RELAXATION OSCILLATIONS OF A VAN DER POL EQUATION WITH LARGE CRITICAL FORCING TERM*

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Abstract. A van der Pol equation with sinusoidal forcing term is analyzed with singular perturbation methods for large values of the parameter. Asymptotic approximations of (sub)harmonic solutions with period $T = 2\pi(2n-1)$, $n = 1, 2, \ldots$ are constructed under certain restricting conditions for the amplitude of the forcing term. These conditions are such that always two solutions with period $T = 2\pi(2n \pm 1)$ coexist.

1. Introduction. In this paper we consider a van der Pol equation for large parameter values with a periodic forcing term of a same order of magnitude:

$$\frac{d^2x}{dt^2} + \nu(x^2-1)\frac{dx}{dt} + x = b(\nu)\cos t, \qquad \nu \gg 1, \tag{1.1}$$

with $b(\nu) = O(\nu)$. This equation was investigated with analytical-topological methods by Littlewood [8], who proved the existence of (sub)harmonic solutions of period

$$T = 2\pi(2n-1), \qquad n = 1, 2, \ldots. \tag{1.2}$$

Littlewood stated that for $b = \alpha\nu$, $\alpha > 2/3$ only globally asymptotically stable solutions of the period 2π are found (see also [9]). The proof of this statement has been given by Lloyd [10]. For decreasing α there also occur solutions of period 6π. As α decreases further the 2π-periodic solution disappears, and α passes alternately intervals where one subharmonic solution of period $T = 2\pi(2n-1)$ exists and intervals where two subharmonic solutions of period $T = 2\pi(2n \pm 1)$ coexist ($n = 1, 2, \ldots$). There are also intervals of more complicated behavior.

In Fig. 1 we give the overlapping domains Ω_n in the $(b/\nu, \nu)$-plane where a solution of period (1.2) with $n \leq 4$ is found. The figure is based on numerical results obtained by Flaherty and Hoppensteadt [2] for $1/\nu > .01$. We see that for $\nu \to \infty$ these domains tend to a common boundary point $(b/\nu, \nu) = (2/3, \infty)$. In this paper we will analyze the local structure of the domains Ω_n near $(b/\nu, \nu) = (2/3, \infty)$. For that purpose we write

$$b = \alpha\nu + \beta. \tag{1.3}$$

Using singular perturbation techniques, we will construct asymptotic approximations of

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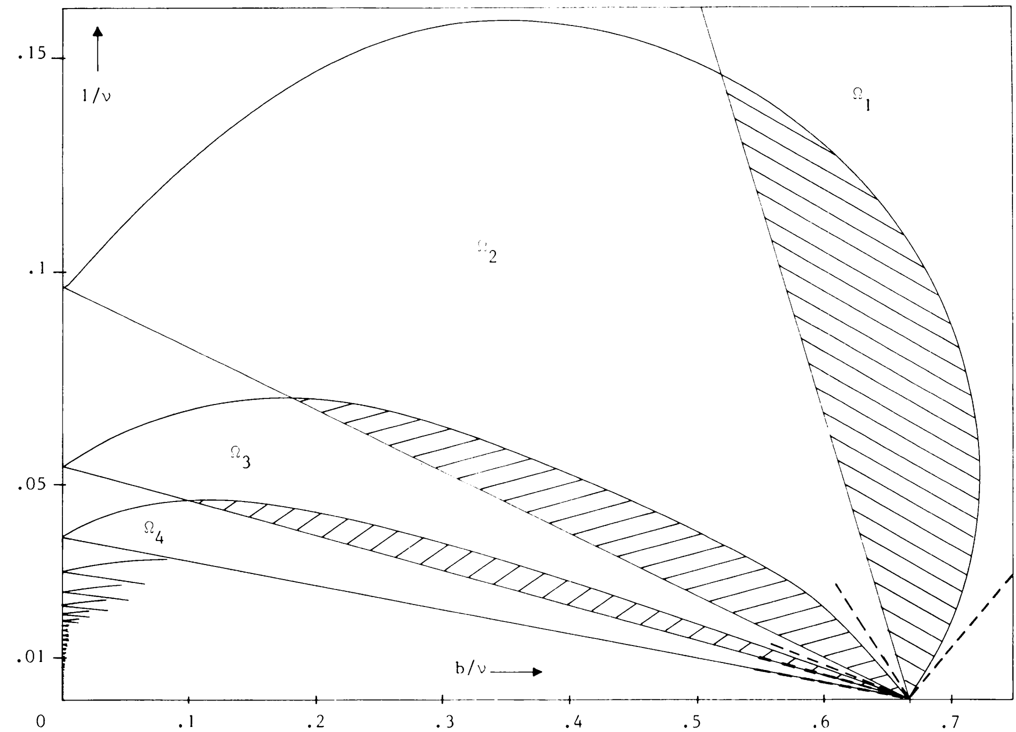

FIG. 1. The domains Ω_n with a periodic solution $T = 2\pi(2n - 1)$.

(sub)harmonic solutions of (1.1), (1.3) with $\alpha = 2/3$. The periods of these solutions satisfy (1.2) with n independent of v. In the process of construction of the approximation we will have to impose conditions upon β of the type

$$\underline{\beta}_n < \beta < \bar{\beta}_n \tag{1.4}$$

to approximate symmetric solutions of period $T = 2\pi(2n - 1)$. It turns out that

$$\underline{\beta}_n < \bar{\beta}_{n+1} = \underline{\beta}_{n-1} < \bar{\beta}_n, \tag{1.5}$$

so that near $(b/v, v) = (2/3, \infty)$ the domains Ω_n overlap as sketched in Fig. 2.

This overlapping of intervals differs slightly from analytical results [5, 6, 8], as we only find intervals with two subharmonic solutions of period $T = 2\pi(2n \pm 1)$. In [3, 4] the case $\alpha = 0$ was also analyzed with asymptotic techniques. There the subharmonics had a period $T = 2\pi n$ with $n = O(v)$. The choice $\alpha = 0$ or $\alpha = 2/3$ leads to solutions with completely different asymptotic behavior, and this makes it necessary to consider them as separate problems. In [3] we met an unusual structure of two-variable expansions matched with boundary layer solutions. We will see here that the case $\alpha = 2/3$ also exhibits an exceptional structure. The global behavior of the solution depends strongly on local conditions: each time the solutions pass a neighborhood of the lines $x = \pm 1$ some quantity is increased by a given value. When this quantity, being an integration constant in the local asymptotic solution, reaches a threshold value the solution enters a phase of rapid change characteristic of a relaxation oscillation. This part of the solution is approximated by a boundary layer type of solution. For the regions sketched in Fig. 3

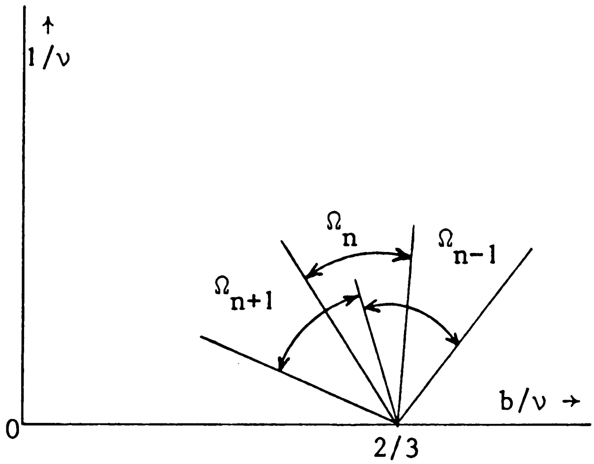

FIG. 2. Local structure of the domains Ω_n near $(b/\nu, \nu) = (2/3, \infty)$ derived from the formal asymptotic analysis. Exact values are given in Fig. 1 (dotted lines) for $n \leq 4$.

separate local approximations have been constructed from the differential equation. Integration constants in these local asymptotic solutions are determined by matching pairs of local solutions of adjacent regions.

Thus, in this paper we investigate the equation

$$\frac{d^2x}{dt^2} + \nu(x^2 - 1)\frac{dx}{dt} + x = (\tfrac{2}{3}\nu + \beta)\cos t. \tag{1.6}$$

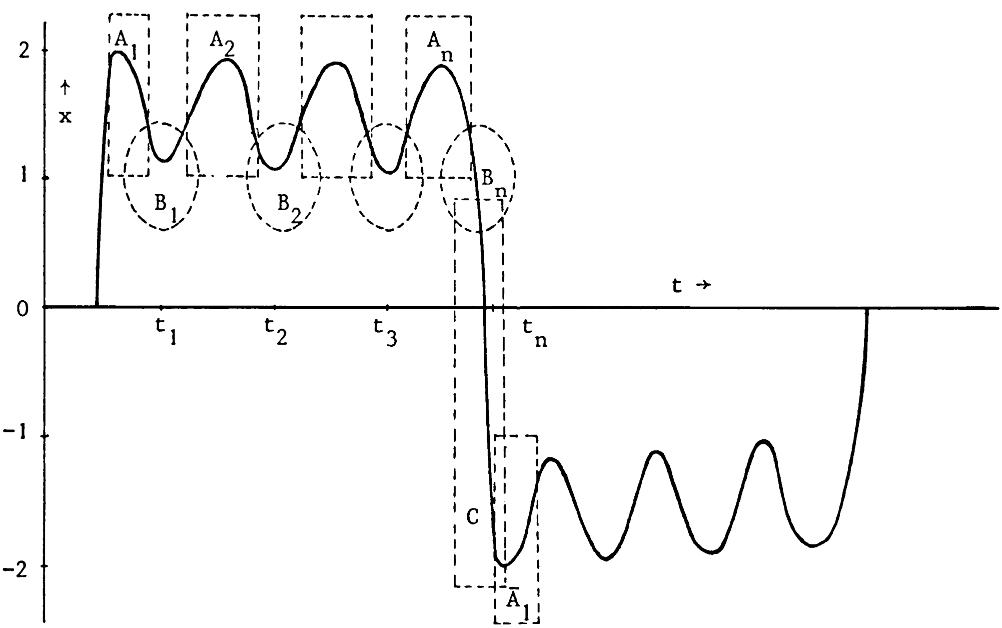

FIG. 3. Characteristic regions for a periodic solution of (1.6).

It is expected that the study of this problem with a critical forcing term may bring us into a position to deal successfully with the more complicated problem of $0 < \alpha < 2/3$. It is anticipated that periodic solutions of this problem have a behavior in which elements of both the case $\alpha = 0$ and $\alpha = 2/3$ are present. It is remarked that the limit cases $\alpha = 0$ and $\alpha = 2/3$ are not covered by the recent analytical studies ([5, 6, 10]), where, respectively, $0 < \alpha < 2/3$ and $\alpha > 2/3$.

2. Asymptotic solutions for the regions A_m. It is supposed that in the regions A_m where $1 < x < 2$ the solution can be expanded as

$$x(t;\varepsilon) = x_{m0}(t) + v^{-1}x_{m1}(t) + \cdots. \tag{2.1}$$

Substituting (2.1) into Eq. (1.6) and equating the terms of order $O(v)$ and $O(1)$, we obtain

$$(x_{m0}^2 - 1)\frac{dx_{m0}}{dt} = \frac{2}{3}\cos t, \tag{2.2}$$

$$(x_{m0}^2 - 1)\frac{dx_{m1}}{dt} + 2x_{m0}x_{m1}\frac{dx_{m0}}{dt} = -\frac{dx_{m0}^2}{dt^2} - x_{m0} + \beta\cos t. \tag{2.3}$$

Integration of Eq. (2.2) gives

$$\tfrac{1}{3}x_{m0}^3 - x_{m0} = \tfrac{2}{3}\sin t + C_0^{(m)}. \tag{2.4}$$

Since in the regions A_m the value of the left-hand side of this equation varies from $-2/3$ to $2/3$, we have to take $C_0^{(m)} = 0$. For this value of $C_0^{(m)}$ the solutions of (2.4) read

$$x_{m0}(t) = 2\cos\{\tfrac{1}{3}(\arccos(\sin t) + 2\pi j)\}, \qquad j = 0, 1, 2. \tag{2.5}$$

As x_{m0} has to be within the interval (1, 2), we select the branch with $j = 0$. Integrating (2.3), while making use of (2.2), we obtain

$$(x_{m0}^2 - 1)x_{m1} = -\frac{2\cos t}{3(x_{m0}^2 - 1)} - \int_{t_{m-1}}^{t} x_{m0}(\tau)\,d\tau + \beta\sin t + C_1^{(m)}, \tag{2.6}$$

$$t_m = 2\pi m - \pi/2. \tag{2.7}$$

When t approaches t_m from below, x_{m0} and x_{m1} behave as

$$x_{m0} \approx 1 - (t - t_m)/\sqrt{3}, \qquad x_{m1} \approx K_m/(t - t_m), \tag{2.8a,b}$$

where

$$K_m = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,(-C_1^{(m)} + I), \qquad I = \int_{t_{m-1}}^{t_m} x_{m0}(t)\,dt = 6\sqrt{3}. \tag{2.9a,b}$$

Thus, for $t \uparrow t_m$ the asymptotic solution (2.1) loses its validity.

3. Asymptotic solution for the regions B_m. We analyze the local behavior of the solution near $(x, t) = (1, t_m)$, $m = 1, 2, \ldots$ by introducing a stretching transformation in both the dependent and independent variable:

$$x = 1 + V_m(\xi)v^{-\gamma}, \qquad t = t_m + \xi v^{-\alpha}. \tag{3.1a,b}$$

Substitution into the differential equation yields

$$\nu^{-\gamma+2\alpha}\frac{d^2V_m}{d\xi^2}+\nu^{1-2\gamma+\alpha}(2V_m+\nu^{-\gamma}V_m^2)\frac{dV_m}{d\xi}+1+V_m\nu^{-\gamma}$$

$$=(\tfrac{2}{3}\nu+\beta)\left(\xi\nu^{-\alpha}-\frac{\xi^3\nu^{-3\alpha}}{3!}+\cdots\right). \quad (3.2)$$

We see that for $\alpha=\gamma=1/2$ the second derivative becomes of the same order of magnitude in ν as the leading terms constituting Eq. (2.2). Multiplying the equation by $\nu^{-1/2}$ and letting ν tend to infinity, we obtain the limit equation

$$\frac{d^2V_{m0}}{d\xi^2}+2V_{m0}\frac{dV_{m0}}{d\xi}=\tfrac{2}{3}\xi. \quad (3.3)$$

The function $V_{m0}(\xi)$ expresses the local limit behavior of the solution for $\nu\to\infty$. In order to match the solution of region A_m it must satisfy

$$V_{m0}(\xi)\approx\frac{\xi}{\sqrt{3}}+\frac{K_m}{\xi} \quad (3.4)$$

for $\xi\to\infty$ (see (2.8)). Such a function indeed exists and has the form

$$V_{m0}(\xi)=a\frac{D'_{K_m}(-a\xi)}{D_{K_m}(-a\xi)}, \qquad a=\sqrt[4]{4/3}, \quad (3.5)$$

where $D_\mu(z)$ is the so-called parabolic cylinder function of order μ (see Whittaker and Watson [11, p. 347]). For $z\to\infty$ we have that

$$D_\mu(z)=\exp(-\tfrac{1}{4}z^2)z^\mu\left\{1-\frac{\mu(\mu-1)}{2z^2}+\cdots\right\},$$

while for $z\to-\infty$

$$D_\mu(z)=\exp(-\tfrac{1}{4}z^2)z^\mu\left\{1-\frac{\mu(\mu-1)}{2z^2}+\cdots\right\}$$

$$-\frac{\sqrt{2\pi}}{\Gamma(-\mu)}\exp(\tfrac{1}{4}z^2)z^{-\mu-1}\left\{1+\frac{(\mu+1)(\mu+2)}{2z^2}+\cdots\right\}.$$

Assuming that $K_m\leq 0$, the function $V_{m0}(\xi)$ will be regular for finite ξ, while for $\xi\to\infty$

$$V_{m0}(\xi)\approx\frac{\xi}{\sqrt{3}}-\frac{K_m+1}{\xi}. \quad (3.6)$$

On the other hand, at region A_{m+1} the solution is approximated by

$$x(t)=1+\frac{(t-t_m)}{\sqrt{3}}+\frac{-\frac{1}{2}+\frac{1}{2}\sqrt{3}\,(C_1^{(m+1)}-\beta)}{(t-t_m)}+O((t-t_m)^{-3})$$

as $t\downarrow t_m$. Consequently, (3.6) matches the local solution for region A_{m+1} if

$$K_m=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}\,(\beta-C_1^{(m+1)})$$

or, using (2.9a),

$$C_1^{(m+1)} = C_1^{(m)} - I \tag{3.7}$$

with $I = 6\sqrt{3}$ (see (2.9b)). Obviously, we will arrive in the situation that for some m, say $m = n$,

$$K_{n-1} \leq 0 < K_n \leq \tfrac{1}{2}\sqrt{3}\, I \tag{3.8}$$

(if $n = 1$, inequality (3.8) reads $K_1 > 0$). The parabolic cylinder function $D_\mu(z)$ with $\mu > 0$ vanishes for certain value(s) of the argument z. Let z_0 be the largest zero. For $\xi \uparrow \xi_0$ with $z_0 = a\xi_0$ we have

$$V_{m0}(\xi) \approx (\xi - \xi_0)^{-1} + \tfrac{1}{3}a^2(\tfrac{1}{4}a^2\xi_0^2 - K_n - \tfrac{1}{2})(\xi - \xi_0), \tag{3.9}$$

so $V_{n0} \to -\infty$ and the local solution at region B_n becomes singular at $\xi = \xi_0$.

4. Asymptotic solution for region C. At this point the solution enters the boundary layer region C with local coordinate

$$\eta = (t - t_n - \xi_0 \nu^{-1/2})\nu. \tag{4.1}$$

We assume that the solution can be expanded as

$$x = W_0(\eta) + \nu^{-1}W_1(\eta) + \nu^{-3/2}W_2(\eta) + \cdots. \tag{4.2}$$

Substituting (4.1) and (4.2) into (1.6) and equating the terms of order $O(\nu^2)$ and $O(\nu)$ we obtain, respectively,

$$d^2W_0/d\eta^2 + (W_0^2 - 1)\, dW_0/d\eta = 0, \tag{4.3}$$

$$d^2W_1/d\eta^2 + (W_0^2 - 1)\, dW_1/d\eta + 2W_0 W_1(dW_0/d\eta) = 0. \tag{4.4}$$

The solution of the first equation matches the local solution for region B_n if

$$W_0(\eta) \approx 1 + 1/\eta \tag{4.5}$$

as $\eta \to -\infty$ (see (3.9)). This condition is satisfied by the class of solutions

$$\frac{1}{1 - W_0} + \frac{1}{3}\log\frac{W_0 + 2}{1 - W_0} = -\eta + H_0, \tag{4.6}$$

where the integration constant H_0 is found from matching with higher-order terms of the asymptotic solution for region B_n. It turns out that

$$H_0 = \tfrac{1}{6}\log \nu + \tfrac{1}{3}\log 3. \tag{4.7}$$

From (3.9) we also deduce that W_1 should behave as

$$W_1(\eta) \approx \tfrac{1}{3}a^2(\tfrac{1}{4}a^2\xi_0^2 - K_n - \tfrac{1}{2})\eta \tag{4.8}$$

for $\eta \to -\infty$, so that the integrated equation (4.4) will have the form

$$\frac{dW_1}{d\eta} + (W_0^2 - 1)W_1 = a^2(\tfrac{1}{4}a^2\xi_0^2 - K_n - \tfrac{1}{2}). \tag{4.9}$$

On the other hand, for $\eta \gg 1/6 \log \nu$ we have

$$W_0(\eta) = -2 + O(\nu^{1/2}e^{-3\eta}), \tag{4.10a}$$

$$W_1(\eta) = \tfrac{1}{3}a^2(\tfrac{1}{4}a^2\xi_0^2 - K_n - \tfrac{1}{2}) + O(e^{-3\eta}). \tag{4.10b}$$

The boundary-layer solution matches the solution for region $\bar{A}_1$ if

$$\bar{x}_{10}(t_n + \xi_0 \nu^{-1/2}) + \nu^{-1}\bar{x}_{11}(t_n + \xi_0 \nu^{-1/2}) = -2 + \tfrac{1}{3}a^2(\tfrac{1}{4}a^2\xi_0^2 - K_n - \tfrac{1}{2})\nu^{-1} + o(\nu^{-1}), \tag{4.11}$$

where $\bar{x}_{1i}(t)$ are the coefficients of an expansion for region $\bar{A}_1$ of the form (2.1).

5. Periodicity conditions. Let us assume that the periodic solutions we are looking for are symmetric in the sense that $x(t) = -x(t - \tfrac{1}{2}T)$. Then we have completed the local approximations. Transposing (4.11) to the complementary phase $t = t_1 - \pi + \xi_0 \nu^{-1/2}$, in region A_1 we have

$$x_{10}(t_1 - \pi + \xi_0 \nu^{-1/2}) + \nu^{-1}x_{11}(t_1 - \pi + \xi_0 \nu^{-1/2}) = 2 - \tfrac{1}{3}a^2(\tfrac{1}{4}a^2\xi_0^2 - K_n - \tfrac{1}{2})\nu^{-1} + o(\nu^{-1}) \tag{5.1}$$

or

$$K_n = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,(\beta + C_1^{(1)} - \tfrac{1}{2}I). \tag{5.2}$$

Using (2.9a), (3.7) and (5.2), we find

$$\beta = \frac{1}{\sqrt{3}}(2K_n + 1) - \tfrac{1}{2}(n - \tfrac{1}{2})I. \tag{5.3}$$

From (3.8) we know that K_n ranges from 0 to 9, so β has to satisfy

$$3\sqrt{3}\,(\tfrac{11}{18} - n) < \beta < 3\sqrt{3}\,(\tfrac{47}{18} - n), \qquad n = 2, 3, \ldots. \tag{5.4}$$

Solutions of period 2π are found for $\beta > -7/6\sqrt{3}$.

6. Some remarks. Our asymptotic results hold for values of β that are independent of ν and satisfy the inequality (5.4). Because of this the domains Ω_{n-1} and Ω_{n+1} are separated near $(b/\nu, \nu) = (2/3, \infty)$ by a sectorial domain Γ_n of thickness $o(1)$ as $\nu \to \infty$. A qualitative analysis of (1.1), (1.3) with $0 < \alpha < 2/3$ by Levi [6] reveals the existence of infinitely many different periodic solutions for uncountably many values of β near β_n and $\bar{\beta}_n$. It is expected that at $\alpha = 2/3$ a similar phenomenon occurs for values of b and ν that are restricted to the domains $\Gamma_{n\pm 1}$. Furthermore, for $0 < \alpha < 2/3$ there are intervals $\bar{\beta}_{n+1} < \beta < \beta_{n-1}$, where only one solution with period $T = 2\pi(2n - 1)$ is found. Clearly, for $\alpha \to 2/3$ these intervals disappear and become part of the small transition intervals described above.

For the case $\alpha = 0$ nonsymmetric solutions with period $T = 4\pi n$ have been constructed in [4], and it is also indicated there how the method can be extended to solutions with a rational rotation number. Trying to derive similar results for $\alpha = 2/3$, we find that the symmetry condition of Sec. 5 is necessary. Thus, nonsymmetric solutions are only expected at the boundaries of the β-intervals. An asymptotic analysis of this problem requires elaborate calculations of higher-order approximations.

Finally, it is remarked that the asymptotic solution of the case $0 < \alpha < 2/3$ should match the present results for $\alpha \to 2/3$ and the outcome of earlier work [3] for $\alpha \to 0$. This problem has not been solved yet. It is expected that it will give rise to some serious difficulties in the construction of the local asymptotic solution of a region of length $O(\nu)$, where the solution has a two time-scale behavior and satisfies asymptotically an equation of the type (2.4) with a slowly varying integration constant.

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