf{v}} = \mathbb{R}_+^*$ and

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We compute $e_{\mathbf{v},\mathbf{v}} = \tilde{\varepsilon}_{\tau,\tau} = -\partial_\tau \tilde{p}$ which is assumed to be positive (we note $\tilde{p}$ the function $p = \tilde{p}(\tau, s)$), hence the eigenvalues of system (20) are indeed 0 and $\pm c$, where $c = \sqrt{-\partial_\tau \tilde{p}}$ (τc is the sound speed).

We might also consider the system in dimension $d = 2$, $\mathbf{u} = (u, v)^T$, and assuming slab symmetry, so that we just add the equation $\partial_t v = 0$ to system (20), and N is the 1×2 matrix $N = (1, 0)$. Then the 2×2 matrix $\mathcal{E}(\mathbf{v})$ has a null eigenvalue, which we have excluded for simplicity. But the variable v can have a decoupled explicit treatment so that the assumption is not really restrictive in this case. If we add a variable advected by the flow, for instance a mass fraction Y, satisfying in Lagrangian coordinates $\partial_t Y = 0$, this is no longer decoupled because the pressure p is now a function of Y, which makes a difference and will imply some difficulties (in proving the Shizuta-Kawashima condition, see remark 6).

1.3.2 MHD

Let us consider the MHD system in Lagrangian coordinates

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t \tau \mathbf{B}_\perp - \partial_x B_x \mathbf{u}_\perp = \mathbf{0} \\ \partial_t u + \partial_x p^* = 0 \\ \partial_t \mathbf{u}_\perp - \partial_x B_x \mathbf{B}_\perp = \mathbf{0}, \\ \partial_t e + \partial_x \big((p^* - B_x^2) u - B_x \mathbf{B}_\perp . \mathbf{u}_\perp \big) = 0, \end{cases} \tag{21}$$

where we have distinguished the longitudinal and transverse components for the velocity variable, if $\mathbf{u} = (u, v, w)^T$, $\mathbf{u}_\perp = (v, w)^T$ and for the magnetic field $\mathbf{B}$, which satisfies $div\mathbf{B} = 0$, if $\mathbf{B} = (B_x, B_y, B_z)^T$, we set $\mathbf{B}_\perp = (B_y, B_z)^T$. We may assume B_x is constant for one-dimensional data, we will assume $B_x \neq 0$. We have $n = 7, d = 3, r = 3$. The total pressure is

$$p^* = p + \frac{1}{2}|\mathbf{B}|^2$$

where the static pressure p is a given function of specific volume τ and internal energy: $p = p(\tau, \varepsilon)$ (for instance $p = (\gamma - 1)\frac{\varepsilon}{\tau}$) or of τ and entropy s, $p = \tilde{p}(\tau, s)$ and $\varepsilon = \tilde{\varepsilon}(\tau, s)$ as a function of (τ, s) satisfies again $\partial_\tau \tilde{\varepsilon} = -p$; the total energy is now

$$e = \frac{1}{2}|\mathbf{u}|^2 + \varepsilon + \frac{1}{2}\tau|\mathbf{B}|^2.$$

If B_x is constant, we may take as 'total pressure'

$$P^* = p^* - \frac{1}{2}B_x^2 = p + \frac{1}{2}|\mathbf{B}_\perp|^2.$$

For smooth solutions, (21) writes

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t \tau \mathbf{B}_\perp - \partial_x B_x \mathbf{u}_\perp = \mathbf{0} \\ \partial_t u + \partial_x P^* = 0 \\ \partial_t \mathbf{u}_\perp - \partial_x B_x \mathbf{B}_\perp = \mathbf{0}, \\ \partial_t s = 0. \end{cases} \tag{22}$$

We have with the above notations, $\mathbf{U} = (\tau, \tau\mathbf{B}_\perp, \mathbf{u}, e)^T$, $\mathbf{v} = (\tau, \tau\mathbf{B}_\perp)$,

$$\epsilon(\mathbf{v}, s) = \tilde{\varepsilon}(\tau, s) + \frac{1}{2}\tau|\mathbf{B}|^2 \tag{23}$$

where ϵ is the total specific internal energy; we compute

$$e_{\mathbf{v}} = (-p + \frac{1}{2}B_x^2 - \frac{1}{2}|\mathbf{B}_\perp|^2, \mathbf{B}_\perp)^T,$$

and the first component $\partial_\tau \epsilon$ is equal to $-P^* + \frac{1}{2}B_x^2$. Defining the matrix

$$N = \begin{pmatrix} 1 & 0 \\ 0 & diag(B_x) \end{pmatrix} \tag{24}$$

which is a 3×3 (square) invertible diagonal matrix (here $diag(B_x)$ is the 2×2 diagonal matrix with only B_x entries on the diagonal) we can indeed write the system (22) in the canonical form (7)

$$\begin{cases} \partial_t \mathbf{v} - N\partial_x \mathbf{u} = \mathbf{0}, \\ \partial_t \mathbf{u} - N\partial_x e_{\mathbf{v}} = \mathbf{0}, \\ \partial_t s = 0, \end{cases} \tag{25}$$

with $n = 7, d = 3$ (see [4]). The 3×3 matrix $e_{\mathbf{v},\mathbf{v}}$ is given by the blocks

$$e_{\mathbf{v},\mathbf{v}} = \begin{pmatrix} -p_\tau + |\mathbf{B}_\perp|^2/\tau & -\mathbf{B}_\perp^T/\tau \\ -\mathbf{B}_\perp/\tau & diag(1/\tau) \end{pmatrix}$$

$p_\tau = \partial_\tau \tilde{p}(\tau, s)$ and we compute the 6 eigenvalues from the square roots of those of $N^T e_{\mathbf{v},\mathbf{v}} N$ which reads

$$N e_{\mathbf{v},\mathbf{v}} N = \begin{pmatrix} -p_\tau + |\mathbf{B}_\perp|^2/\tau & -B_x\mathbf{B}_\perp^T/\tau \\ -B_x\mathbf{B}_\perp/\tau & diag(B_x^2/\tau) \end{pmatrix}. \tag{26}$$

Thus, system (22) has only one null eigenvalue, the other three pairs correspond to what are called the magnetosonic waves $\pm c_s$ (slow) and $\pm c_f$ (fast) and Alfven waves $\pm c_a$, where

$$c_a^2 = \frac{B_x^2}{\tau}, \tag{27}$$

and c_f^2, c_s^2 are the roots of the quadratic polynomial $X^2 - X(-p_\tau + \frac{|\mathbf{B}|^2}{\tau}) - p_\tau \frac{B_x^2}{\tau} = 0$. The Alfven waves give linearly degenerate fields. □

For the system in Eulerian coordinates, from (15)(16), and $\epsilon_{\mathbf{v}} = (-p^* + B_x^2, B_\perp)$, we get $\overline{U} = (\varrho, B_\perp)$, $\underline{U} = (\varrho u, \varrho u_\perp)$ and $\overline{f}(U) = (\varrho u, -B_x u_\perp + uB_\perp)$, $\underline{f}(U) = (\varrho u^2 + p^* - B_x^2, -B_x B_\perp + \varrho u u_\perp)$ which gives the system

$$\begin{cases} \partial_t \varrho + \partial_x \varrho u = 0 \\ \partial_t \mathbf{B}_\perp + \partial_x(u\mathbf{B}_\perp - B_x\mathbf{u}_\perp) = \mathbf{0} \\ \partial_t \varrho u + \partial_x(\varrho u^2 + p^*) = 0 \\ \partial_t \varrho \mathbf{u}_\perp + \partial_x(\varrho u \mathbf{u}_\perp - B_x\mathbf{B}_\perp) = \mathbf{0}, \\ \partial_t \varrho s + \partial_x \varrho s u = 0. \end{cases} \tag{28}$$

The energy flux is then $q^E = \varrho u e - \mathbf{u}N^T\epsilon_{\mathbf{v}}$ and since from the computation of $\epsilon_{\mathbf{v}}$, we have $-\mathbf{u}N^T\epsilon_{\mathbf{v}} = u(p^* - B_x^2) - B_x\mathbf{u}_\perp.\mathbf{B}_\perp$, we get $q^E = (\varrho e + p^*)u - B_x\mathbf{u}.\mathbf{B}$.

2 A relaxation model

In order to approximate the nonlinear fluid system (1), we start from its simple form (7), equivalent for smooth solutions. Let us stress at this stage that a straightforward correction will be added later on in order to extend the approach to discontinuous solutions of (1). We introduce a larger *relaxation* system, obtained by introducing new dependent variables $\mathcal{V}$ meant to 'relax' to the state variables $\mathbf{v}$ as some relaxation parameter goes to 0 (in fact we write the parameter in the form $\frac{1}{\lambda}$ and thus $\lambda \to \infty$), and such that the limit of the relaxation system (as $\lambda \to \infty$) 'reduces' to (7). Such a relaxation procedure is now classical for Euler system (for instance [15], [17], [3]).

Let us recall shortly the even simpler case of a scalar equation,

$$\partial_t u + \partial_x f(u) = 0$$

approximated by the Jin-Xin relaxation scheme

$$\left\{ \begin{array}{l} \partial_t u + \partial_x v = 0 \\ \partial_t v + a^2 \partial_x u = \lambda(f(u) - v), \end{array} \right. \tag{29}$$

where λ is some 'relaxation coefficient'. The homogeneous part of the system is linear, with eigenvalues $\pm a$. Whitham's stability condition (see [38]) requires that these velocities should bound the exact velocity, which writes $-a < f'(u) < a$. Under this assumption, smooth solutions $(u, v)_\lambda$ of (29) have been proved to converge as $\lambda \to \infty$ to an equilibrium $(u, v = f(u))$, and u is solution of the conservation law (see [30], [33]). A Chapman-Enskog expansion, introducing a first order corrector so that $v_\lambda = f(u_\lambda) + \lambda^{-1} v_1 + \mathcal{O}(\lambda^{-2})$, and neglecting terms of order larger than one wrt. λ^{-1}, gives that u_λ satisfies the PDE

$$\partial_t u_\lambda + \partial_x f(u_\lambda) = \lambda^{-1} \partial_x (a^2 - (f'(u_\lambda))^2) \partial_x u_\lambda) \tag{30}$$

and Whitham's condition ensures that the right hand side is dissipative, hence u_λ appears as a viscous approximation of u. A simple numerical scheme for the scalar equation consists in the Godunov's scheme (here it coincides with the upwind scheme) for the homogeneous part of the linear relaxation system, followed by an instantaneous projection step on the equilibrium manifold $v = f(u)$. The convergence of this scheme has also been proven (see [12] for a simple convergence proof, and the references therein).

We will follow a similar approach for approximating our fluid system, relaxing only a part of the nonlinearity in the model. The approximation will lead to an efficient numerical scheme if the relaxation system is indeed easy to solve, at least for the Riemann problem, so that we may use Godunov's scheme and also if stability results can justify the relaxation limit. Several stability conditions exist (see [6], [8]) and we will examine some of these in detail below. We also note that some trick is needed to justify the procedure for discontinuous solutions of (1), this will also come later on.

For simplicity we consider first the isentropic case, $s = s_0$, the extension to the full system is rather straightforward and will be presented in section 4.3 below.

2.1 The relaxation system

We consider the nonlinear system (7) in the isentropic case. In order to simplify the notations, we set $\epsilon = \epsilon(\mathbf{v})$, $e_{\mathbf{v}} = \epsilon'$, thus (7) reads

$$\left\{ \begin{array}{l} \partial_t \mathbf{v} - N \partial_x \mathbf{u} = \mathbf{0} \\ \partial_t \mathbf{u} - N^T \partial_x \epsilon'(\mathbf{v}) = \mathbf{0}. \end{array} \right. \tag{31}$$

Thanks to the last equation in (1) which expresses the conservation of energy, the total energy now takes the place of a mathematical entropy since it is a convex function of $(\mathbf{v}, \mathbf{u})$, provided the matrix $\epsilon_{\mathbf{v},\mathbf{v}}(\mathbf{v}) = \epsilon''(\mathbf{v})$ is (symmetric) positive-definite.

For the numerical approximation of system (31), we follow a relaxation approach, introducing some kind of linearization of the nonlinear term $\epsilon'(\mathbf{v})$ at the price of introducing a new variable $\mathcal{V}$ which at *equilibrium* coincides with the state variable $\mathbf{v}$. We thus consider the larger *relaxation* system of $2r+d$ equations

$$\begin{cases} \partial_t \mathbf{v} - N\partial_x \mathbf{u} = \mathbf{0} \\ \partial_t \mathbf{u} - N^T \partial_x \mathcal{W} = \mathbf{0} \\ \partial_t \mathcal{V} = \lambda(\mathbf{v} - \mathcal{V}) \end{cases} \tag{32}$$

with

$$\mathcal{W} = \mathcal{W}(\mathbf{v}, \mathcal{V}) = \epsilon'(\mathcal{V}) + \theta'(\mathbf{v}) - \theta'(\mathcal{V}), \tag{33}$$

where $\mathcal{V} \in \Omega_{\mathcal{V}} \subset \mathbb{R}^r$, and $\theta : \mathbb{R}^r \to \mathbb{R}$ is some $\mathcal{C}^2$ mapping on which we will make more assumptions hereafter. The definition (33) of $\mathcal{W}$ supposes that ϵ is defined on $\Omega_{\mathcal{V}}$ this will be precised when necessary. We now note $\mathbf{U} = (\mathbf{v}, \mathbf{u}, \mathcal{V})^T$, and $\mathcal{U} \subset \Omega_{\mathbf{v}} \times \mathbb{R}^d \times \mathbb{R}^r$ the set of states $\mathbf{U}$. In the sequel, we will partition $(2r+d) \times (2r+d)$ matrices in blocks relative to the above decomposition of system (32): $\mathbb{R}^{2r+d} = \mathbb{R}^r \times \mathbb{R}^d \times \mathbb{R}^r$. Also, we note either $\nabla_{\mathbf{v}}\theta, \theta_{\mathbf{v}}$ or θ' the derivative of θ (similarly for ϵ). For the system with entropy, all the computations could be done in exactly the same way, only replacing the derivatives by partial derivatives (for instance $\theta = \theta(\mathbf{v}, s)$ would depend on s).

Formally, as the relaxation parameter $\lambda \to \infty$, $\mathbf{v} - \mathcal{V} \to 0$ and at *equilibrium*, $\mathcal{W}(\mathbf{v}, \mathbf{v}) = \epsilon'(\mathbf{v})$, we recover system (31) (recall we have assumed we are in the isentropic case). We will denote by $\mathcal{U}_{eq}$ the manifold of *equilibrium states* $\mathbf{U}_{eq} = (\mathbf{v}, \mathbf{u}, \mathbf{v})^T$ i.e. states satisfying $\mathcal{V} = \mathbf{v}$. We need to make assumptions in order that this 'relaxation procedure' is indeed stable. Note that for a state at equilibrium, $\mathcal{V} \in \Omega_{\mathbf{v}}$, otherwise $\mathcal{V}$ is only supposed to lay in a neighborhood of $\Omega_{\mathbf{v}}$ in $\mathbb{R}^r$.

The projection operator is denoted when necessary by $\mathcal{P} : \mathbf{U} = (\mathbf{v}, \mathbf{u}, \mathcal{V})^T \mapsto (\mathbf{v}, \mathbf{u})$ and the *equilibrium mapping* by $\mathcal{M} : (\mathbf{v}, \mathbf{u}) \mapsto (\mathbf{v}, \mathbf{u}, \mathbf{v})$. The source term is written $\lambda \mathbf{G}(\mathbf{U})$ with

$$\mathbf{G}(\mathbf{U}) = \begin{pmatrix} 0 \\ 0 \\ \mathbf{v} - \mathcal{V} \end{pmatrix}, \tag{34}$$

thus satisfying $\mathcal{P}\mathbf{G}(\mathbf{U}) = (0,0)^T$.

From now on, we assume that θ, as ϵ, is *convex* i.e., we also assume $\theta''(\mathcal{V})$ is symmetric positive definite, we will make below a more precise assumption. We often identify these forms with matrices and note equivalentlty $(\theta''(\mathcal{V})\mathbf{v}, \mathbf{v})$ or $\mathbf{v}^T\theta''(\mathcal{V})\mathbf{v}$.

Remark 1. For example, we may take a quadratic θ, $\theta(\mathbf{v}) = \frac{1}{2}(\mathbf{v}, \Lambda\mathbf{v})$ with Λ a positive definite $r \times r$ constant matrix, so that $\theta''(\mathcal{V}) = \Lambda$ is constant and the system is linear. We will indeed make this assumption later on (as in the example of Euler system and more generally in section 3.1.2 for the proof of stability properties or for the effective numerical scheme in section 4.1 below). Even in the simplest case, Λ may be diagonal with positive entries, and this may lead to the Jin-Xin relaxation model (see [26]) when all the entries are the same. □

With these convexity assumptions, the left hand side of system (32) is hyperbolic. Indeed, the Jacobian matrix is

$$\mathbf{A}(\mathbf{U}) = \begin{pmatrix} 0 & -N & 0 \\ -N^T\theta''(\mathbf{v}) & 0 & -N^T(\epsilon - \theta)''(\mathcal{V}) \\ 0 & 0 & 0 \end{pmatrix}. \tag{35}$$

As for the matrix (4), an eigenvalue μ (outside 0 which has multiplicity $r = n - d - 1$) is such that there exists Y, $N^T\theta''(\mathbf{v})NY = \mu^2 Y$ and this $d \times d$ symmetric matrix has d real positive eigenvalues with a basis of eigenvectors. We will ask below that the (modulus of these) wave speeds dominate the (modulus of the) initial wave speeds,which is the so-called subcharacteristic condition.

Example: Euler system (revisited).

Let us again explicit the above relations for the isentropic Euler system ($p-$system)

$$\left\{ \begin{array}{l} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \end{array} \right. \tag{36}$$

where $p = \tilde{p}(\tau)$ satisfies $p'(\tau) < 0$. Then $\mathbf{v}$ reduces to τ, and, $\epsilon'(\mathbf{v}) = -\tilde{p}(\tau)$, $\mathbf{U} = (\tau, u)^T$, N is a scalar, $N = 1$. System (32) in this case simply writes

$$\left\{ \begin{array}{l} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x \Pi = 0, \\ \partial_t \mathcal{T} = \lambda(\tau - \mathcal{T}), \end{array} \right. \tag{37}$$

where, following notations used in [3], we have rather noted by Π the function $-\mathcal{W}$ and by $\mathcal{T}$ the new variable $\mathcal{V}$. For (33), we take a quadratic θ: $\theta(\mathcal{T}) = a^2 \frac{\mathcal{T}^2}{2}$ so that $\Lambda \equiv \theta'' = a^2$ is constant and

$$\Pi(\tau, \mathcal{T}) = \tilde{p}(\mathcal{T}) + a^2(\mathcal{T} - \tau). \tag{38}$$

The eigenvalues are 0 and $\pm a$, we assume that a satisfies Whitham's (or subcharacteristic) condition $a^2 > \max(-\tilde{p}'(s))$. Besides, note that under this condition, the mapping $h : \xi \to \tilde{p}(\xi) + a^2\xi$ defined on $]0, \infty[$ is invertible so that $\mathcal{T} = h^{-1}(\Pi + a^2\tau)$. □

Remark 2. Let us give a hint on how the pressure law (38) is chosen. The only nonlinearity in the $p-$system lies in the non linear pressure law. The idea is thus to introduce a linearization of $\tilde{p}$, however it must be done at the expanse of introducing a new variable, say $\mathcal{T}$, otherwise the approximation would be poor. Moreover, this new pressure law, $\Pi = \Pi(\tau, \mathcal{T})$ should coincide with p at equilibria, and the relaxation system should relax to the $p-$system. Thus, $\partial_\tau \Pi$ is a negative constant say $= -a^2$ and $\Pi = -a^2\tau + h(\mathcal{T}) \to p$ when the system 'relaxes'. We have chosen the simplest way to achieve all these constraints, introducing as new variable $\mathcal{T}$ a kind of extended volume fraction τ, and $\mathcal{T} \to \tau$ so that $h(\tau) = p(\tau) + a^2\tau$, moreover $\mathcal{T}$ satisfies the simplest possible PDE with a relaxation right-hand side.

The other way round would be to introduce the new dependent variable Π, to replace the second equation by $\partial_t u + \partial_x \Pi = 0$, then to observe that p satisfies $\partial_t p - \tilde{p}'(\tau)\partial_x u = 0$ and, by linearization, we replace $\tilde{p}'(\tau)$ by a constant $-a^2$, which gives

$$\partial_t \Pi + a^2 \partial_x u = \lambda RHS.$$

The right-hand side 'relaxation term' should ensure that $\Pi \to p$ as $\lambda \to \infty$, and thus include some $p - \Pi$ term. This alternative approach is very similar to the Jin-Xin approximation (29) where the nonlinear term is now $p(\tau)$ instead of $f(u)$, and the additional variable is Π instead of v. This is the approach followed for instance in [13]. If we start from (37) and identify the right-hand sides, we find $RHS = \frac{1}{a^2}(\tilde{p}'(\mathcal{T}) + a^2)(p - \Pi)$.

This generalizes easily to our fluid system. We want $\mathcal{W} = \theta'(\mathbf{v}) + h(\mathcal{V})$, when θ'' is constant, $\mathcal{W}$ is indeed a linearization of the 'pressure' $\epsilon'(\mathbf{v})$, with a correction h function of a new variable $\mathcal{V}$ extension of the state variable $\mathbf{v}$. If $\mathcal{W} \to \epsilon'(\mathbf{v})$ when $\mathcal{V} \to \mathbf{v}$ then $h(\mathbf{v}) = (\epsilon' - \theta')(\mathbf{v})$ and we choose $\mathcal{V}$ to satisfy the simplest possible PDE. Note that smooth solutions of (32) then satisfy

$$\partial_t \mathcal{W} - \theta''(\mathbf{v}) N \partial_x \mathbf{u} = \lambda(\epsilon - \theta)''(\mathcal{V})(\mathbf{v} - \mathcal{V})$$

and we might want to take $\mathcal{W}$ as dependent variable (see Remark 7 below). □

This kind of relaxation system, introduced by Suliciu [35] as an approximation for the equations of isothermal viscoelasticity, has been studied in several recent papers, see for instance [36], [28], [25], [15], and both [13], [17] use Yong's results [39]. In fact, in [13], the authors do work with variables (τ, u, Π) so that the last equation is replaced by $\partial_t \Pi + a^2 \partial_x u = \mu(p - \Pi)$, which together with the first equation gives the same system provided $\mu = \lambda(\tilde{p}' + a^2)/a^2$. In the appendix of [13], the authors prove under some stronger assumption, $a^2 > \Gamma \geq \max(-\tilde{p}'(s)) \geq \gamma > 0$, and for smooth initial data, the convergence (as $\mu \to \infty$) of the global solution of the relaxation system (37) to a local in time solution of the $p-$system (36). Reference [17] also addresses the full system with energy (written in Eulerian coordinates) for which an existence result is given for the relaxation system (near a data at equilibrium) together with the proof of convergence to a solution of the Euler system. Moreover in [17], a numerical procedure, very close to the one we will develop below, is introduced.

All those results nequire that the relaxation system satisfies some stability property, which we are now going to atudy.

2.2 Stability of the relaxation model

In order to justify the relaxation procedure, we first investigate the entropy extension condition (see [8] where the relations between different stability conditions is clearly stated). We can indeed select an entropy extension for the relaxation system, which is convex on the equilibrium manifold, however it is not convex on the entire set of states.

2.2.1 Energy dissipation

We look for a mathematical entropy (which will be an energy) $\Sigma(\mathbf{v}, \mathbf{u}, \mathcal{V})$, which is dissipated, coincides with the entropy of the system at equilibrium, i.e. the total energy e of the system at equilibrium, which writes

$$\Sigma(\mathbf{v}, \mathbf{u}, \mathbf{v}) = \frac{1}{2}|\mathbf{u}|^2 + \epsilon(\mathbf{v}),$$

together with some Gibb's minimization principle. Define

$$\Sigma(\mathbf{v}, \mathbf{u}, \mathcal{V}) = \frac{1}{2}|\mathbf{u}|^2 + \epsilon(\mathcal{V}) + \theta(\mathbf{v}) - \theta(\mathcal{V}) + ((\epsilon' - \theta')(\mathcal{V}), \mathbf{v} - \mathcal{V}). \tag{39}$$

Lemma 2.1. *Let the function Σ be defined by (39); smooth solutions of (32) satisfy*

$$\partial_t \Sigma - \partial_x(\mathbf{u}, N^T \mathcal{W}) = \lambda((\epsilon'' - \theta'')(\mathcal{V})(\mathbf{v} - \mathcal{V}), \mathbf{v} - \mathcal{V}). \tag{40}$$

Proof. Though we already gave the expression (39), we will construct the function Σ progressively, so that it mimics the total energy. First, in order to identify the kinetic energy term, we take the scalar product of the second equation in (32) with $\mathbf{u}$, and we get

$$\partial_t \frac{|\mathbf{u}|^2}{2} - \partial_x(\mathbf{u}, N^T \mathcal{W}) + (N^T \mathcal{W}, \partial_x \mathbf{u}) = 0.$$

Now, using the first equation,

$$(N^T \mathcal{W}, \partial_x \mathbf{u}) = (\mathcal{W}, N \partial_x \mathbf{u}) = (\mathcal{W}, \partial_t \mathbf{v}) = ((\epsilon' - \theta')(\mathcal{V}), \partial_t \mathbf{v}) + (\theta'(\mathbf{v}), \partial_t \mathbf{v}),$$

we have

$$(\theta'(\mathbf{v}), \partial_t \mathbf{v}) = \partial_t(\theta(\mathbf{v}))$$

and we write

$$((\epsilon' - \theta')(\mathcal{V}), \partial_t \mathbf{v}) = \partial_t((\epsilon' - \theta')(\mathcal{V}), \mathbf{v}) - (\partial_t(\epsilon' - \theta')(\mathcal{V}), \mathbf{v}).$$

The last equation in (32) yields

$$\partial_t((\epsilon' - \theta')(\mathcal{V})) = \lambda(\epsilon'' - \theta'')(\mathcal{V})(\mathbf{v} - \mathcal{V})$$

and taking the scalar product with $\mathbf{v}$ gives

$$(\partial_t(\epsilon' - \theta')(\mathcal{V}), \mathbf{v}) = \lambda((\epsilon'' - \theta'')(\mathcal{V})(\mathbf{v} - \mathcal{V}), \mathbf{v}),$$

hence we obtain

$$\partial_t\big(\frac{|\mathbf{u}|^2}{2} + \theta(\mathbf{v}) + ((\epsilon' - \theta')(\mathcal{V}), \mathbf{v})\big) - \partial_x(\mathbf{u}, N^T\mathcal{W}) = \lambda((\epsilon'' - \theta'')(\mathcal{V})(\mathbf{v} - \mathcal{V}), \mathbf{v}).$$

We take Σ in the form $\Sigma = \frac{|\mathbf{u}|^2}{2} + \theta(\mathbf{v}) + ((\epsilon' - \theta')(\mathcal{V}), \mathbf{v}) + \Phi(\mathcal{V})$ and choose Φ in order that $\Sigma = e$ when $\mathcal{V} = \mathbf{v}$. We have

$$\partial_t(\frac{|\mathbf{u}|^2}{2} + \theta(\mathbf{v}) + \Phi(\mathcal{V}) + ((\epsilon' - \theta')(\mathcal{V}), \mathbf{v}) - \partial_x(\mathbf{u}, N^T\mathcal{W}) = \lambda(\Phi'(\mathcal{V}) + ((\epsilon'' - \theta'')(\mathcal{V})(\mathbf{v} - \mathcal{V}), \mathbf{v}),$$

thus we take

$$\Phi(\mathcal{V}) = \epsilon(\mathcal{V}) - \theta(\mathcal{V}) - ((\epsilon' - \theta')(\mathcal{V}), \mathcal{V})$$

which results in the formula (39): $\Sigma(\mathbf{v}, \mathbf{u}, \mathcal{V}) = \frac{1}{2}|\mathbf{u}|^2 + \epsilon(\mathcal{V}) + \theta(\mathbf{v}) - \theta(\mathcal{V}) + ((\epsilon' - \theta')(\mathcal{V}), \mathbf{v} - \mathcal{V})$. Finally, noticing that

$$\Phi'(\mathcal{V}) = -((\epsilon'' - \theta'')(\mathcal{V}), \mathcal{V}),$$

we get

$$\partial_t\Sigma - \partial_x(\mathbf{u}, N^T\mathcal{W}) = \lambda((\epsilon'' - \theta'')(\mathcal{V})(\mathbf{v} - \mathcal{V}), \mathbf{v} - \mathcal{V})$$

as announced in (40). □

In order to check the entropy dissipation, since the right hand side in (40) is negative if the matrix $\theta'' - \epsilon''$, which is obviously symmetric, is positive definite, we shall thus make the (rough) assumption

(H) $\qquad \theta''(\mathcal{V}) - \epsilon''(\mathcal{V})$ is positive definite

for all $\mathcal{V}$ under consideration.

The first precise assumption for the set on which (H) is valid concerns naturally the set of state variables $\Omega_{\mathbf{v}}$. Let us assume: $\exists\alpha > 0$, $\exists\delta > 0$, $\forall\mathbf{v} \in \Omega_{\mathbf{v}}$, $\forall X \in \mathbb{R}^r$, $(\epsilon''(\mathbf{v})X, X) \geq \alpha|X|^2$ and

(**H1**) $\qquad ((\theta'' - \epsilon'')(\mathbf{v})X, X) \geq \delta|X|^2.$

Then, we can prove

Lemma 2.2. *Assuming (**H1**), the energy $\Sigma(\mathbf{U})$ defined by (39) is strictly convex on the set $\mathcal{U}_{eq}$ of equilibrium states $\mathbf{U}_{eq} = (\mathbf{v}, \mathbf{u}, \mathbf{v})^T$.*

Proof. Indeed, we compute the derivative $\nabla_{\mathbf{U}}\Sigma = \Sigma'$ (wrt. the decomposition $\mathbf{U} = (\mathbf{v}, \mathbf{u}, \mathcal{V})^T$)

$$\Sigma'(\mathbf{U}) = (\theta'(\mathbf{v}) + (\epsilon' - \theta')(\mathcal{V}), \mathbf{u}, ((\epsilon'' - \theta'')(\mathcal{V}), \mathbf{v} - \mathcal{V}))^T \tag{41}$$

and denoting $\nabla_{\mathbf{U},\mathbf{U}} = \Sigma''$

$$\Sigma''(\mathbf{U}_{eq}) = \begin{pmatrix} \theta''(\mathbf{v}) & 0 & (\epsilon'' - \theta'')(\mathbf{v}) \\ 0 & I & 0 \\ (\epsilon'' - \theta'')(\mathbf{v}) & 0 & (\theta'' - \epsilon'')(\mathbf{v}) \end{pmatrix}, \tag{42}$$

so that, for a vector $\mathbf{V}$ in the form $(X_{\mathbf{v}}, Y_{\mathbf{u}}, Z_{\mathcal{V}}) \in \mathbb{R}^r \times \mathbb{R}^d \times \mathbb{R}^r$, we have

$$(\Sigma''(\mathbf{U}_{eq})\mathbf{V}, \mathbf{V}) = (\epsilon''(\mathbf{v})X, X) + ((\theta'' - \epsilon'')(\mathbf{v})(X - Z), X - Z) + |Y|^2.$$

Assuming (H1), in which we may assume that $\delta \leq \frac{\alpha}{2}$, we then have $\forall \mathbf{U}_{eq} \in \mathcal{U}_{eq}$, $\forall \mathbf{V} \in \mathbb{R}^{2r+d}$,

$$(\Sigma''(\mathbf{U}_{eq})\mathbf{V}, \mathbf{V}) \geq \alpha|X|^2 + \delta|X - Z|^2 + |Y|^2 \geq \frac{2\delta}{3}|X|^2 + \frac{\delta}{4}|Z|^2 + |Y|^2,$$

which gives $\forall \mathbf{U}_{eq} \in \mathcal{U}_{eq}$, $\forall \mathbf{V} \in \mathbb{R}^{2r+d}$,

$$(\Sigma''(\mathbf{U}_{eq})\mathbf{V}, \mathbf{V}) \geq c|\mathbf{V}|^2 \tag{43}$$

for $c = \min(\frac{\delta}{4}, 1)$. □

Then Σ is not necessarily convex on the whole set $\mathcal{U} \subset \Omega_{\mathbf{v}} \times \mathbb{R}^d \times \mathbb{R}^r$, because $\mathcal{V}$ is not supposed to lay in $\Omega_{\mathbf{v}}$, but Σ is still convex in a compact neighborhood K of any equilibrium state $\mathbf{U}_{eq}$. Moreover, under the above assumption H, we have a minimization principle.

More precisely, assume that for a given $\overline{\mathbf{v}} \in \Omega_{\mathbf{v}}$, $\exists K(\overline{\mathbf{v}})$ compact neighborhood of $\overline{\mathbf{v}}$ in $\mathbb{R}^r$, $\exists \delta > 0$, for all $\mathcal{V} \in K(\overline{\mathbf{v}})$, $\forall X \in \mathbb{R}^r$

(H2) $$((\theta'' - \epsilon'')(\mathcal{V})X, X) \geq \delta|X|^2.$$

Then, we can prove

Lemma 2.3. *Assuming (**H2**) for fixed $\overline{\mathbf{v}} \in \Omega_{\mathbf{v}}$, we have for all $\mathbf{u} \in \mathbb{R}^d$*

$$e(\overline{\mathbf{v}}, \mathbf{u}) = \min_{\mathcal{V} \in K} \Sigma(\overline{\mathbf{v}}, \mathbf{u}, \mathcal{V})$$

i.e., the minimum of Σ is attained at equilibrium $\mathcal{V} = \mathbf{v}$.

Proof. We have computed the first order derivative of Σ wrt. $\mathcal{V}$ at point $(\overline{\mathbf{v}}, \mathbf{u}, \mathcal{V})$

$$\Sigma_{\mathcal{V}}(\overline{\mathbf{v}}, \mathbf{u}, \mathcal{V}) = (\epsilon'' - \theta'')(\mathcal{V})(\overline{\mathbf{v}} - \mathcal{V}).$$

If Σ attains an extremum in the interior of K, $\Sigma_{\mathcal{V}}$ vanishes, thus $\mathcal{V} = \overline{\mathbf{v}}$ and it is a minimum because at this point $\mathcal{V} = \overline{\mathbf{v}}$ and

$$\Sigma_{\mathcal{V},\mathcal{V}}(\overline{\mathbf{v}}, \mathbf{u}, \overline{\mathbf{v}}) = (\theta'' - \epsilon'')(\overline{\mathbf{v}})$$

is positive definite by (H2). If $\mathcal{V}_0$ is a point on the boundary of K, we set $\mathcal{V} = \overline{\mathbf{v}} + y(\mathcal{V}_0 - \overline{\mathbf{v}})$ where $y \in [0, 1]$, then the function

$$\sigma(y) = \Sigma(\overline{\mathbf{v}}, \mathbf{u}, \overline{\mathbf{v}} + y(\mathcal{V}_0 - \overline{\mathbf{v}}))$$

is such that $\sigma'(y)$ vanishes only at $y = 0$ since

$$\sigma'(y) = \Sigma_{\mathcal{V}}(\overline{\mathbf{v}}, \mathbf{u}, \mathcal{V})(\mathcal{V}_0 - \overline{\mathbf{v}}) = ((\epsilon'' - \theta'')(\mathcal{V})(\overline{\mathbf{v}} - \mathcal{V}), \mathcal{V}_0 - \overline{\mathbf{v}}) = y((\theta'' - \epsilon'')(\mathcal{V})(\mathcal{V}_0 - \overline{\mathbf{v}}), \mathcal{V}_0 - \overline{\mathbf{v}})$$

and

$$\sigma''(0) = ((\theta'' - \epsilon'')(\mathcal{V})(\mathcal{V}_0 - \overline{\mathbf{v}}), \mathcal{V}_0 - \overline{\mathbf{v}}) \geq \delta|\mathcal{V}_0 - \overline{\mathbf{v}}|^2$$

is positive so that $y = 1$ cannot be a minimum and Σ cannot attain its minimum on the boundary. □

Now, the source term cannot ensure *total dissipation* but still ensures some partial dissipation. Assuming (**H2**), we obtain the system is (locally, near the equilibrium manifold) *entropy dissipative* (in the sense of [25], [29]); again we use the notation $\nabla_{\mathbf{U}}\Sigma(\mathbf{U}) = \Sigma'(\mathbf{U})$.

Lemma 2.4. *Assume that (**H2**) is verified for some fixed $\overline{\mathbf{v}} \in \Omega_{\mathbf{v}}$. Then, for all $\mathbf{U} = (\mathbf{v}, \mathbf{u}, \mathcal{V}) \in \mathcal{U}$, with $\mathcal{V} \in K(\overline{\mathbf{v}})$, and for all $\mathbf{U}_{eq} \in \mathcal{U}_{eq}$*

$$(\Sigma'(\mathbf{U}) - \Sigma'(\mathbf{U}_{eq})).\mathbf{G}(\mathbf{U}) \leq -\delta|\mathbf{G}(\mathbf{U})|^2.$$

Proof. We have by definition of $\mathcal{U}_{eq}$, $\forall \mathbf{U}_{eq} \in \mathcal{U}_{eq}$, $\mathbf{G}(\mathbf{U}_{eq}) = \mathbf{0}$. Then, using the above computations, we get from (41)

$$\Sigma'(\mathbf{U}_{eq}) = (\theta'(\mathbf{v}), \mathbf{u}, 0)^T$$

and thus, $\forall \mathbf{U} \in \mathcal{U}$,

$$\Sigma'(\mathbf{U}_{eq}).\mathbf{G}(\mathbf{U}) = 0. \tag{44}$$

Hence, $\forall \mathbf{U} = (\mathbf{v}, \mathbf{u}, \mathcal{V}) \in \mathcal{U}$,

$$\Sigma'(\mathbf{U}).\mathbf{G}(\mathbf{U}) = (\Sigma'(\mathbf{U}) - \Sigma'(\mathbf{U}_{eq})).\mathbf{G}(\mathbf{U}) = ((\epsilon'' - \theta'')(\mathcal{V})(\mathbf{v} - \mathcal{V}), \mathbf{v} - \mathcal{V}),$$

which under (**H2**) gives the result. □

2.2.2 Chapman-Enskog dissipativity

Let us now consider a Chapman-Enskog type expansion, we identify a first order corrector term

$$\mathcal{V}_\lambda = \mathbf{v}_\lambda + \frac{1}{\lambda}\mathcal{V}_\lambda^{(1)} + ...$$

where we now emphasize the dependence on λ of the solution of (32), then plug it in the equations. Keeping the right-hand side of the relaxation system, we get

$$\partial_t \mathbf{v}_\lambda = -\mathcal{V}_\lambda^{(1)}$$

at the order $\mathcal{O}(\frac{1}{\lambda})$, hence with the first equation of (32)

$$\mathcal{V}_\lambda^{(1)} = -N\partial_x \mathbf{u}_\lambda$$

then

$$\mathcal{W}(\mathbf{v}_\lambda, \mathcal{V}_\lambda) = \epsilon'(\mathbf{v}_\lambda) - \frac{1}{\lambda}(\theta''(\mathbf{v}_\lambda) - \epsilon''(\mathbf{v}_\lambda))\mathcal{V}_\lambda^{(1)} + \mathcal{O}(\frac{1}{\lambda^2}),$$

and using the second equation in (32) and retaining the first order term, we obtain

$$\partial_t \mathbf{u}_\lambda - N^T\partial_x \epsilon'(\mathbf{v}_\lambda) = \frac{1}{\lambda}\partial_x(N^T(\theta''(\mathbf{v}_\lambda) - \epsilon''(\mathbf{v}_\lambda))N\partial_x \mathbf{u}_\lambda).$$

Hence (now droping the dependence on λ), if $(\mathbf{v}, \mathbf{u})$ is solution of (32), it satisfies (at least formally)

$$\begin{cases} \partial_t \mathbf{v} - N\partial_x \mathbf{u} = \mathbf{0} \\ \partial_t \mathbf{u} - N^T\partial_x \epsilon'(\mathbf{v}) = \frac{1}{\lambda}\partial_x(N^T(\theta''(\mathbf{v}) - \epsilon''(\mathbf{v}))N\partial_x \mathbf{u}) \end{cases} \tag{45}$$

at the order $\mathcal{O}(\frac{1}{\lambda^2})$. Under hypothesis (**H1**), (45) is indeed a diffusive approximation of system (31).

Remark 3. The second-order operator

$$D = \begin{pmatrix} 0 & 0 \\ 0 & N^T(\theta'' - \epsilon'')(\mathbf{v}))N \end{pmatrix}$$

is *symmetrically entropy dissipative* (as defined in [6]), i.e. the matrix De'' is symmetric nonnegative. Also, if we compute $\Sigma''(\mathbf{U}_{eq})\mathbf{A}(\mathbf{U}_{eq})$ where the Jacobian matrix is given in (35) and Σ'' in (42), we get

$$\Sigma''\mathbf{A}(\mathbf{U}_{eq}) = \begin{pmatrix} 0 & -\theta''(\mathbf{v})N & 0 \\ -N^T\theta''(\mathbf{v}) & 0 & N^T(\theta - \epsilon)''(\mathbf{v})) \\ 0 & (\theta - \epsilon)''(\mathbf{v})N & 0 \end{pmatrix}$$

which is symmetric, so that the *reduced stability condition* holds.

The *subcharacteristic condition* is then a consequence of the previous condition, as proved in [6]. Denote by $\mathbf{a}(\mathbf{v}, \mathbf{u})$ the Jacobian of system (31), and let $\mu_k(\mathbf{a}(\mathbf{v}, \mathbf{u}))$, $k = 1, ..., r + d$ denote its ordered eigenvalues (respectively $\mu_k(\mathbf{A}(\mathcal{M}(\mathbf{v}, \mathbf{u})))$, $1 \leq k \leq 2r + d$, the ordered eigenvalues of the Jacobian $\mathbf{A}$ of the relaxation system) repeated with multiplicities. The subcharacteristic condition says that, for equilibrium states, the eigenvalues are *interlaced*: for any $1 \leq k \leq 2r + d$

$$\mu_k(\mathbf{A}(\mathcal{M}(\mathbf{v}, \mathbf{u})) \leq \mu_k(\mathbf{a}(\mathbf{v}, \mathbf{u})) \leq \mu_{k+r}(\mathbf{A}(\mathcal{M}(\mathbf{v}, \mathbf{u}))).$$

This condition means that the equilibrium (reduced) system cannot propagate signals faster than the relaxation system. The reduced stability condition yields weak entropy inequalities for the approximate solver which we will construct below (cf. again [6], Theorem 3.4 and [7]). □

We will later use the notation

$$\mathcal{E} = N^T\epsilon''N, \Theta = N^T\theta''N \tag{46}$$

(note that the notation is coherent with (9) in the isentropic case) and precise the set of states for which $\Theta - \mathcal{E}$ is positive definite.

2.3 Example: MHD

We revisit the MHD system in the isentropic case. We have $r = 3$, $d = 3$. Let again $\mathbf{v} = (\tau, \tau B_\perp)$, assume $p = \tilde{p}(\tau)$, from (26), we have by (23)

$$\epsilon(\mathbf{v}) = \tilde{\varepsilon}(\tau) + \frac{1}{2}\tau|\mathbf{B}|^2 = \tilde{\varepsilon}(\tau) + \frac{1}{2}\tau B_x^2 + \frac{1}{2\tau}|\tau\mathbf{B}_\perp|^2$$

with $\tilde{\varepsilon}'(\tau) = -p$, then $\mathcal{V}$ is a new variable which relaxes to $\mathbf{v} = (\tau, \tau\mathbf{B}_\perp)$, and by (33)

$$\mathcal{W}(\mathbf{v}, \mathcal{V}) = \epsilon'(\mathcal{V}) + \theta'(\mathbf{v}) - \theta'(\mathcal{V})$$

relaxes to

$$\epsilon'(\mathbf{v}) = (-p + \frac{1}{2}B_x^2 - \frac{1}{2}|\mathbf{B}_\perp|^2, \mathbf{B}_\perp)^T.$$

Thus ϵ'' is given by

$$\epsilon''(\mathbf{v}) = \begin{pmatrix} -\tilde{p}'(\tau) + |\mathbf{B}_\perp|^2/\tau & -\mathbf{B}_\perp^T/\tau \\ -\mathbf{B}_\perp/\tau & diag(1/\tau) \end{pmatrix}$$

and $\mathcal{E}$ (see (45) and (46))

$$\mathcal{E}(\mathbf{v}) = N^T\epsilon''(\mathbf{v})N = \begin{pmatrix} -\tilde{p}' + |\mathbf{B}_\perp|^2/\tau & -B_x\mathbf{B}_\perp^T/\tau \\ -B_x\mathbf{B}_\perp/\tau & diag(B_x^2/\tau) \end{pmatrix}. \tag{47}$$

We take $\Theta = N^T\theta''N$ in the same block form.

Lemma 2.5. *A necessary and sufficient condition for a matrix of the form*

$$\begin{pmatrix} b_0 & -b_1 & -b_2 \\ -b_1 & d & 0 \\ -b_2 & 0 & d \end{pmatrix} \tag{48}$$

to be positive definite is $b_0 > 0$, $b_0 d > b_1^2 + b_2^2$. Then its eigenvalues are positive and given by d and $\frac{1}{2}(b_0 + d \pm \sqrt{(b_0 - d)^2 + 4(b_1^2 + b_2^2)}$.

Proof. The second inequality ensures $b_0 d > b_1^2$ and if $b_0 > 0$ imposes $d > 0$. Since $d > 0$, we can write $d = c^2$. We shall set $B = (b_1, b_2)^T$ so that $b_1^2 + b_2^2 = |B|^2$. □

For the matrix $\mathcal{E}(\mathbf{v})$ in (47), we have $b_0 = -\tilde{p}'(\tau) + |\mathbf{B}_\perp/\sqrt{\tau}|^2$, $c = B_x/\sqrt{\tau}$, $B = B_x \mathbf{B}_\perp/\tau$. Recall that $\tilde{p}'(\tau) < 0$, noting $c_L^2 = -\tilde{p}'(\tau)$, c_L is the Lagrangian sound speed. Also, by (27), $c_a^2 = B_x^2/\tau$, where $\pm c_a$ is the velocity of an Alfven wave. It is easy to check that c_a^2 is an eigenvalue of $\mathcal{E}(\mathbf{v})$ (computations already done in section 1.3.2, see (26)) .

For $\Theta = N^T \theta'' N$, we may mimic the form of $\mathcal{E}(\mathbf{v})$ and take $b_0 = c_0^2 + |C|^2$ and $B = cC$,

$$\Theta = \begin{pmatrix} c_0^2 + |C|^2 & -cC^T \\ -cC & diag(c^2) \end{pmatrix}$$

and even we can set $c = mB_x$, which gives, with the definition of N in (24) that

$$\theta'' = \begin{pmatrix} c_0^2 + |C|^2 & -mC^T \\ -mC & diag(m^2) \end{pmatrix} \tag{49}$$

Θ is positive definite if and only if $c_0^2 > 0$ and $m^2 > 0$. The difference $\Theta - \mathcal{E}$ has the same form (48).

Proposition 2.6. *A necessary and sufficient condition for (H) to be satisfied, where θ'' is given by (49), is*

$$c_0^2 \geq -\tilde{p}'(\tau),\ m^2 > \varrho = \frac{1}{\tau},\ |C - m\mathbf{B}_\perp|^2 \leq (c_0^2 + \tilde{p}'(\tau))(\tau m^2 - 1) \tag{50}$$

for all quantities $\tau, \mathbf{B}_\perp$ under consideration.

Proof. The condition

$$c_0^2 \geq c_L^2 = -\tilde{p}'(\tau),\ c^2 \geq c_a^2 = B_x^2/\tau \Leftrightarrow m^2 > \varrho = 1/\tau$$

is clearly necessary, $\Theta - \mathcal{E}$ is positive (semi-definite) if and only if

$$|cC - c_a \mathbf{B}_\perp/\sqrt{\tau}|^2 \leq (c_0^2 + |C|^2 - c_L^2 - |\mathbf{B}_\perp|^2/\tau)(c^2 - c_a^2)$$

which is equivalent to

$$|c_a C - c\mathbf{B}_\perp/\sqrt{\tau}|^2 \leq (c_0^2 - c_L^2)(c^2 - c_a^2)$$

and can be written

$$|C - m\mathbf{B}_\perp|^2 \leq (c_0^2 - c_L^2)(\tau m^2 - 1),$$

We interpret (50) as follows: c_0 is a bound for the sound speed, this is the analogous of Whitham's condition for the relaxation of the Euler system, m^2 for the density, the vector C/m should approach the transverse magnetic field. □

Remark 4. In [9], a very close relaxation system is considered for the MHD equations in Eulerian coordinates (with energy) and the authors introduce analogue conditions (see (2.23)) for the entropy dissipation. □

If c_0, C and m are constant, the matrix θ'' is constant and we have $\theta(\mathbf{v}) = (c_0^2 + |C|^2)\tau^2/2 + m^2\tau^2|B_\perp|^2/2 - m\tau^2(C, B_\perp)$.

To micmic $\mathbf{v} = (\tau, \tau B_\perp)$, let us note the components of $\mathcal{V}$, by $\mathcal{V} = (\mathcal{T}, \mathcal{B}) \in \mathbb{R} \times \mathbb{R}^2$, and for $\mathcal{W}$, $\mathcal{W} = (-\Pi, \Pi_\perp) \in \mathbb{R} \times \mathbb{R}^2$ where

$$\Pi = \tilde{p}^*(\mathcal{T}) + \theta_\tau(\mathcal{V}) - \theta_\tau(\mathbf{v}),$$

then the 9×9 relaxation system (32) for isentropic MHD writes

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t \tau \mathbf{B}_\perp - \partial_x B_x \mathbf{u}_\perp = \mathbf{0} \\ \partial_t u + \partial_x \Pi = 0, \\ \partial_t \mathbf{u}_\perp - \partial_x B_x \Pi_\perp = \mathbf{0}, \\ \partial_t \mathcal{T} = \lambda(\tau - \mathcal{T}), \\ \partial_t \mathcal{B} = \lambda(\tau \mathbf{B}_\perp - \mathcal{B}). \end{cases} \tag{51}$$

For instance, if $C = 0$ and c_0 are constant in the matrix θ'' given by (49)

$$\Pi = \tilde{p}^*(\mathcal{T}) + c_0^2(\mathcal{T} - \tau),$$

and we find the the extension to MHD of the pressure relaxation already introduced for Euler (see [3], [20]).

The eigenvalues of (51) are 0 with muliplicity 3 (corresponding to the last equations) and three pairs of symmetric eigenvalues.

Remark 5. The results of lemma 2.5 extend easily to a larger matrix of the same structure

$$\begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ -b_1 & d & 0 & 0 \\ -b_2 & 0 & d & 0 \\ -b_3 & 0 & 0 & d \end{pmatrix} \tag{52}$$

□

For the numerical scheme, we need not solve a general Cauchy problem for (32), only Riemann problems for the extracted first-order system

$$\begin{cases} \partial_t \mathbf{v} - N \partial_x \mathbf{u} = \mathbf{0} \\ \partial_t \mathbf{u} - N^T \partial_x \mathcal{W} = \mathbf{0} \\ \partial_t \mathcal{V} = 0. \end{cases} \tag{53}$$

We will see that it is very simple at least when θ is quadratic, because all the characteristic fields of (53) are linearly degenerate (cf. section 4.1 below). However, it is interesting to see how the general theory for systems with relaxation applies in our case, which means how the solutions of the relaxation system approach solutions of our original fluid system as $\lambda \to \infty$.

3 Relaxation approximation

Approximation results are proved in the pioneering work of [18], however, under the assumption that the entropy of the relaxation system is convex. Since Σ is not necessarily convex in the entire set of states, we cannot apply the results in [18]. As in [17], which concerns the Euler system, we want to apply Yong's existence results stated in [39] which yield existence of classical solutions of the relaxation system near a smooth equilibrium initial data and convergence as $\lambda \to \infty$. We will be first interested by applying [40] on the existence of a global in time smooth solution of the relaxation system under initial data close to a constant equilibrium state, with a continuity result, similar results can be found in [25].

3.1 Existence of a solution

3.1.1 General result

In order to apply Yong's global existence result, we have to check that the assumptions of [40] are satisfied. Let us first recall the general framework. Consider a system written in the general form

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \lambda \mathbf{Q}(\mathbf{U}) \tag{54}$$

with $\mathbf{U} = (u, v)^T \in \mathbb{R}^{q+r}$, $\mathbf{Q}(\mathbf{U}) = (0, g(u, v))^T \in \mathbb{R}^{q+r}$, the set of states is denoted G, an equilibrium state $\mathbf{U}_e$ by definition satisfies $\mathbf{Q}(\mathbf{U}_e) = \mathbf{0} \Leftrightarrow g(u, v) = 0$. The assumptions which ensure the existence of a (unique global smooth) solution to (54), under an initial data close to an equilibrium state $\mathbf{U}_e$ are

1. $g_v(\mathbf{U}_e)$ is invertible on $\mathbb{R}^r$
2. $\exists\, \eta$ a strictly convex smooth entropy, defined in a convex compact neighborhood K of $\mathbf{U}_e$, such that $\eta_{\mathbf{UU}}(\mathbf{U})\mathbf{F}'(\mathbf{U})$ is symmetric for all $\mathbf{U} \in K_e$
3. $\exists\, c_G > 0, \forall \mathbf{U} \in K, \big(\eta_{\mathbf{U}}(\mathbf{U}) - \eta_{\mathbf{U}}(\mathbf{U}_e)\big)\mathbf{Q}(\mathbf{U}) \leq -c_G|\mathbf{Q}(\mathbf{U})|^2$
4. the kernel of $\mathbf{Q}_{\mathbf{U}}(\mathbf{U}_e)$ contains no eigenvector of the Jacobian matrix $\mathbf{F}'(\mathbf{U}_e)$.

The precise existence result is Theorem 3.1 stated in [40]. Its application gives in our case

Theorem 3.1. *Let $s \geq 5/2$ and $\mathbf{U}_{eq}$ a constant equilibrium state; assume (**H2**) is valid. Then, there are two constants c_1, c_2 such that if $\mathbf{U}_0 \in H^s(\mathbb{R})$ satisfies $||\mathbf{U}_0 - \mathbf{U}_{eq}|| \leq c_1$, the relaxation system (32) with initial condition $\mathbf{U}_0$ has a unique global solution $\mathbf{U} = (\mathbf{v}, \mathbf{u}, \mathcal{V}) \in C(0, \infty; H^s(\mathbb{R}))$ satisfying*

$$||\mathbf{U}(., T) - \mathbf{U}_{eq}||_s^2 + \lambda^2 \int_0^T ||(\mathbf{v} - \mathcal{V})(., t)||_s^2 + \int_0^T ||\nabla \mathbf{U}(., t)||_{s-1}^2 dt \leq c_2 ||\mathbf{U}_0 - \mathbf{U}_{eq}||_s^2$$

for any $T > 0$.

3.1.2 Checking the structural properties

In order to prove Theorem 3.1, we check precisely all the required items. Our system writes

$$\begin{cases} \partial_t \mathbf{v} - N \partial_x \mathbf{u} = \mathbf{0} \\ \partial_t \mathbf{u} - N^T \partial_x \mathcal{W} = \mathbf{0} \\ \partial_t \mathcal{V} = \lambda(\mathbf{v} - \mathcal{V}) \end{cases} \tag{55}$$

with (33): $\mathcal{W} = \mathcal{W}(\mathbf{v}, \mathcal{V}) = \epsilon'(\mathcal{V}) + \theta'(\mathbf{v}) - \theta'(\mathcal{V})$. The source term is $\lambda \mathbf{G}(\mathbf{U}) = \lambda(0, 0, \mathbf{v} - \mathcal{V})^T$ with $\mathbf{G}'$ given in block form by

$$\mathbf{G}'(\mathbf{U}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & -I \end{pmatrix} \tag{56}$$

and does not depend on the state $\mathbf{U}$. The dimension of (55) is $2r+d$, with the general notations, $q = r+d$.

We will assume from now on that N is a square $r \times r$ invertible matrix, we have $n = 2d + 1$, and $r = d$, which is verified for Euler and MHD. For other examples it means that we have been able to reduce the system as explained before in (10). The set of states G is then contained in $\mathbb{R}^{3r}$ and we will write a vector of $\mathbb{R}^{3r}$ representing a state $\mathbf{U}$ in the form $(\mathbf{v}, \mathbf{u}, \mathcal{V})$.

We also assume that θ is *quadratic*, thus θ'' is a constant matrix Λ (see remark 1) which is positive definite and such that (H) is satisfied on some set of states that we will precise.

System (55) can be written in the general form (54). We have $\mathbf{U} = (u, v)^T$ with $u = (\mathbf{v}, \mathbf{u}), v = \mathcal{V}$, the source term is $\lambda \mathbf{G}(\mathbf{U}) = (0, g(u, v))^T$, with $g(u, v) = \lambda(\mathbf{v} - \mathcal{V})$.

1.$q_v(\mathbf{U}_{eq}) = -\lambda I$ is indeed invertible. An equilibrium state satisfies $\mathcal{V} = \mathbf{v}$.
2. The entropy η is our energy Σ defined in (39)

$$\Sigma(\mathbf{v}, \mathbf{u}, \mathcal{V}) = \frac{1}{2}|\mathbf{u}|^2 + \epsilon(\mathcal{V}) + \theta(\mathbf{v}) - \theta(\mathcal{V}) + ((\epsilon' - \theta')(\mathcal{V}), \mathbf{v} - \mathcal{V}),$$

it coincides with the classical energy $e(\mathbf{v}, \mathbf{u}) = \frac{1}{2}|\mathbf{u}|^2 + \epsilon(\mathbf{v})$ on the equilibrium manifold. The Jacobian matrix of (55) is given by (35)

$$\mathbf{A}(\mathbf{U}) = \begin{pmatrix} 0 & -N & 0 \\ -N^T\theta''(\mathbf{v}) & 0 & -N^T(\epsilon - \theta)''(\mathcal{V}) \\ 0 & 0 & 0 \end{pmatrix}.$$

We have computed (see Remark 3)

$$\Sigma''\mathbf{A}(\mathbf{U}) = \begin{pmatrix} 0 & -\theta''(\mathbf{v})N & 0 \\ -N^T\theta''(\mathbf{v}) & 0 & -N^T(\epsilon - \theta)''(\mathcal{V}) \\ 0 & -(\epsilon - \theta)''(\mathcal{V})N & 0 \end{pmatrix}$$

which is indeed symmetric. It expresses that the following system is symmetric

$$\partial_t \Sigma'(\mathbf{U}) + \Sigma''\mathbf{F}'(\mathbf{U})\partial_x \mathbf{U} = \Sigma''\mathbf{G}(\mathbf{U}), \text{ with } \Sigma'(\mathbf{U}) = (\mathcal{W}, \mathbf{u}, (\epsilon - \theta)''(\mathcal{V})(\mathbf{v} - \mathcal{V}))$$

which results from simple manipulations of the equations.
3. We have

$$\Sigma(\mathbf{v}, \mathbf{u}, \mathcal{V}) = e(\mathbf{v}, \mathbf{u}) + (\epsilon - \theta)(\mathcal{V}) - (\epsilon - \theta)(\mathbf{v}) - ((\epsilon' - \theta')(\mathcal{V}), \mathcal{V} - \mathbf{v}),$$

and if we write

$$(\theta - \epsilon)(\mathbf{v}) = (\theta - \epsilon)(\mathcal{V}) + (\theta' - \epsilon')(\mathcal{V}), \mathbf{v} - \mathcal{V}) + \big(\int_0^1 (\theta'' - \epsilon'')(\mathbf{v} + s(\mathbf{v} - \mathcal{V}))sds)(\mathbf{v} - \mathcal{V}), \mathbf{v} - \mathcal{V}\big),$$

we get

$$\Sigma(\mathbf{v}, \mathbf{u}, \mathcal{V}) = e(\mathbf{v}, \mathbf{u}) + \big(\int_0^1 (\theta'' - \epsilon'')(\mathbf{v} + s(\mathbf{v} - \mathcal{V}))sds)(\mathbf{v} - \mathcal{V}), \mathbf{v} - \mathcal{V}\big).$$

We will assume (**H2**), i.e. that (H) holds true in a convex compact neighborhood K of a given equilibrium state, then, given $\mathbf{v}$, for $|\mathbf{v}-\mathcal{V}|$ small enough and $s\in(0,1)$,

$$\big((\theta''-\epsilon'')(\mathbf{v}+s(\mathbf{v}-\mathcal{V}))(\mathbf{v}-\mathcal{V}),\mathbf{v}-\mathcal{V}\big)\geq\delta_K|\mathbf{v}-\mathcal{V}|^2.$$

We compute, for $\mathbf{U}=(\mathbf{v},\mathbf{u},\mathcal{V})$, $\mathbf{U}_{eq}=\mathcal{M}(\mathbf{U})=(\mathbf{v},\mathbf{u},\mathbf{v})$, and $\eta=\Sigma$

$$\eta_{\mathbf{U}}(\mathbf{U})-\eta_{\mathbf{U}}(\mathbf{U}_{eq})=\Sigma'(\mathbf{U})-\Sigma'(\mathbf{U}_{eq})=(0,0,(\epsilon''-\theta'')(\mathcal{V})(\mathbf{v}-\mathcal{V}))$$

(see (41)) hence, using moreover the identity $|\mathbf{G}(\mathbf{U})|^2=|\mathbf{v}-\mathcal{V}|^2$, we get (see lemma 2.4)

$$\big(\eta_{\mathbf{U}}(\mathbf{U})-\eta_{\mathbf{U}}(\mathbf{U}_{eq})\big).\mathbf{G}(\mathbf{U})=-\lambda\big((\theta''-\epsilon'')(\mathcal{V})(\mathbf{v}-\mathcal{V}),\mathbf{v}-\mathcal{V}\big)\leq-\lambda\delta_K|\mathbf{G}(\mathbf{U})|^2.$$

4. A vector (U_1,U_2,U_3) is in the kernel of $\mathbf{G}_{\mathbf{U}}(\mathbf{U}_{eq})=\mathbf{G}'$ (see (56)) if and only if $U_3=U_1$. Such a vector is an eigenvector of the Jacobian matrix (35)

$$\mathbf{F}'(\mathbf{U}_{eq})=\begin{pmatrix}0&-N&0\\-N^T\theta''(\mathbf{v})&0&-N^T(\epsilon-\theta)''(\mathbf{v})\\0&0&0\end{pmatrix}$$

associated to an eigenvalue μ if and only if U_1,U_2 satisfy

$$\begin{cases}-NU_2=\mu U_1\\-N^T\theta''(\mathbf{v})U_1-N^T(\epsilon-\theta)''(\mathbf{v})U_1=\mu U_2\\\mu U_1=\mathbf{0}.\end{cases}$$

The last equation gives either

-$\mu\neq0$, $U_1=\mathbf{0}$, then if N is invertible, the first equation gives $U_2=\mathbf{0}$ and this is impossible for an eigenvector,

-or $\mu=0$, then the first equation gives again $U_2=\mathbf{0}$, and the second $\epsilon''(\mathbf{v})U_1=\mathbf{0}$, so that $U_1=\mathbf{0}$ and this is impossible too.

This ends the proof of Theorem 3.1.

Remark 6. Condition 4 is known as Shizuta-Kawashima condition.When N is not invertible, which can occur if we add a variable convected by the flow, for instance a mass fraction Y, in Lagrangian coordinates $\partial_tY=0$, the Shizuta-Kawashima condition cannot be satisfied without additionnal assumption. Indeed, even in the simplest case of the $p-$system (Euler system (20) without energy equation), with $p=p(\tau,Y)$, $p=cst$ does not imply Y and τ constant. We must assume $Y=Y_0$ constant (independent of x). See [29] for more general results concerning systems violating Shizuta-Kawashima condition. □

3.2 Relaxation and approximation

We now refer to the *local* results of [39] which are a little different and investigate the limit problem as $\lambda\to\infty$. In order to apply this local (in time) existence result, we check that the assumptions of [39] are satisfied, this will be done assuming moreover that $\theta''\equiv\nabla_{\mathbf{U},\mathbf{U}}\theta$ is a constant matrix.

3.2.1 Structural stability condition

We are going to derive some more estimates linked to the existence of a symmetrizer for the hyperbolic part of (32) and to the particular form of the source term (34) which cannot ensure *total dissipation* but still ensures some partial dissipation. These sharpened properties are obtained in order to apply Yong's

result in the next section. The source term is $\lambda \mathbf{G}(\mathbf{U}) = \lambda(0, 0, \mathbf{v} - \mathcal{V})^T$ with $\mathbf{G}'$ given by (56). Moreover, $\mathbf{G}'$ is diagonalizable. Define

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_r \end{pmatrix}, P_\Omega = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\Omega & 0 & \Omega \end{pmatrix} \tag{57}$$

for any square invertible $r \times r$ matrix $\Omega = \Omega(\mathbf{U})$ so that P_Ω is invertible. Note that the square upper identity block comes from the fact that when multiplying the system by $P(\mathbf{U}_{eq})$ the two first lines must give the equilibrium system (see (1.6) in [39]). Then,

$$-JP_\Omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Omega & 0 & -\Omega \end{pmatrix} = P_\Omega \mathbf{G}'$$

and thus $P_\Omega \mathbf{G}' P_\Omega^{-1} = -J$ is a block diagonal matrix with a lower strictly dissipative block. Note that J, defined on $\mathbb{R}^r \times \mathbb{R}^d \times \mathbb{R}^r$ maps $(X_\mathbf{v}, Y_\mathbf{u}, Z_\mathcal{V})$ on $(0, 0, Z_\mathcal{V})$.

The matrix $\Sigma''(\mathbf{U}) = \nabla_{\mathbf{U},\mathbf{U}}\Sigma(\mathbf{U})$ cannot be taken as a symmetrizer since in (42), the Hessian matrix is computed on an equilibrium state $\mathbf{U}_{eq} = (\mathbf{v}, \mathbf{u}, \mathbf{v})$, and only the expressions of $\nabla_{\mathbf{v},\mathbf{v}}$ and $\nabla_{\mathbf{u},\mathbf{u}}$, $\nabla_{\mathbf{v},\mathbf{u}}$, $\nabla_{\mathcal{V},\mathbf{u}}$ would remain the same if it were computed on a state $\mathbf{U} = (\mathbf{v}, \mathbf{u}, \mathcal{V})$, not those involving other partial derivatives wrt. to $\mathcal{V}$, which leads to the fact that Σ is not necessarily convex. We may have however the idea of considering the nearby expression

$$S_0(\mathbf{U}) = \begin{pmatrix} \theta''(\mathbf{v}) & 0 & -(\theta - \epsilon)''(\mathcal{V}) \\ 0 & I & 0 \\ -(\theta - \epsilon)''(\mathcal{V}) & 0 & (\theta - \epsilon)''(\mathcal{V}) \end{pmatrix} \tag{58}$$

which coincides with $\Sigma'' = \nabla_{\mathbf{U},\mathbf{U}}\Sigma$ on the equilibrium manifold $\mathcal{U}_{eq}$.

Lemma 3.2. *Assume that $\theta'' \equiv \nabla_{\mathbf{U},\mathbf{U}}\theta$ is a constant matrix. Under hypothesis (**H2**), for $\mathbf{U} = (\mathbf{v}, \mathbf{u}, \mathcal{V})^T$, $\mathcal{V} \in K(\overline{\mathbf{v}})$, the symmetric matrix $S_0(\mathbf{U})$ defined by (58) is positive definite and $S_0(\mathbf{U})A(\mathbf{U})$ is symmetric.*

Proof. For a vector $\mathbf{V}$ in the form $(X_\mathbf{v}, Y_\mathbf{u}, Z_\mathcal{V})$, we compute

$$(S_0(\mathbf{U})\mathbf{V}, \mathbf{V}) = (\theta''(\mathbf{v})X, X) + |Y|^2 - 2((\theta'' - \epsilon'')(\mathcal{V})X, Z) + ((\theta'' - \epsilon'')(\mathcal{V})Z, Z).$$

If θ'' is constant (θ is quadratic), we may write

$$(S_0(\mathbf{U})\mathbf{V}, \mathbf{V}) = (\epsilon''(\mathcal{V})X, X) + ((\theta'' - \epsilon'')(\mathcal{V})(X - Z), X - Z) + |Y|^2.$$

We assume moreover

$$\forall \mathcal{V} \in K(\mathbf{v}),\ (\epsilon''(\mathcal{V})X, X) \geq \alpha |X|^2, \tag{59}$$

(till now, the assumption on ϵ'' only concerned state variables $\mathbf{v}$ in $\Omega_\mathbf{v}$), then

$$(S_0(\mathbf{U})\mathbf{V}, \mathbf{V}) \geq \alpha |X|^2 + |Y|^2 + \delta |X - Z|^2$$

and we conclude as in the proof of Lemma 2.2.

Then, setting

$$\Delta(\mathcal{V}) = (\theta'' - \epsilon'')(\mathcal{V}), \tag{60}$$

we have

$$S_0(\mathbf{U})A(\mathbf{U}) = \begin{pmatrix} 0 & -\theta''(\mathbf{v})N & 0 \\ -N^T\theta''(\mathbf{v}) & 0 & N^T\Delta \\ 0 & \Delta N & 0 \end{pmatrix} \tag{61}$$

where with shorthand notations, $\Delta \equiv \Delta(\mathcal{V})$, so that $S_0(\mathbf{U})$ is indeed a symmetrizer for $A(\mathbf{U})$. □

Lemma 3.3. *Assume that (**H2**) is verified. Define in (57) the matrix Ω by the square root of Δ,*

$$\Omega^2(\mathcal{V}) = (\theta'' - \epsilon'')(\mathcal{V}) \tag{62}$$

and note by P the matrix associated to this $\Omega = \Delta^{1/2}$ by (57). Then we have for all $\mathcal{V} \in K(\mathbf{v})$,

$$S_0(\mathbf{U}).\mathbf{G}' = \mathbf{G}'^T.S_0(\mathbf{U}) = P^T J P = -\begin{pmatrix} \Delta & 0 & -\Delta \\ 0 & 0 & 0 \\ -\Delta & 0 & \Delta \end{pmatrix}(\mathcal{V}).$$

This result expresses a kind of coupling between the hyperbolic part and the source term.

Proof. The matrix $S_0(\mathbf{U}).\mathbf{G}'(\mathbf{U})$ is symmetric and

$$S_0(\mathbf{U}).\mathbf{G}' = -\begin{pmatrix} \Delta & 0 & -\Delta \\ 0 & 0 & 0 \\ -\Delta & 0 & \Delta \end{pmatrix}(\mathcal{V})$$

while for any Ω,

$$P_\Omega^T J P_\Omega = -\begin{pmatrix} -\Omega^T\Omega & 0 & \Omega^T\Omega \\ 0 & 0 & 0 \\ \Omega^T\Omega & 0 & -\Omega^T\Omega \end{pmatrix}.$$

If (H2) is verified, Δ is (symmetric) positive definite and we can indeed define Ω by $\Omega^T\Omega = \Delta(\mathcal{V})$ which once plugged in the expression of P_Ω (simply denoted by P) gives

$$S_0(\mathbf{U}).\mathbf{G}' = P^T J P.$$

Note also that we have a strict dissipation since, for any vector $\mathbf{V}$ in the form $(X_\mathbf{v}, Y_\mathbf{u}, Z_\mathcal{V})$,

$$(P^T J P\mathbf{V}, \mathbf{V}) = (JP\mathbf{V}, P\mathbf{V}) = -(\Delta(X - Z), X - Z) \leq -\delta|X - Z|^2.$$

Let us notice that if, instead of (H2), (H1) is satisfied, we have the result for equilibrium states: $\forall \mathbf{v} \in \Omega_\mathbf{v}$, $S_0(\mathbf{U}_{eq}).\mathbf{G}'(\mathbf{U}_{eq}) = P^T J P(\mathbf{v})$. □

3.2.2 An approximation result

We now recall the *local* results of [39]. Consider a system written in the general form

$$\begin{cases} \partial_t \mathbf{W} + \mathbf{A}(\mathbf{W})\partial_x(\mathbf{W}) = \lambda \mathbf{Q}(\mathbf{W}) \\ \mathbf{W}(x, 0) = \mathbf{W}_0^\lambda(x) \end{cases} \tag{63}$$

with $\mathbf{W} \in G$, the equilibrium manifold G_e is the set of states $\mathbf{W}_e \in G$ satisfying $\mathbf{Q}(\mathbf{W}_e) = \mathbf{0}$. The assumptions done in [39], which are referred to as the *stability conditions* are

1. $\exists P(\mathbf{W}_e)$ an $(q + r) \times (q + r)$ invertible matrix, and $S(\mathbf{W}_e)$ an $r \times r$ invertible matrix defined on G_e, $\mathbf{Q}'(\mathbf{W}_e) = P^{-1}(\mathbf{W}_e) J_S(\mathbf{W}_e) P(\mathbf{W}_e)$ where

$$J_S(\mathbf{W}_e) = \begin{pmatrix} 0 & 0 \\ 0 & S(\mathbf{W}_e) \end{pmatrix}$$

2. for $\mathbf{W} \in G$, $\exists A_0(\mathbf{W})$ a positive definite matrix such that $A_0(\mathbf{W})A(\mathbf{W})$ is symmetric

3. for $\mathbf{W}_e \in G_e$, $A_0(\mathbf{W}_e)\mathbf{Q}'(\mathbf{W}_e) + \mathbf{Q}'(\mathbf{W}_e)^T A_0(\mathbf{W}_e) \leq -P^T(\mathbf{W_e})JP(\mathbf{W_e})$ where $J = J_{I_r}$

These assumptions ensure the existence of a unique local smooth solution to (63), for any initial (smooth) data. The precise theorem is stated in [39] (Theorem 6.2) under some additional assumptions and provides an asymptotic expansion, see also its application to the Euler system in [17].

Theorem 3.4. *Assume that (**H2**) holds. Let $s \geq 2$, and consider a (periodic) initial data $\mathbf{U}_0 = (\mathbf{v}_0, \mathbf{u}_0, \mathcal{V}_0) \in H^{s+2}(\mathbb{T})$ that take values in a compact neighborhood of an equilibrium state. Then there exists a time $T > 0$ such that:*

-$\forall \lambda \geq 1$ there exists a unique solution $\mathbf{U}_\lambda = (\mathbf{v}_\lambda, \mathbf{u}_\lambda, \mathcal{V}_\lambda) \in C([0,T), \mathbb{H}^s(\mathbb{T}))$ of (32) with initial data $\mathbf{U}_0$,

-the system (31) admits a unique solution $(\overline{\mathbf{v}}, \overline{\mathbf{u}}) \in C([0,T), \mathbb{H}^{s+2}(\mathbb{T}))$ with initial data $(\mathbf{v}_0, \mathbf{u}_0)$

-$(\mathbf{v}_\lambda, \mathbf{u}_\lambda)$ converges towards $(\mathbf{v}, \mathbf{u}) \in C([0,T), \mathbb{H}^s)$ as $\lambda \to \infty$ and $\mathcal{V}_\lambda$ converges to $\overline{\mathbf{v}}$ in $\mathbb{L}^1(0,T;\mathbb{H}^s)$ as $\lambda \to \infty$.

Let us come back to our case and take $\mathbf{W} = \mathbf{U}$ i.e. $(\mathbf{v}, \mathbf{u}, \mathcal{V})$, we have again $q = r + d$, the system is given by (32) with $\mathbf{A}(\mathbf{U}) = \mathbf{F}'(\mathbf{U})$, $\mathbf{Q}(\mathbf{W}) = \mathbf{G}(\mathbf{U}) = (0, 0, \mathbf{v} - \mathcal{V})$ and $\mathbf{G}'(\mathbf{U}) = \mathbf{G}'$ given in block form by (56). Following the items of the general framework listed above:
1. Let $S(\mathbf{W}_e) = I_r$, $J_S = J$ and $P(\mathbf{U})$ be defined by (57) we have indeed $P(\mathbf{U}_{eq})\mathbf{G}' = -JP(\mathbf{U}_{eq})$.
2. We take $A_0 = \frac{1}{2}S_0$ defined by (58). We assume (H2) is satisfied for all $\mathcal{V}$ in a compact neighborhood and use the result of lemma 3.2 which yields that $A_0(\mathbf{U})A(\mathbf{U})$ is symmetric.

3. We have computed in lemma 3.3 $\frac{1}{2}\big(S_0(\mathbf{U}_{eq})\mathbf{G}'(\mathbf{U}_{eq}) + \mathbf{G}'(\mathbf{U}_{eq})^T S_0(\mathbf{U}_{eq})\big)$. We now compute $-P^T(\mathbf{U_e})JP(\mathbf{U_e}) = P^T P(\mathbf{U_e})\mathbf{G}'$

$$P^T P(\mathbf{U_e})\mathbf{G}' = \begin{pmatrix} -\Omega^T\Omega & 0 & \Omega^T\Omega \\ 0 & 0 & 0 \\ \Omega^T\Omega & 0 & -\Omega^T\Omega \end{pmatrix}.$$

The condition is thus

$$-(\theta - \epsilon)''(\mathbf{v}) \leq -\Omega^T\Omega \;\Leftrightarrow\; \Omega^T\Omega \leq (\theta - \epsilon)''(\mathbf{v}). \tag{64}$$

Let $\Omega(\mathbf{U}) = \Omega(\mathcal{V})$ be the square root of $(\epsilon'' - \theta'')(\mathcal{V})$ as in (62), then we have equality in (64) and assumption 3 is satisfied.

Now, still following the approach in [39], we have to consider the system of ordinary differential equations characterizing the *limiting inner problem* which reads

$$\frac{d\mathbf{I}}{ds} = \mathbf{G}(\mathbf{I}),\ \mathbf{I}(x,0) = \mathbf{U}_0^0(x)$$

where $\mathbf{U}_0^\lambda(x)$ is the initial condition for (63). This system is very simple in our case and has an explicit global solution. If $\mathbf{U}_0^0(x) = (\mathbf{v}_0(x), \mathbf{u}_0(x), \mathcal{V}_0(x))$, then $\mathbf{I}(x,s) = (\mathbf{v}_0(x), \mathbf{u}_0(x), \mathcal{V}(x,s))$ with

$$\mathcal{V}(x,s) = \exp(-s)\mathcal{V}_0(x) + (1 - \exp(-s))\mathbf{v}_0(x).$$

As $s \to \infty$, $\mathbf{I}(x,s)$ converges exponentially to some limit which is indeed, as required, an equilibrium state $\mathbf{U}_{eq}(x) = (\mathbf{v}_0(x), \mathbf{u}_0(x), \mathbf{v}_0(x))$. Thus we obtain the existence of a solution $\mathbf{U}^\lambda$ to (63) associated to $\mathbf{U}_0^\lambda(x)$. The asymptotic expansion wrt. λ of $\mathbf{U}^\lambda$ has a zero order term $\mathbf{U}$ which is solution of the reduced system

$$\begin{cases} \mathbf{G}(\mathbf{U}) = \mathbf{0} \\ P^I\big(\partial_t \mathbf{U} + \mathbf{A}(\mathbf{U})\partial_x(\mathbf{U})\big) = \mathbf{0} \\ \mathbf{U}(x,0) = \mathbf{U}_{eq}(x) \end{cases} \tag{65}$$

where P^I denotes the matrix consisting of the q (with our notations $q = d + r$) first rows of P, this is the relaxed or equilibrium system.

In our case, P^I is the identity matrix and (65) gives a solution of (31) associated to the initial condition $(\mathbf{v}_0(x), \mathbf{u}_0(x))$. The leading term (in λ^{-1}) of the *initial layer correction* is $\mathbf{I}(x, \lambda t) - \mathbf{U}_{eq}(x) = (\mathbf{0}, \mathbf{0}, \exp(-\lambda t)(\mathcal{V}_0(x) - \mathbf{v}_0(x))$. It vanishes if the initial data is at equilibrium.

Remark 7. As far as assumption 2 is concerned, we may think of another symmetrizer for our system. Recall (33)

$$\mathcal{W} = \mathcal{W}(\mathbf{v}, \mathcal{V}) = (\epsilon - \theta)'(\mathcal{V}) + \theta'(\mathbf{v}),$$

thus

$$\partial_t \mathcal{W} = (\epsilon - \theta)''(\mathcal{V})\partial_t \mathcal{V} + \theta''(\mathbf{v})\partial_t \mathbf{v} = \theta''(\mathbf{v}) N \partial_x \mathbf{u}.$$

Choosing the set of variables $\mathbf{W} = (\mathbf{u}, \mathcal{W}, \mathcal{V})$, we may consider the relaxation system

$$\begin{cases} \partial_t \mathbf{u} - \partial_x N^T \mathcal{W} = \mathbf{0} \\ \partial_t \mathcal{W} - \theta''(\mathbf{v}) N \partial_x \mathbf{u} = \lambda(\epsilon - \theta)''(\mathcal{V})(\mathbf{v} - \mathcal{V}) \\ \partial_t \mathcal{V} = \lambda(\mathbf{v} - \mathcal{V}). \end{cases} \tag{66}$$

This corresponds to the choice of the equation for Π (relaxation pressure) in the case of Euler system. It is easy to prove that the above system (66) without source term is symmetrizable. However it happens that though system (66) in $(\mathbf{u}, \mathcal{W}, \mathcal{V})$ seems at first glance very simple because (almost) already in symmetric form, the computations for checking the structural properties are not straightforward. □

4 The numerical approximation

We come to the numerical approximation procedure for the fluid system, which may be described by three steps: a reconstruction step (the *equilibrium mapping*), an evolution step involving explicit solutions of Riemann problems for the relaxation system, and a projection step on the *equilibrium manifold*, this last step needs a little tricky complement for the full system with energy, however, the resulting numerical scheme is indeed very simple. The way it is built enables us to obtain good stability properties. We describe the approach and give the results in the much simpler Lagrangian framework; thanks to the equivalence results between the two frameworks, the results can then be directly stated for the fluid system in Eulerian coordinates.

4.1 The Riemann problem

In the numerical procedure, we need to solve the Riemann problems for (53) i.e. for (32) with $\lambda = 0$ where $\mathcal{W}$ is given by (33)

$$\mathcal{W} = \mathcal{W}(\mathbf{v}, \mathcal{V}) = \epsilon'(\mathcal{V}) + \theta'(\mathbf{v}) - \theta'(\mathcal{V}).$$

Recall that the d eigenvalues of the matrix (9), which we assume to be non null, give the d pairs of opposite non null eigenvalues of the system (31). With the notations (46), the hypothesis (H) is satisfied if and only if $\Theta - \mathcal{E}$ is positive definite thus the eigenvalues of $\Theta(\mathbf{v}) = N^T \theta''(\mathbf{v}) N$ are also non null.

Lemma 4.1. *Assume that θ'' is constant. Then all the characteristic fields of system (53) are linearly degenerate.*

Proof. We assume θ'' is constant and thus Θ is constant too. All solutions of the system (53) satisfy also

$$\partial_t N^T \mathcal{W} - \Theta \partial_x \mathbf{u} = \mathbf{0}. \tag{67}$$

If we consider this equation in place of the first equation in (53), since all the matrices are constant, the resulting system is linear. The mapping $\mathbf{v} \to \mathcal{W}(\mathbf{v}, \mathcal{V}) = \epsilon'(\mathcal{V}) + \theta'(\mathbf{v}) - \theta'(\mathcal{V})$ is linear (θ'' is constant) and invertible from the set of state variables ($\mathbf{v}$ is a state variable) on its range since $\mathcal{W}_{\mathbf{v}} = \theta''$ is positive definite. Hence $(N^T\mathcal{W}, \mathbf{u}, \mathcal{V})$ defines an admissible change of variables, and thus all the fields of system (53) are also linearly degenerate. □

From now on, we assume that θ'' is *constant*. Since the wave curves are integral curves of the field of eigenvectors, these may be found by writing the Rankine–Hugoniot relations across a discontinuity propagating with speed σ, which gives

$$\begin{cases} -\sigma[\mathbf{v}] - N[\mathbf{u}] = \mathbf{0}, \\ -\sigma[\mathbf{u}] - N^T[\mathcal{W}] = \mathbf{0}, \\ -\sigma[\mathcal{V}] = 0. \end{cases} \tag{68}$$

We then also have from (67)

$$-\sigma N^T[\mathcal{W}] - \Theta[\mathbf{u}] = \mathbf{0}.$$

In the above formulas, σ may be either 0 and then $\mathbf{u}$ and $N^T\mathcal{W}$ are continuous, or a non zero eigenvalue and $\mathcal{V}$ is continuous. We assume that the eigenvalue zero has multiplicity r exactly (equivalently we assume $d = r$ since we have seen in section 1.1 that for the equilibrium system, the multiplicity is $r - d$ in the isentropic case), associated to the last equation $\partial_t \mathcal{V} = 0$. The quantities $\mathbf{u}, N^T\mathcal{W}$ are Riemann invariants associated to 0, $\mathcal{V}$ is a strong Riemann invariant associated to all $\sigma \neq 0$. From the equations

$$\begin{cases} -\sigma[\mathbf{u}] - N^T[\mathcal{W}] = \mathbf{0}, \\ -\sigma N^T[\mathcal{W}] - \Theta[\mathbf{u}] = \mathbf{0}, \end{cases}$$

we get

$$\sigma^2[\mathbf{u}] = -\sigma N^T[\mathcal{W}] = \Theta[\mathbf{u}],$$

hence $[\mathbf{u}]$ appears as an eigenvector of Θ associated to σ^2, then the first equation in (68) will give $[\mathbf{v}]$ across $x = \sigma t$. The matrix Θ is positive definite and has d positive eigenvalues $\sigma_i^2 > 0$, an orthonormal basis of eigenvectors, say $\mathbf{s}_i, 1 \leq i \leq d$. The linear system

$$\begin{cases} \partial_t N^T\mathcal{W} - \Theta\partial_x \mathbf{u} = \mathbf{0} \\ \partial_t \mathbf{u} - N^T\partial_x \mathcal{W} = \mathbf{0}, \end{cases} \tag{69}$$

with matrix

$$\begin{pmatrix} 0 & -\Theta \\ -Id & 0 \end{pmatrix}$$

is diagonalizable in a basis of the form $(\pm\sigma_i\mathbf{s}_i, -\mathbf{s}_i), 1 \leq i \leq d$. For the full system (53) or equivalently (69) completed with $\partial_t \mathcal{V} = \mathbf{0}$,

$$\begin{cases} \partial_t N^T\mathcal{W} - \Theta\partial_x \mathbf{u} = \mathbf{0} \\ \partial_t \mathbf{u} - N^T\partial_x \mathcal{W} = \mathbf{0} \\ \partial_t \mathcal{V} = \mathbf{0} \end{cases} \tag{70}$$

setting $\mathbf{V} = (N^T\mathcal{W}, \mathbf{u}, \mathcal{V})$, we have, with this choice of dependent variables, the Jacobian matrix

$$\begin{pmatrix} 0 & -\Theta & 0 \\ -Id & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with eigenvectors $(\pm\sigma_i\mathbf{s}_i, -\mathbf{s}_i, 0), 1 \leq i \leq d$, each associated to an eigenvalue $\pm\sigma_i$. And, associated to 0, are the vectors $\mathbf{r}_i = (0, 0, e_i), 1 \leq i \leq r$, with $e_i = (0, .., 1, 0, .., 0)^T$ and 1 in ith position. We

call all the eigenvectors $\mathbf{R}_i$ (they form a basis) and let $\mathbf{l}_i$ be a dual basis. For a given initial condition $\mathbf{V}_0 = (N^T\mathcal{W}_0, \mathbf{u}_0, \mathcal{V}_0)$, the explicit solution of the Cauchy problem is given by

$$\mathbf{V}(x,t) = \sum_i (\mathbf{l}_i, \mathbf{V}_0)(x - \sigma_i t)\mathbf{R}_i.$$

We can define $\mathbf{v}$ from $\mathcal{W}$, and thus $\mathbf{U}(x,t)$ from $\mathbf{V}(x,t)$.

Practically, the Riemann problem will be solved for given left and right states at equilibrium, which defines immediately $\mathcal{V}$ everywhere since it jumps only across the field $\mu = 0$; and for the determination of the $d+r$ remaining components of the $2r$ intermediate states, we use that both $N\mathbf{u}+\sigma\mathbf{v}$ and $\sigma\mathbf{u}+N^T\mathcal{W}$ are continuous across the wave of speed σ (it gives $(r+d)\times 2r$ equations minus $2d$ equations for the continuity of the σ_i's), and naturally the fact that $\mathbf{u}$ and $N^T\mathcal{W}$ are continuous across the field $\mu = 0$ ($2d$ equations).

On the specific example of Euler system, the computations are indeed simple.

Example: Euler system (revisited). We consider system (37) with (38), and recall the notations $\mathcal{V} \equiv \mathcal{T}, \mathcal{W} \equiv -\Pi$, and $\theta'' = a^2$, satisfying Whitham's condition $a^2 > \max(-\tilde{p}'(s))$ for all states under consideration. The eigenvalues are $\mu_1 = -a < \mu_2 = 0 < \mu_3 = a$. We note $\mathbf{U} = (\tau, u, \mathcal{T})$ and assume the data at equilibrium $\mathcal{T}_{\ell,r} = \tau_{\ell,r}$.

Proposition 4.2. *Given two constant states $\mathbf{U}_\ell, \mathbf{U}_r$, the solution of the Riemann problem for (37), (38) consists of three contact discontinuities, each propagating with a characteristic speed $\mu_i, i = 1,2,3$, separating $\mathbf{U}_\ell$, two intermediate states $\mathbf{U}_\ell^*, \mathbf{U}_r^*$ and $\mathbf{U}_r$; the states $\mathbf{U}_\ell^*, \mathbf{U}_r^*$ are respectively characterized by $(u^*, \Pi^*; \mathcal{T}_\ell, \tau_\ell^*)$ and $(u^*, \Pi^*; \mathcal{T}_r, \tau_r^*)$ with*

$$\begin{aligned}
u^* &= \frac{u_\ell + u_r}{2} - \frac{\Pi_r - \Pi_\ell}{2a},\\
\Pi^* &= \frac{\Pi_\ell + \Pi_r}{2} - \frac{a}{2}(u_r - u_\ell),\\
\tau_\ell^* &= \tau_\ell - \frac{1}{2a^2}(\Pi_r - \Pi_\ell - a(u_r - u_\ell)),\\
\tau_r^* &= \tau_r + \frac{1}{2a^2}(\Pi_r - \Pi_\ell + a(u_r - u_\ell)).
\end{aligned} \tag{71}$$

We assume with Whitham's condition that a is large enough so that all intermediate states in the solution of the Riemann problem have a finite density ($\tau > 0$). Now, the fact that the intermediate states belong to Ω_τ, i.e., $\tau_\ell^* > 0, \tau_r^* > 0$ is equivalent to the ordering of the three waves in the solution of the Riemann problem for the Eulerian formulation

$$\mu_1(\mathbf{U}_\ell) = u_\ell - a\tau_\ell < u^* < \mu_3(\mathbf{U}_r) = u_r + a\tau_r,$$

since

$$u^* - \mu_1(\mathbf{U}_\ell) = a\tau_\ell^*,\ \mu_3(\mathbf{U}_r) - u^* = a\tau_r^*.$$

4.2 The relaxation solver

In the global numerical procedure for approximating the solutions of (7), we will need assumption (**H2**). We use an operator splitting method and solve in a first step (70) on a time interval of length Δt, then in a second step, we solve a system of ordinary differential equations

$$\begin{cases} \partial_t \mathbf{v} = \mathbf{0} \\ \partial_t \mathbf{u} = \mathbf{0} \\ \partial_t \mathcal{V} = \lambda(\mathbf{v} - \mathcal{V}) \end{cases} \tag{72}$$

with $\lambda \to \infty$ (instantaneous relaxation) which can be understood as the projection on the equilibrium manifold consisting of the states satisfying $\mathcal{V} = \mathbf{v}$.

More precisely, in the first step, we solve (70) with given piecewise constant initial data $\mathbf{U}_\Delta(x, t_n)$ at equilibrium (equivalently for given $(\mathbf{v}_j^n, \mathbf{u}_j^n, \mathcal{V}_j^n = \mathbf{v}_j^n), j \in \mathbb{Z}$) and for $t \in]t_n, t_n + \Delta t]$. It gives for $t = t_{n+1}-$, the function $\mathbf{U}(x, t_{n+1}^-) = (\mathbf{v}, \mathbf{u}, \mathcal{V})(x, t_{n+1}^-)$. Hence after the second step, and pointwise projection on the equilibrium manifold, we have $\mathbf{v}(x, t_{n+1}) = \mathbf{v}(x, t_{n+1}^-)$, $\mathbf{u}(x, t_{n+1}) = \mathbf{u}(x, t_{n+1}^-)$, and $\mathcal{V}(x, t_{n+1}) = \mathbf{v}(x, t_{n+1})$, then $(\mathbf{v}_\Delta, \mathbf{u}_\Delta, \mathcal{V}_\Delta)(x, t_{n+1})$ after the usual L^2 projection on piecewise constant functions.

Recall that Σ is not convex on the entire set of states. However, for proving the entropy dissipation, notice that since the homogeneous system is totally linearly degenerate, its weak solutions also satisfy the energy conservation

$$\partial_t \Sigma - \partial_x(\mathbf{u}, N^T \mathcal{W}) = 0.$$

Following the usual approach to compute the flux of Godunov's method, we may integrate this last equation on a cell $C_i \times [t_n, t_{n+1}[$ and compute

$$\frac{1}{\Delta x}\int_{C_i} \Sigma(t_{n+1}^-)dx - \frac{1}{\Delta x}\int_{C_i} \Sigma(t_n)dx - \frac{\Delta t}{\Delta x}\big((\mathbf{u}, N^T\mathcal{W})_{i+1/2}) - (\mathbf{u}, N^T\mathcal{W})_{i-1/2}\big) = 0 \qquad (73)$$

with Σ defined in (39)

$$\Sigma(\mathbf{v}, \mathbf{u}, \mathcal{V}) = \frac{1}{2}|\mathbf{u}|^2 + \theta(\mathbf{v}) + \epsilon(\mathcal{V}) - \theta(\mathcal{V}) + ((\epsilon' - \theta')(\mathcal{V}), \mathbf{v} - \mathcal{V}),$$

and the notation $i + 1/2$ denotes the Godunov flux at interface $x = x_{i+1/2}$. The data at time t_n is at equilibrium, thus $\Sigma(t_n) = (\frac{1}{2}|\mathbf{u}|^2 + \epsilon)(t_n) = e(t_n)$. In the first step, $\mathcal{V}$ is kept constant in time, say $\mathcal{V} = \mathbf{v}(t_n)$ which we note for short $\mathbf{v}_0$ in the following lines, hence we have

$\Sigma(t_{n+1}^-) = (\frac{1}{2}|\mathbf{u}|^2 + \theta(\mathbf{v}))(t_{n+1}) + \epsilon(\mathbf{v}_0) - \theta(\mathbf{v}_0) + ((\epsilon' - \theta')(\mathbf{v}_0), \mathbf{v}_1 - \mathbf{v}_0)$ (with $\mathbf{v}_1 = \mathbf{v}(t_{n+1})$) thus we obtain

$$\begin{aligned}
&\tfrac{1}{\Delta x}\textstyle\int_{C_i}(\tfrac{1}{2}|\mathbf{u}|^2 + \theta(\mathbf{v}))(t_{n+1})dx - \tfrac{1}{\Delta x}\textstyle\int_{C_i}(\tfrac{1}{2}|\mathbf{u}|^2 + \theta(\mathbf{v}))(t_n)dx \\
&-\tfrac{\Delta t}{\Delta x}\big((\mathbf{u}, N^T\mathcal{W})_{i+1/2} - (\mathbf{u}, N^T\mathcal{W})_{i-1/2}\big) = -\tfrac{1}{\Delta x}\textstyle\int_{C_i}((\epsilon' - \theta')(\mathbf{v}_0), \mathbf{v}_1 - \mathbf{v}_0)dx
\end{aligned}$$

which we write

$$\begin{aligned}
&\tfrac{1}{\Delta x}\textstyle\int_{C_i}(\tfrac{1}{2}|\mathbf{u}|^2 + \epsilon(\mathbf{v}))(t_{n+1})dx \\
&-\tfrac{1}{\Delta x}\textstyle\int_{C_i}(\tfrac{1}{2}|\mathbf{u}|^2 + \epsilon(\mathbf{v}))(t_n)dx - \tfrac{\Delta t}{\Delta x}\big((\mathbf{u}, N^T\mathcal{W})_{i+1/2} - (\mathbf{u}, N^T\mathcal{W}_{i-1/2}\big) \\
&= \tfrac{1}{\Delta x}\textstyle\int_{C_i}\big(\epsilon(\mathbf{v}) - \epsilon(\mathbf{v}_0) - (\theta(\mathbf{v}) - \theta(\mathbf{v}_0)) - ((\epsilon' - \theta')(\mathbf{v}_0), \mathbf{v}_1 - \mathbf{v}_0)\big)dx,
\end{aligned}$$

and the right hand side is negative under the hypothesis (H) since

$$(\epsilon - \theta)(\mathbf{v}) - (\epsilon - \theta)(\mathbf{v}_0) - ((\epsilon' - \theta')(\mathbf{v}_0), \mathbf{v} - \mathbf{v}_0) = (\int_0^1 (\epsilon'' - \theta'')(\mathbf{v}_0 + s(\mathbf{v} - \mathbf{v}_0))(\mathbf{v} - \mathbf{v}_0)ds, \mathbf{v} - \mathbf{v}_0).$$

Since after the pointwise relaxation step, the state is again at equilibrium, we get by lemma 2.3 that the energy satisfies $e(t_{n+1}) \le \Sigma(t_{n+1-})$ which yields an 'entropy inequality' for the numerical scheme

$$\frac{1}{\Delta x}\int_{C_i} e(t_{n+1})dx - \frac{1}{\Delta x}\int_{C_i} e(t_n)dx - \frac{\Delta t}{\Delta x}\big((\mathbf{u}, N^T\mathcal{W})_{i+1/2}) - (\mathbf{u}, N^T\mathcal{W})_{i-1/2}\big) \le 0.$$

4.3 The relaxation solver for the full system

In fact we want an entropy scheme for the original system (1) with energy. The extension is rather straightforward, but for the exchange between the entropy and energy equations. Let us give some details to complete the previous isentropic case.

4.3.1 The relaxation system for the fluid system with energy

We use the entropy formulation (7), which we write again for convenience

$$\left\{\begin{array}{l}\partial_t \mathbf{v} - N\partial_x \mathbf{u} = \mathbf{0}\\ \partial_t \mathbf{u} - N^T \partial_x \phi(\mathbf{v}, s) = \mathbf{0}\\ \partial_t s = 0\end{array}\right. \tag{74}$$

with $e = \epsilon + \frac{1}{2}|\mathbf{u}|^2$, and $\phi(\mathbf{v}, s) \equiv e_{\mathbf{v}}(\mathbf{v}, s) = \epsilon_{\mathbf{v}}(\mathbf{v}, s)$. We consider the larger *relaxation* system of $2r + d + 1$ equations

$$\left\{\begin{array}{l}\partial_t \mathbf{v} - N\partial_x \mathbf{u} = \mathbf{0}\\ \partial_t \mathbf{u} - N^T \partial_x \mathcal{W} = \mathbf{0}\\ \partial_t s = 0\\ \partial_t \mathcal{V} = \lambda(\mathbf{v} - \mathcal{V})\end{array}\right. \tag{75}$$

with now

$$\mathcal{W} = \mathcal{W}(\mathbf{v}, \mathcal{V}, s) = \phi(\mathcal{V}, s) + \theta'(\mathbf{v}) - \theta'(\mathcal{V}), \tag{76}$$

and θ quadratic. Like (53), the homogeneous system is linearly degenerate, with the same distinct eigenvalues as before, only the multiplicity of the eigenvalue 0 has been increased by 1.

Hypothesis (H) will now write

(H) $\qquad \theta''(\mathcal{V}) - \epsilon_{\mathbf{vv}}(\mathcal{V}, s)$ is positive definite

for all $\mathcal{V}$ under consideration.

For instance, for Euler system with energy, (H) resumes to Whitham's condition (see the analogue of (38)) which writes $a^2 > \max(-\partial_\tau \tilde{p}(\mathcal{T}, s))$.

Define

$$\Sigma(\mathbf{v}, \mathbf{u}, s, \mathcal{V}) = \frac{1}{2}|\mathbf{u}|^2 + \epsilon(\mathcal{V}, s) + \theta(\mathbf{v}) - \theta(\mathcal{V}) + ((\epsilon'(\mathcal{V}, s) - \theta'(\mathcal{V}))(\mathbf{v} - \mathcal{V}). \tag{77}$$

We have the analogue of lemma 2.1: smooth solutions of (69) satisfy

$$\partial_t \Sigma - \partial_x(\mathbf{u}, N^T \mathcal{W}) = \lambda\Big(\big(\epsilon_{\mathbf{vv}}(\mathcal{V}, s) - \theta''(\mathcal{V})\big)(\mathbf{v} - \mathcal{V}), \mathbf{v} - \mathcal{V}\Big), \tag{78}$$

and thus weak solutions of (75) for $\lambda = 0$ also satisfy the energy conservation

$$\partial_t \Sigma - \partial_x(\mathbf{u}, N^T \mathcal{W}) = 0.$$

4.3.2 The global relaxation solver

We now define the resulting global relaxation scheme, using again Godunov's scheme for the relaxation system. In this section, we will note $\mathbf{V}$ a full state $\mathbf{V} = (\mathbf{v}, \mathbf{u}, s, \mathcal{V})$ and $\mathbf{U} = (\mathbf{v}, \mathbf{u}, e)$ a state for the fluid model. We also define the operators $\mathcal{M}(\mathbf{U}) = (\mathbf{v}, \mathbf{u}, s, \mathbf{v})$ where s is defined by $s = s(\mathbf{U})$ and $\mathcal{P}(\mathbf{V}) = (\mathbf{v}, \mathbf{u})$ which are rather natural extensions of the operators defined in section 2.1 in the isentropic case). For any quantity φ_0, we set $\varphi_j^0 = \frac{1}{\Delta x}\int_{C_j} \varphi_0(x)dx$.

Given an initial data $\mathbf{U}_0 = (\mathbf{v}_0, \mathbf{u}_0, e_0)$ for (74), define $s_0 = s(\mathbf{U}_0)$ and $\mathbf{U}_j^0$. Then at time t_n, from $\mathbf{U}_j^n = (\mathbf{v}, \mathbf{u}, e)_j^n$, we want to define the update state $\mathbf{U}_j^{n+1}$, which we do following some steps.
- reconstruction step 1: define for $j \in \mathbb{Z}$, $\mathbf{V}_j^n = \mathcal{M}(\mathbf{U}_j^n) = (\mathbf{v}, \mathbf{u}, s, \mathbf{v})_j^n$ which is a state at equilibrium, and associate a piecewise constant data.
- evolution step 2: solve (under CFL $1/2$) a noninteracting juxtaposition of Riemann problems of the homogeneous linearly degenerate system (75) to compute $\mathbf{W}_R(x/t; \mathbf{V}_j^n, \mathbf{V}_{j+1}^n)$. Note $\mathbf{V}(x, t_{n+1}-)$ the value at t_{n+1} of the solution obtained after this evolution step; this state is not at equilibrium.
- pointwise relaxation step 3: $\mathbf{W}_R^{eq} = \mathcal{M}(\mathbf{W}_R)$
- projection step 4: define the updated piecewise constant value as usual for a Godunov solver by integrating the solution on a cell

$$\mathbf{V}_j^{n+1-} = \frac{1}{\Delta x}\Big(\int_{-\Delta x/2}^{0} \mathbf{W}_R^{eq}(\frac{x}{\Delta t}; \mathbf{V}_j^n, \mathbf{V}_{j+1}^n)dx + \int_{0}^{\Delta x/2} \mathbf{W}_R^{eq}(\frac{x}{\Delta t}; \mathbf{V}_{j-1}^n, \mathbf{V}_j^n)dx\Big),$$

keep the two first block-components to define $(\mathbf{v}, \mathbf{u})_j^{n+1} \equiv \mathcal{P}(\mathbf{V}_j^{n+1-})$ which provides a scheme consistent with the equations of conservation of $\mathbf{v}$ and $\mathbf{u}$, with numerical flux at the interface $x = x_{j+1/2}$ noted $\mathcal{G}_{\mathbf{v},j+1/2}^n$ and $\mathcal{G}_{\mathbf{u},j+1/2}^n$, giving by the scheme

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \mu\big(\mathcal{G}_{\mathbf{u},j+1/2}^n - \mathcal{G}_{\mathbf{u},j-1/2}^n\big), \tag{79}$$

and a similar equation for $\mathbf{v}$. The third one gives an equation for the entropy, with numerical flux denoted by $\mathcal{G}_{s,j+1/2}^n$:

$$s_j^{n+1-} = s_j^n - \mu(\mathcal{G}_{s,j+1/2}^n - \mathcal{G}_{s,j-1/2}^n), \tag{80}$$

however, s_j^{n+1-} is not the updated entropy, and we define s_j^{n+1} below. In fact the entropy flux in Lagrangian coordinates is trivial, $\mathcal{G}_s = 0$, we have kept a general notation, just for generality.
- equilibrium energy step 5: in this step, we want to recover a state $\mathbf{U}_j^{n+1}$ by projection on the equilibrium manifold; a careful analysis as in [19] enables us to reverse the role of entropy conservation and energy inequality and turn back to a scheme for defining $\mathbf{U}_j^{n+1}$, with an equation for the energy conservation, which moreover satisfies an entropy inequality. We thus assert that the resulting global Godunov relaxation solver is conservative in $(\mathbf{v}, \mathbf{u}, e)$:

Proposition 4.3. *Under CFL 1/2, the resulting global relaxation scheme can be written*

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n - \mu\left(\mathbf{G}_{j+1/2}^n - \mathbf{G}_{j-1/2}^n\right), \quad j \in \mathbb{Z}, n \geq 0, \tag{81}$$

with the three components of the flux $\mathbf{G}_{j+1/2}^n$ given by $(-N\mathbf{u}, -N^T\mathcal{W}, (\mathbf{u}, N^T\mathcal{W}))(\mathbf{W}_{j+1/2}^n)$ (i.e. evaluated on state $\mathbf{W}_{j+1/2}^n$) where

$$\mathbf{W}_{j+1/2}^n = \mathbf{W}_R(0+; \mathcal{M}(\mathbf{U}_j^n), \mathcal{M}(\mathbf{U}_{j+1}^n)). \tag{82}$$

and $\mathbf{W}_R$ denotes the solution of the Riemann problem for (75). Moreover, the global relaxation solver satisfies a discrete entropy inequality

$$s_j^{n+1} \leq s_j^n - \mu(\mathcal{G}_{s,j+1/2}^n - \mathcal{G}_{s,j-1/2}^n), \quad j \in \mathbb{Z}. \tag{83}$$

Note that in (83), s_j^{n+1} is defined by $s(\mathbf{U}_j^{n+1})$ and not by the third component s_j^{n+1-} of the updated value after the projection step 4.

Proof. We have to define the updated energy. Since the system is linearly degenerate, the solution $(\mathbf{v}, \mathbf{u}, s, \mathcal{V})$ of the evolution step 2 defined above, also satisfies the entropy conservation law (78) (for $\lambda = 0$)

$$\partial_t \Sigma - \partial_x(\mathbf{u}, N^T \mathcal{W}) = 0. \tag{84}$$

Then, we know that for Godunov's scheme, the formula for the updated value can be obtained by integrating this equation on a mesh cell $C_j \times (t_n, t_{n+1})$ under CFL 1/2. We have, whatever the flux function say $\mathcal{G}$

$$\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \mathcal{G}(\mathbf{W}_R(0+))dt = \mathcal{G}(\mathbf{W}_R(0+)).$$

Now, $\mathbf{W}_R(0+; \mathbf{V}_j^n, \mathbf{V}_{j+1}^n)$ is such that $\mathcal{G}(\mathbf{W}_R(0+))$ is continuous even if $\mathbf{W}_R$ is discontinuous at 0, so that we define for the energy flux function $\mathcal{G}_\Sigma(\mathbf{V}) = (\mathbf{u}, N^T\mathcal{W})$ the energy numerical flux by (see again (84)

$$\mathcal{G}_{\Sigma,j+1/2}^n \equiv \mathcal{G}_\Sigma(\mathbf{W}(0\pm; \mathbf{V}_j^n, \mathbf{V}_{j+1}^n)). \tag{85}$$

where recall that $\mathbf{V}_j^n = \mathcal{M}(\mathbf{U}_j^n)$. Thus, integrating the energy conservation law (84), and using the relation $\frac{1}{\Delta x}\int_{C_j} \Sigma(x, t_n)dx = \Sigma(\mathbf{U}_j^n) \equiv e_j^n$, since the state $\mathbf{U}_j^n$ is at equilibrium, we get after the evolution step 2

$$\frac{1}{\Delta x} \int_{C_j} \Sigma(\mathbf{V}(x, t_{n+1}-))dx = e_j^n - \mu\big(\mathcal{G}_{\Sigma,j+1/2}^n - \mathcal{G}_{\Sigma,j-1/2}^n\big). \tag{86}$$

Define the final updated energy e_j^{n+1} by the right-hand side of (86)

$$e_j^{n+1} = e_j^n - \mu\big(\mathcal{G}_{\Sigma,j+1/2}^n - \mathcal{G}_{\Sigma,j-1/2}^n\big). \tag{87}$$

We note that the discrete energy flux is consistent with the continuous one. Indeed, if $\mathbf{V}_j^n = \mathcal{M}(\mathbf{U})$, for some state $\mathbf{U}$, we have

$$\mathcal{G}_\Sigma(\mathbf{W}_R(0+; \mathcal{M}(\mathbf{U}), \mathcal{M}(\mathbf{U})) = \mathcal{G}_\Sigma(\mathcal{M}(\mathbf{U})) = (\mathbf{u}, N^T\phi(\mathbf{U}, s)).$$

Now, we want to prove some discrete entropy inequalities. As already emphasized, Σ is not convex and we cannot directly use Jensen's inequality, we must use a refined argument. Note $\mathbf{V}(x, t_{n+1}-)$ the value at t_{n+1} of the solution obtained after the evolution step. For the equilibrium projection step, assume for a while that the projection on the equilibrium manifold is done pointwise in x, i.e., define $\mathbf{V}(x, t_{n+1}^-)$ by

$$\left\{\begin{array}{l} \mathbf{v}(x, t_{n+1}^-) = \mathbf{v}(x, t_{n+1}-), \\ \mathbf{u}(x, t_{n+1}^-) = \mathbf{u}(x, t_{n+1}-), \\ s(x, t_{n+1}^-) = s(x, t_{n+1}-), \\ \mathcal{V}(x, t_{n+1}^-) = \mathbf{v}(x, t_{n+1}-). \end{array}\right. \tag{88}$$

We use the minimization principle of lemma 2.3 which says that the maximal dissipation of entropy is attained for equilibrium states (K is a compact set such that it contains all possible values of $\mathcal{V}$ obtained at the evolution step 2)

$$\Sigma(\mathbf{V}^{eq}) = \min_{\mathcal{V}\in K} \Sigma(\mathbf{V}),\ \mathbf{V}^{eq} = (\mathbf{v}, \mathbf{u}, s, \mathbf{v}).$$

After the pointwise relaxation, we have $\mathbf{V}^{eq} = \mathbf{V}(x, t_{n+1}^-)$, and we deduce from the analogue of lemma 2.3,

$$\frac{1}{\Delta x} \int_{C_j} \Sigma(\mathbf{V}(x, t_{n+1}^-))dx \le \frac{1}{\Delta x} \int_{C_j} \Sigma(\mathbf{V}(x, t_{n+1}-))dx = e_j^{n+1}, \tag{89}$$

the last equation coming from the definition (87) of e_j^{n+1}. Now, we also have for a state at equilibrium

$$\Sigma(\mathbf{V}(x,t_{n+1}^-)) = e(\mathbf{V}(x,t_{n+1}^-)). \tag{90}$$

We notice that the three first components $(\mathbf{u},\mathbf{v},s)$ of $\mathbf{U}(x,t_{n+1}-)$ are kept unchanged during the equilibrium projection step, whether pointwise (88) or globally defined, $\mathbf{u}_j^{n+1} = \frac{1}{\Delta x}\int_{C_j}\mathbf{u}(x,t_{n+1}-)dx$ is given by the scheme (79), similarly for $\mathbf{v}_j^{n+1}$. Now, since the function $(\mathbf{u},\mathbf{v},s) \to e$ is convex and $e(x,t_{n+1}-) = e(\mathbf{u},\mathbf{v},s)(x,t_{n+1}-)$ we can apply Jensen's inequality and get, together with (89)(90)

$$e_j^{n+1-} \equiv e(\mathbf{u}_j^{n+1},\mathbf{v}_j^{n+1},s_j^{n+1-}) \leq \frac{1}{\Delta x}\int_{C_j} e(x,t_{n+1}-)dx \leq e_j^{n+1}.$$

Recall that we assumed at the beginning that $s_e = \partial_e s < 0$, where $e = \varepsilon(\mathbf{v},s) + |\mathbf{u}|^2/2$, we get that s is decreasing wrt. the variable e. Then having defined

$$s_j^{n+1} = s(\mathbf{U}_j^{n+1}) = s(\mathbf{v}_j^{n+1},\mathbf{u}_j^{n+1},e_j^{n+1})$$

which is the definition of the entropy of state $\mathbf{U}_j^{n+1}$, we deduce since $e_j^{n+1-} \leq e_j^{n+1}$

$$s_j^{n+1} \leq s(\mathbf{v}_j^{n+1},\mathbf{u}_j^{n+1},e_j^{n+1-}) = s_j^{n+1-}$$

and with (80)

$$s_j^{n+1} \leq s_j^n - \mu\big(\mathcal{G}_{\varrho s,j+1/2}^n - \mathcal{G}_{\varrho s,j-1/2}^n\big),$$

which ends the proof. □

We have moreover the formal Lax-Wendroff-type convergence result, where $\mathbf{U}_\Delta$ denotes the piecewise constant function associated in a classical way to scheme (81), and $\mathbf{U}_0$ is the given initial data.

Proposition 4.4. *Assume (**H2**). Assume moreover that $\mathbf{U}_\Delta$ is bounded in $\mathbb{L}^\infty$ and that the scheme converges in the sense that $\mathbf{U}_\Delta \to \mathbf{U}$ in $\mathbb{L}^1_{loc}(\mathbb{R}+;\mathbb{L}^1_{loc}(\mathbb{R}))$ and a.e. Then the limit $\mathbf{U}$ is a weak solution of the conservative fluid system (1) with initial condition $\mathbf{U}_0$.*

Since we have can prove a similar result for the entropy inequality, the limit solutions satisfies the entropy inequality

4.4 Relaxation model in Eulerian coordinates

4.4.1 The Eulerian relaxation model

We now use the results of section 1.2 to write our relaxation model in Eulerian coordinates. We state without proof the analogue of proposition 1.1.

Proposition 4.5. *Let $(\mathbf{v},\mathbf{u},s,\mathcal{V})$ where $\mathbf{v} = (\tau,v_2,...,v_r)^T$, $\mathbf{u} = (u,u_2..,u_d)^T$ be a solution of (75)*

$$\begin{cases} \partial_t\mathbf{v} - N\partial_y\mathbf{u} = \mathbf{0} \\ \partial_t\mathbf{u} - N^T\partial_y\mathcal{W} = \mathbf{0} \\ \partial_t s = 0 \\ \partial_t\mathcal{V} = \lambda(\mathbf{v}-\mathcal{V}) \end{cases}$$

with

$$\mathcal{W} = \mathcal{W}(\mathbf{v},\mathcal{V},s) = \phi(\mathcal{V},s) + \theta'(\mathbf{v}) - \theta'(\mathcal{V}),$$

where N is a rectangular $r \times d$ constant matrix with first line equal to $(1, 0, ...0)$, $\phi : \mathbb{R}^{r+1} \to \mathbb{R}^d$ and $\theta : \mathbb{R}^r \to \mathbb{R}$ are smooth mappings. Assume that $\tau > 0$. Defining $(\overline{U}, \underline{U}) \in \mathbb{R}^r \times \mathbb{R}^d$ by

$$\overline{U}_1 = \varrho \equiv \frac{1}{\tau},\ \overline{U}_i = \varrho v_i, i = 2, ...r,\ \underline{U}_1 = \varrho u,\ \underline{U}_i = \varrho u_i, i = 2, ..., d \tag{91}$$

setting $U = (\overline{U}, \underline{U}, , \varrho s, \varrho \mathcal{V}) \in \mathbb{R}^{n+r}$; defining $(\overline{f}(U), \underline{f}(U)) \in \mathbb{R}^r \times \mathbb{R}^d$ by

$$\overline{f}_1(U) = \varrho u,\ \overline{f}_i(U) = -(N\frac{\underline{U}}{\varrho})_i + u\overline{U}_i, i = 2, ..., r,\ \underline{f}_i(U) = -(N^T\check{\mathcal{W}})_i + u\underline{U}_i, i = 1, ..., d \tag{92}$$

where

$$\check{\mathcal{W}} = \check{\mathcal{W}}(\overline{U}, \mathcal{V}, s) = \mathcal{W}(\mathbf{v}, \mathcal{V}, s) = \phi(\mathcal{V}, s) + \theta'(\mathbf{v}) - \theta'(\mathcal{V}),$$

and setting $f(U) = (\overline{f}(U), \underline{f}(U), \varrho su, \varrho u\mathcal{V})$, then U is solution of the system

$$\partial_t U + \partial_x f(U) = 0,$$

which writes equivalently

$$\begin{cases} \partial_t \varrho + \partial_x \varrho u = 0 \\ \partial_t \overline{U}_i + \partial_x(-\frac{1}{\varrho}(N\underline{U})_i + u\overline{U}_i) = 0,\ i = 2, \dots, r, \\ \partial_t \varrho u + \partial_x(\varrho u^2 - (N^T\check{\mathcal{W}})_1 = 0, \\ \partial_t \underline{U}_i + \partial_x(-(N^T\check{\mathcal{W}})_i + u\underline{U}_i) = 0,\ i = 2, \dots, d \\ \partial_t \varrho s + \partial_x \varrho s u = 0 \\ \partial_t \varrho \mathcal{V} + \partial_x \varrho u \mathcal{V} = \lambda \varrho(\mathbf{v} - \mathcal{V}). \end{cases} \tag{93}$$

Note that we have stated the result with a separate treatment of the variable $\mathcal{V}$ for clarity; it might have been incorporated in some augmented $\overline{U}$.

Let us recall the expression of the energy (77) and define $\check{\Sigma}(U) = \Sigma(\mathbf{v}, \mathbf{u}, s, \mathcal{V})$ where U is given in terms of $(\mathbf{v}, \mathbf{u}, s, \mathcal{V})$ in proposition 4.5, which writes

$$\check{\Sigma}(U) = \frac{1}{2}|\mathbf{u}|^2 + \epsilon(\mathcal{V}, s) + \theta(\mathbf{v}) - \theta(\mathcal{V}) + ((\epsilon'(\mathcal{V}, s) - \theta'(\mathcal{V}))(\mathbf{v} - \mathcal{V}). \tag{94}$$

In Eulerian coordinates, smooth solutions of (93) satisfy (dropping the notation $\check{s}$)

$$\partial_t \varrho\Sigma + \partial_x\big(- (\mathbf{u}, N^T\mathcal{W}) + \varrho u\Sigma\big) = \lambda\Big(\big(\epsilon_{\mathbf{vv}}(\mathcal{V}, s) - \theta''(\mathcal{V})\big)(\mathbf{v} - \mathcal{V}), \mathbf{v} - \mathcal{V}\Big),$$

and thus weak solutions of (93) for $\lambda = 0$ also satisfy the energy conservation equation

$$\partial_t \varrho\Sigma + \partial_x\big(- (\mathbf{u}, N^T\mathcal{W}) + \varrho u\Sigma\big) = 0. \tag{95}$$

Now, let us check that linearly degenerate fields in a Lagrangian frame give linearly degenerate fields in Eulerian coordinates.

Lemma 4.6. *All the characteristic fields of the relaxation system (93) are linearly degenerate.*

Proof. The correspondence between eigenvalues can be found in [32]. The particular form of the relaxation system has been detailed above, because it was important to see how the relaxation model was transformed in Eulerian coordinates, it is not necessary for this lemma. Thus we write (93) in the general form (11), and only distinguish, as in (12) the first coordinate, which for convenience we write $u_1 = \varrho$ and similarly, $f_1(U) = \varrho u$. Then we can perform an admissible change of variables in order to write the

Lagrangian system (12) in variables $U = (\varrho, u_2, ...u_n)^T$, it gives (dropping in what follows the notation tilde for functions: $\phi(y,t)$ should read $\tilde{\phi}(y,t)$)

$$\left\{ \begin{array}{l} \frac{1}{\varrho^2}\partial_t \varrho + \partial_y u = 0, \\ -\frac{u_i}{\varrho^2}\partial_t \varrho + \frac{1}{\varrho}\partial_t u_i + \partial_y\big(f_i(U) - uu_i\big) = 0, \quad i = 2, ..., n. \end{array} \right. \tag{96}$$

Then, substituting the term $\partial_t \varrho = -\varrho^2 \partial_y u$ in the second equation, we get

$$\left\{ \begin{array}{l} \partial_t \varrho + \varrho\partial_y \varrho u - \varrho u \partial_y \varrho = 0, \\ \partial_t u_i + \varrho\partial_y f_i(U) - \varrho u \partial_y u_i = 0, \quad i = 2, ..., n. \end{array} \right. \tag{97}$$

This proves that the Jacobian matrix of system (12) (say $G'(V)$, with $V = (\frac{1}{\varrho}, \frac{u_i}{\varrho})^T$) is equivalent to the matrix $B(U)$ of the nonconservative system (97) which writes

$$B(U) = \varrho f'(U) - \varrho u I. \tag{98}$$

Thus the eigenvalues are linked by $\mu_k(V;G) = \varrho\mu_k(U;f) - \varrho u$ where $\mu_k(V;G)$ (resp. $\mu_k(U;f)$) denotes the $k-$th eigenvalue of the Jacobian matrix $G'(V)$ (resp. $f'(U)$). Thus

$$\mu_k(U;f) = \frac{1}{\varrho}(f_1 + \mu_k(V;G)) = u + \frac{1}{\varrho}\mu_k(V;G).$$

Now, for our relaxation system, the eigenvalues $\mu_k(V;G)$ are in fact constant, say $\mu_k(V;G) = \overline{\mu}_k$, and

$$\mu_k(U;f) = \frac{f_1}{\varrho} + \frac{1}{\varrho}\overline{\mu}_k = u + \tau\overline{\mu}_k,$$

so that $\mu_k'(U;f) = \frac{1}{\varrho}f_1'(U) - \frac{1}{\varrho^2}(f_1 + \overline{\mu}_k)(1,0,0,0)^T$, and

$$\mu_k'(U;f).\mathbf{r} = \frac{1}{\varrho}f_1'(U).\mathbf{r} - \frac{1}{\varrho^2}(f_1 + \overline{\mu}_k)r_1$$

whatever the vector $\mathbf{r} = (r_1, r_2, .., r_n)^T \in \mathbb{R}^n$. If we choose $\mathbf{r} = \mathbf{r}_k$, the $k-$th eigenvector of $f'(U)$, we have the identity $f_1'(U).\mathbf{r}_k = \mu_k(U;f)r_{k,1} = \frac{1}{\varrho}(f_1 + \overline{\mu}_k)r_{k,1}$ so that $\mu_k'(U;f) = 0$ which says that the corresponding field is also linearly degenerate for the system in Eulerian coordinates. □

Proposition 4.7. *Given two constant states U_ℓ, U_r, the solution $W_R(x/t; U_\ell, U_r)$ of the Riemann problem for (93) consists of (at most) $2r+1$ contact waves propagating at speed $u \pm \tau\sigma_i, i = 1, ...r$ and u (with multiplicity $d+1$), where the σ_i^2 are the eigenvalues of the matrix $\Theta = N^T\theta''N$; these waves separate constant states which coincide with those of the Riemann problem for (4.5) when expressed in terms of the same variable U.*

Proof. The first statement is a direct consequence of lemma 4.6. If some eigenvalues of Θ coincide, the number of waves will not be $2r+1$. For instance, as we have already noticed in remark 1, if $\theta''(\mathcal{V}) = \Lambda$, for the diagonal matrix $\Lambda = \sigma^2 I$ (leading to the Jin-Xin relaxation model, [26]). We also note from (98) that $B(U)$ and $f'(U)$ have the same eigenvectors, hence the wave curves which are integral curves of the vector field lead to the same computations of the solution of the Riemann problem the intermediate states coincide when both solutions of the Riemann problem, say $w_R(f)$, $w_R(G)$ are expressed in terms of variables U, but one will be a function of $\frac{x}{t}$ the other of $\frac{y}{t}$. It can be written formally, with eigenvalues ranked in increasing order $\overline{\mu}_1 = -\sigma_r < \cdots < \overline{\mu}_r = -\sigma_1 < \overline{\mu}_{r+1} = 0 = \overline{\mu}_{r+d} < \overline{\mu}_{r+d+1} = \sigma_1 < \cdots < \overline{\mu}_{n+r} = \sigma_r$: if $\overline{\mu}_k < \frac{y}{t} < \overline{\mu}_{k+1}$, then $w_R(\frac{x}{t}; U_\ell, U_r; f) = w_R(\frac{y}{t}; U_\ell, U_r, g)$ if $\mu_k(f) < \frac{x}{t} < \mu_{k+1}(f)$. □

We illustrate the result on Euler system.

Example: Euler system (revisited).

For the full Euler system, from the Lagrangian computations and the quasi decoupling of the entropy variable seen in section 4.3, we deduce from (93) that the relaxation system writes

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) & = 0 \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + \Pi) & = 0 \\ \partial_t \varrho s + \partial_x(\varrho s u) & = 0 \\ \partial_t(\varrho \mathcal{T}) + \partial_x(\varrho \mathcal{T} u) & = \lambda \varrho(\tau - \mathcal{T}) \end{cases} \tag{99}$$

with (see (76))

$$\Pi = \tilde{\Pi}(\tau, s, \mathcal{T}) \equiv \tilde{p}(\mathcal{T}, s) + a^2(\mathcal{T} - \tau), \tag{100}$$

and that the eigenvalues of the Jacobian matrix of (99) are $\mu_1(\mathbf{U}) = u - a\tau < \mu_2 = \mu_3(\mathbf{U}) = u < \mu_4(\mathbf{U}) = u + a\tau$. The solution of the Riemann problem follows similarly from propositions 4.2, 4.7.

Proposition 4.8. *Given two constant states $\mathbf{U}_\ell, \mathbf{U}_r$, the solution of the Riemann problem for (99) consists of three contact discontinuities, each propagating with a characteristic speed $\mu_i, i = 1, 2-3, 4$, separating $\mathbf{U}_\ell$, two intermediate states $\mathbf{U}_\ell^*, \mathbf{U}_r^*$ and $\mathbf{U}_r$. The states $\mathbf{U}_\ell^*, \mathbf{U}_r^*$ are respectively characterized by $(u^*, \Pi^*; \mathcal{T}_\ell, s_\ell, \tau_\ell^*)$ and $(u^*, \Pi^*; \mathcal{T}_r, s_r, \tau_r^*)$ with*

$$\begin{aligned} u^* &= \frac{u_\ell + u_r}{2} - \frac{\Pi_r - \Pi_\ell}{2a}, \\ \Pi^* &= \frac{\Pi_\ell + \Pi_r}{2} - \frac{a}{2}(u_r - u_\ell), \\ \tau_\ell^* &= \tau_\ell - \frac{1}{2a^2}(\Pi_r - \Pi_\ell - a(u_r - u_\ell)), \\ \tau_r^* &= \tau_r + \frac{1}{2a^2}(\Pi_r - \Pi_\ell + a(u_r - u_\ell)). \end{aligned} \tag{101}$$

Proof. The formula (101) naturally coincide with those of (71), the solution is now expressed as a function of (x, t) while it was function of (y, t). For the full system in Lagrangian coordinates, the entropy changes across the second (double) eigenvalue $u = 0$, similarly in Eulerian coordinates it is discontinuous across the second (double) eigenvalue, here $\mu_{2,3} = u^*$. In the above statement, the characterisitic speeds are more precisely given by $\mu_1 = u_\ell - a\tau_\ell = u^* - a\tau_\ell^*$ and $\mu_4 = u_r + a\tau_r = u^* + a\tau_r^*$ □

4.4.2 Global relaxation solver in Eulerian coordinates

Now, in order to approximate the solutions of the fluid system in Eulerian coordinates, which means (17), in which the entropy conservation equation is replaced by the energy one

$$\partial_t \varrho e + \partial_x(\varrho u e - (\mathbf{u}, N^T \Phi(\mathbf{v}, s)) = 0 \tag{102}$$

we can follow two paths. Either we consider a direct global relaxation approach on the relaxation system for the Eulerian formulation, which means that we use a Godunov solver for (93), an instantaneous relaxation step and the same exchange trick between energy and entropy. Or we use the correspondence between solvers at the discrete level as in [23] (this correspondence is used again in [16]). Anyhow, we deduce the properties from the equivalence between the two Lagrangian/Eulerian formulation at the continuous or discrete level.

Let us note here $U = (\overline{U}, \underline{U}, \varrho e)$ the state variable of the fluid system, $V = (\overline{U}, \underline{U}, \varrho s, \varrho \mathcal{V})$ the state variable of the augmented relaxation system. We note again $\mathcal{M}$ the *equilibrium mapping* $\mathcal{M}(U) = V = (\overline{U}, \underline{U}, \varrho s, \overline{U})$ which allows to reconstruct an equilibrium state, where $s = s(U)$.

We can state the analogue of proposition 4.3.

Proposition 4.9. *Under CFL 1/2, the resulting global relaxation scheme can be written*

$$U_j^{n+1} = U_j^n - \mu \left(G_{j+1/2}^n - G_{j-1/2}^n \right), \quad j \in \mathbb{Z}, n \geq 0, \tag{103}$$

with the components of the flux $G_{j+1/2}^n$ given by the corresponding flux components of (93), (95) evaluated on state $W_{j+1/2}^n$, where

$$W_{j+1/2}^n = W_R(0+; \mathcal{M}(U_j^n), \mathcal{M}(U_{j+1}^n)), \tag{104}$$

and W_R denotes the solution of the Riemann problem for (93) given by proposition 4.7. Moreover, the global relaxation solver satisfies a discrete entropy inequality

$$(\varrho s)_j^{n+1} \leq (\varrho s)_j^n - \mu(\mathcal{G}_{\varrho s, j+1/2}^n - \mathcal{G}_{\varrho s, j-1/2}^n), \quad j \in \mathbb{Z}. \tag{105}$$

These results are valid for the natural multi-d extension of the scheme.

5 Numerical illustrations

We now present some numerical results obtained by the relaxation solver, in order to illustrate its accuracy and its ability to deal with complex test cases. We have chosen Saint-Venant equations for shallow-water flows The Saint-Venant system models free surface water flows, under the assumption of a small average depth with respect to the length of the spatial domain. In 2D, the flow is described by the height of water h and by the horizontal velocity field (u, v) (u is the x-component and v the y-component of the velocity) and satisfies the equations

$$\begin{aligned} &\partial_t h + \partial_x(hu) + \partial_y(hv) = 0, \\ &\partial_t(hu) + \partial_x(hu^2 + gh^2/2) + \partial_y(huv) = 0, \\ &\partial_t(hv) + \partial_x(huv) + \partial_y(hv^2 + gh^2/2) = 0, \end{aligned} \tag{106}$$

where g denotes the gravity constant. One of the main difficulties about this system is the possibility of the presence of dry areas in the domain of computation. Therefore, the numerical methods must be in agreement with the requirement $h \geq 0$. Though it is not addressed in the paper, relaxation solvers may comply with positivity preservation, as described for instance in [7]. We present a 2D numerical test, performed by the above described relaxation solver for isentropic Euer system, naturally extended in 2D, on an unstructured grid which contains 20964 triangles. The domain size is 10×10 and the mesh is represented in figure 1(a). The initial condition (see figure 1(b)) is

$$\forall (x, y) \in [0, 10]^2, \quad \begin{aligned} h(0, x, y) &= 3H(x - 4), \\ u(0, x, y) &= 0, \\ v(0, x, y) &= 0, \end{aligned} \tag{107}$$

where H is the Heaviside function and wall conditions are set on the whole boundary. This test corresponds to a dam-break problem over a complex domain, inducing many reflecting waves. Moreover, during the first instants, the solution is a sonic rarefaction wave over a dry area. For this case, it is crucial that the solver be positivity preserving and entropy satisfying: in figure 1(c), one may check that the solution is very well approximated. in figures 1(d) and (e), more complex flooding and reflected waves can be shown, with a good resolution. The last figure 1(f) corresponds to a large time result, which is a constant surface with a null velocity field.

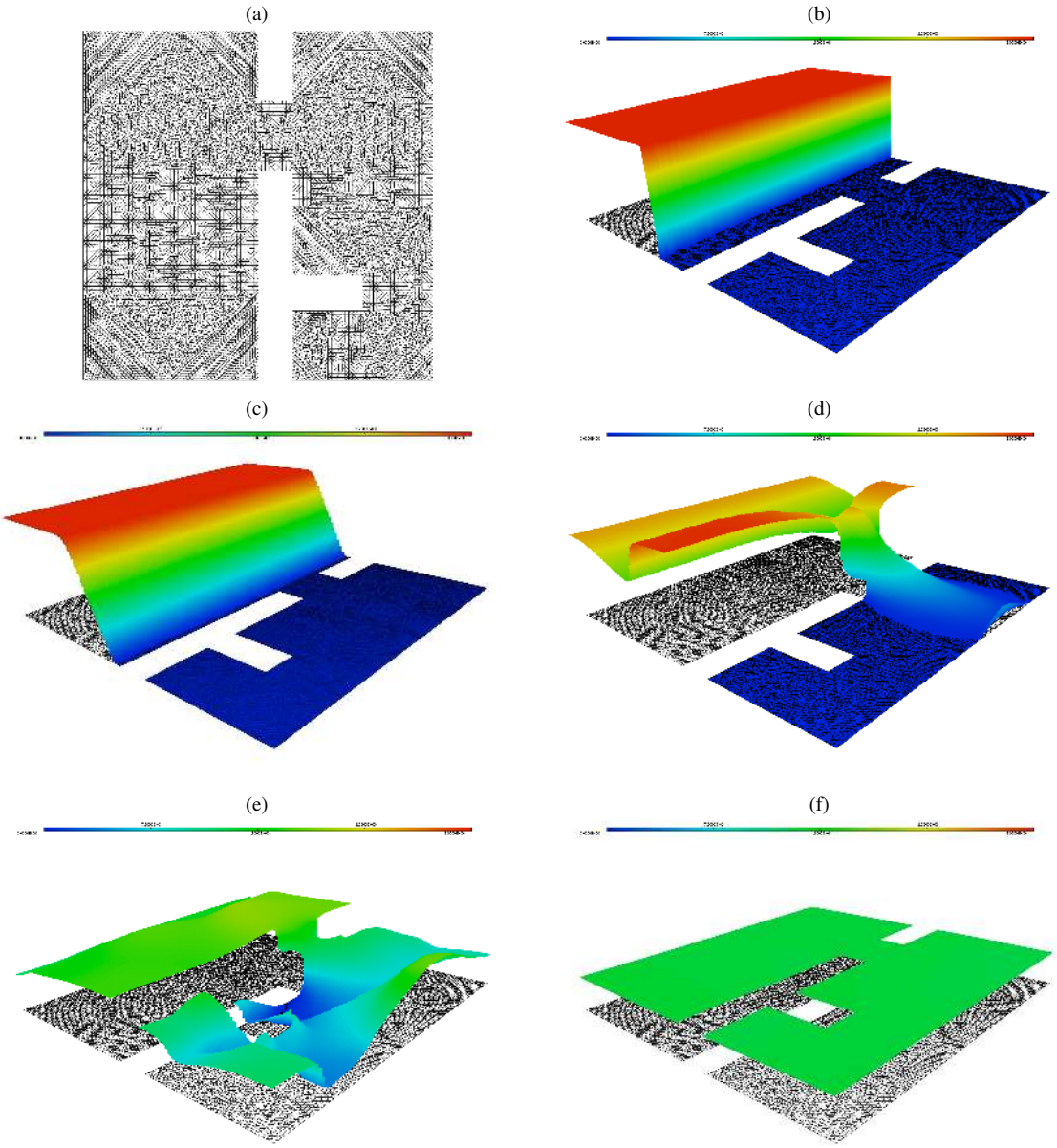

Figure 1: (a) The domain and the mesh; (b) the initial data; (c) the surface at $t = 0.1$; (d) the surface at $t = 0.8$; (e) the surface at $t = 4$; (f) the surface at $t = 30$.

6 Conclusion

We have described a numerical approach for computing numerically the (entropy weak) solutions of general fluid systems. This approach involves a relaxation system which may be considered as an extension of the Suliciu relaxation system for the Euler equations of gas dynamics [35]. The Godunov solver for the homogeneous relaxation system results in an HLLC-type solver for the equilibrium system, i.e. the fluid system. Note that the resulting scheme is indeed simple because the homogeneous relaxation system is totally linearly degenerate, and the solution of the Riemann problem is explicit.

Though the analysis may seem rather technical, and sometimes even daunting, we have chosen to present it with some details since the underlying PDE relaxation system brings an interesting light on why these schemes possess so many good properties, in particular they satisfy naturally discrete entropy inequalities. Moreover the equivalence relation between the systems written in Lagrangian and Eulerian frames (see [37]) is inherited by the relaxation systems and then by the numerical schemes. We mentionned some approximation properties which are interesting by themselves.

The numerical tests we ran on Euler system in 1D and 2D have illustrated the good properties and we hope that the extension to other systems, such as MHD which are addressed to in [9] [10], which we did not yet perform, will be as convincing.

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