Relaxation time of processes driven by multiplicative noise

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We consider systems described by nonlinear stochastic differential equations with multiplicative noise. We study the relaxation time of the steady-state correlation function as a function of noise parameters. We consider the white- and nonwhite-noise case for a prototype model for which numerical data are available. We discuss the validity of analytical approximation schemes. For the white-noise case we discuss the results of a projector-operator technique. This discussion allows us to give a generalization of the method to the non-white-noise case. Within this generalization, we account for the growth of the relaxation time as a function of the correlation time of the noise. This behavior is traced back to the existence of a non-Markovian term in the equation for the correlation function.

I. INTRODUCTION

Recent interest in multiplicative stochastic processes[1–16] has been motivated by the study of instabilities in nonequilibrium systems with state-dependent fluctuations. Optical systems are a typical example in which these fluctuations are relevant.[3,5,6,16] These general type of systems are usually described by stochastic differential equations (SDE) with a multiplicative noise term. The origin of this noise term can be either the existence of intrinsic fluctuations or a way of modeling fluctuations in a control parameter of the system. An important quantity to characterize the behavior of these systems is the correlation function in a nonequilibrium steady state and the associated relaxation time T. In this paper we discuss the dependence of this relaxation time on the noise parameters. It is generally assumed that the noise term can be well represented by a Gaussian white noise. The white-noise assumption has been eliminated in several studies.[1,4,5,10,13,15] A characteristic behavior of the correlation function in a dye-laser system has been recently explained in terms of a nonwhite noise.[16] For a particular prototype model first studied by Stratonovich[17] and lately by many others[1,7–9,11,14] the relaxation time has been calculated both for the white and nonwhite cases by means of a numerical simulation of the SDE in Ref. 1. The purpose of this paper is to account for these numerical results through an analytical calculation of T. For the white-noise case we discuss the validity of already known approximation schemes. For the non-white-noise case we present an approximation scheme to calculate T that reproduces the main features of the numerical data. We do not know of any other explicit calculation of T for a nonlinear SDE with nonwhite noise. (For linear problems see Refs. 18 and 19.)

We will consider the following SDE for a relevant macroscopic variable q:

$$\dot{q}(t)=v(q(t))+g(q(t))\xi(t) , \tag{1.1}$$

where $v(q)$ and $g(q)$ are general nonlinear functions of q and $\xi(t)$ is the random force which we take to be an Ornstein-Uhlenbeck process: a Gaussian noise with zero mean and correlation function

$$\langle \xi(t)\xi(t')\rangle=\frac{D}{\tau}\exp\left[-\frac{|t-t'|}{\tau}\right] . \tag{1.2}$$

The parameter D measures the noise intensity and τ is the correlation time of the noise. The white-noise limit is obtained by taking $\tau\to 0$ with D fixed:

$$\lim_{\tau\to 0}\langle \xi(t)\xi(t')\rangle=2D\delta(t-t') . \tag{1.3}$$

The particular model that we will consider[1,7–9,11,14,17] is given by (1.1) with

$$v(q)=\alpha q-q^3 , \tag{1.4}$$

$$g(q)=q . \tag{1.5}$$

This model has a deterministic instability point at $\alpha=0$. In this paper we set $\alpha=1$ so that we are always beyond the instability point. In the case of additive fluctuations [$g(q)$=const] it corresponds to a bistable situation. In our case the multiplicative fluctuations imply the existence of a boundary at $q=0$ which destroys the bistable character of the stationary distribution. The conditions for the existence of a stationary distribution for a general model of the form (1.1) in the white-noise limit are given, for example, in Ref. 20. These conditions are satisfied for the prototype model of (1.4) and (1.5), and its stationary distribution is[1,17]

$$P_{\rm st}^0(q)=2\left[\frac{1}{2D}\right]^{1/2D}\Gamma^{-1}\left[\frac{1}{2D}\right]q^{(-1+1/D)}\times\exp\left[-\frac{1}{2D}q^2\right] . \tag{1.6}$$

This distribution is discussed in detail in Ref. 17. For $D<1$ it has a single maximum at a finite value of q and $P_{\rm st}^0(q=0)=0$. For $D>1$ the single maximum is at $q=0$ where $P_{\rm st}^0(q=0)=\infty$. For any D, $P_{\rm st}^0(q\to\infty)\to 0$. The properties of the stationary distribution of the model of (1.4) and (1.5) when $\tau\neq 0$ have been studied analytically and numerically in Ref. 1. The main difference with the white-noise limit is the appearance for $D>1$ of a new relative maximum of $P_{\rm st}(q)$ when τ increases from zero. This maximum becomes preponderant as $\tau\to\infty$.

For a general model for which a stationary state exists, the stationary correlation function is defined by

$$C(s)=\langle\delta q(t+s)\delta q(t)\rangle_{\rm st}=\lim_{t\to\infty}\langle\delta q(t+s)\delta q(t)\rangle , \tag{1.7}$$

where

$$\delta q(t)=q(t)-\langle q(t)\rangle . \tag{1.8}$$

The relaxation time is defined by

$$T=\int_0^\infty ds\, C^0(s) , \tag{1.9}$$

where $C^0(s)$ is a normalized correlation function

$$C^0(s)=\frac{C(s)}{\langle(\delta q)^2\rangle_{\rm st}} . \tag{1.10}$$

We note that in our parametrization of the SDE, D and τ are the only independent parameters left in the model. We want precisely to study the dependence of T on D and τ. From the numerical simulation in Ref. 1 it is known that for the model of (1.4) and (1.5) there exists a slowing down phenomenon in the sense that T increases monotonically both as a function of D and τ. To give an analytical discussion of this behavior we consider separately the white- and non-white-noise cases. In the first case the problem is Markovian. The special difficulty of the second case is due to the non-Markovian character of $q(t)$.[21] The retardation in the decay of fluctuations when D increases already exists in the white-noise limit. This effect is due to the multiplicative character of the noise: increasing the noise intensity has an opposite effect to that for the additive noise case. This can be understood in terms of a noise-dependent potential as explained below. The slowing down as a function of τ is of a different physical nature and it is clearly associated with memory effects present in a non-Markovian dynamics. A slowing down of this sort also exists in linear processes with additive or multiplicative nonwhite noise.[19] For the general case (1.1) and also for our particular model of (1.4) and (1.5) this effect is mixed with the problem of nonlinearities.

We discuss the white-noise case in Sec. II. The model of (1.4) and (1.5) has been already studied in this case by Fujisaka and Grossman[7] using a projector-operator method leading to a continued-fraction expansion (see also Ref. 8). This method is a generalization of the well-known Zwanzig-Mori formalism[22,23] to nonequilibrium systems whose dynamical evolution is given by a Fokker-Planck operator. From a truncation in first order of the continued-fraction expansion it follows that T grows with D for small D passing through a maximum at $D\simeq 0.3$ and becoming a decreasing function of D for larger D. This behavior is in disagreement with the numerical simulation of Ref. 1. We discuss this calculation in the light of the numerical results and also of the knowledge of the exact spectrum of the associated eigenvalue problem.[9] We give an interpretation of the general scheme used in Ref. 7 in connection with a decoupling ansatz of Stratonovich.[17] The exact value of T to order D, in an expansion of T in powers of D, is recovered from the first-order truncation of the continued-fraction expansion. We also show that a higher-order approximation in the continued-fraction expansion extends the range of values of D for which a qualitative agreement with the numerical simulation exists. This discussion allows us to introduce the basic ideas used in the calculation for the non-white-noise case. Such a calculation is presented in Sec. III. We introduce a generalization for nonwhite noise of the scheme used in Ref. 7. This generalization accounts for the main effects of being $\tau\neq 0$. The main difference in the value of T with respect to the white-noise case is traced back to the existence of a term of non-Markovian origin in the equation for the correlation function. This term produces a shift of order τ of the value of T as $D\to 0$. For $D\neq 0$ this effect can be disentangled from other contributions which appear due to the nonlinear character of the problem.

II. WHITE NOISE

In this section we discuss approximation methods that have been proposed to calculate the relaxation time in the white-noise case (1.3). These methods will be one of our essential ingredients in the calculation for $\tau\neq 0$ of Sec. III. In the white-noise case the process (1.1) is completely described by the associated Fokker-Planck equation for the probability density $P(q,t)$:

$$\frac{\partial}{\partial t}P(q,t)=LP(q,t) , \tag{2.1}$$

$$L=-\frac{\partial}{\partial q}v(q)+D\frac{\partial}{\partial q}g(q)\frac{\partial}{\partial q}g(q) . \tag{2.2}$$

The operator L determines the dynamics of the process. In terms of the adjoint operator $L^\dagger$ of (2.2) the correlation function (1.7) is

$$C(s)=\int dq\, P_{\rm st}(q)\delta q\exp(L^\dagger s)\delta q , \tag{2.3}$$

where $P_{\rm st}(q)$ is the stationary solution of (2.1). The Laplace transform of $C^0(s)$ [defined in (1.10)] is

$$C^0(w)=\int_0^\infty ds\exp(-ws)C^0(s)=\frac{1}{(\delta q\mid\delta q)}\left[\frac{1}{w-L^\dagger}\delta q\middle|\delta q\right] , \tag{2.4}$$

where the scalar product $(A \mid B)$ in (2.4) is defined by

$$(A \mid B) = \int dq\, P_{st}(q) A(q) B(q) = \langle AB \rangle_{st} . \tag{2.5}$$

From (2.4) it is possible to obtain a continued-fraction expansion for $C^0(w)$. The method used to obtain such an expansion has been reviewed elsewhere.[23,24,8] We summarize here the main steps and results. The basic idea of the method is to identify δq as the relevant slow variable. The nonlinear dynamics is taken into account through memory terms in the evolution of δq, which represent the effect of the remaining fast variables. This idea is put into practice by introducing a projector operator P which projects on the subspace associated with the variable δq:

$$P = 1 - Q = \mid \delta q) \frac{1}{(\delta q \mid \delta q)} (\delta q \mid . \tag{2.6}$$

A formal manipulation of the resolvent $(w - L^\dagger)^{-1}$ in (2.4) with the projector operators P and Q leads to[23,24,8]

$$C^0(w) = [w + \gamma^0 - K^0(w)]^{-1} , \tag{2.7}$$

where

$$\gamma^0(w) = -(L^\dagger \delta q \mid \delta q)/(\delta q \mid \delta q) , \tag{2.8}$$

$$K^0(w) = \frac{1}{(\delta q \mid \delta q)} \left(\frac{1}{w - QL^\dagger Q} QL^\dagger \delta q \,\middle|\, QL^\dagger \delta q \right) . \tag{2.9}$$

The physical contents of (2.7) are better understood inverting the Laplace transform:

$$\frac{d}{ds} C^0(s) = -\gamma^0 C^0(s) + \int_0^s ds' K^0(s') C^0(s - s') . \tag{2.10}$$

The effect of the Q subspace on the dynamics of δq is contained in the memory kernel $K^0(s')$. The nonlinearity of the problem is now hidden in $K^0(s')$. For a linear problem there is no contribution from the Q subspace and then $K^0(s)=0$. If the memory effects are completely neglected in (2.10), $C^0(s)$ relaxes exponentially with a correlation time $T = (\gamma^0)^{-1}$:

$$C^0(s) = \exp(-\gamma^0 s) . \tag{2.11}$$

In order to take into account the memory effects one can now consider $QL^\dagger \delta q$ as a slow variable in Q space. Since $K^0(w)$ has the same formal structure for $QL^\dagger \delta q$ as does $C^0(w)$ for δq, we can proceed as above introducing a projector operator

$$P_1 = Q - Q_1 = \mid QL^\dagger \delta q) \frac{1}{(QL^\dagger \delta q \mid QL^\dagger \delta q)} (QL^\dagger \delta q \mid . \tag{2.12}$$

We can select again the slow variable in the Q_1 subspace by means of another projector operator.

$$P_2 = Q_1 - Q_2 = \mid Q_1 L^\dagger QL^\dagger \delta q) \frac{1}{(Q_1 L^\dagger QL^\dagger \delta q \mid Q_1 L^\dagger QL^\dagger \delta q)} (Q_1 L^\dagger QL^\dagger \delta q \mid . \tag{2.13}$$

Iterating this procedure we arrive at a continued-fraction expansion for $C^0(w)$:

$$C^0(w) = \left[w + \gamma^0 - \cfrac{K_1}{w + \gamma^1 - \cfrac{K_2}{w + \gamma^2 - \cdots}} \right]^{-1} , \tag{2.14}$$

where

$$\gamma^1 = -(L^\dagger QL^\dagger \delta q \mid QL^\dagger \delta q)/(QL^\dagger \delta q \mid QL^\dagger \delta q) , \tag{2.15}$$

$$\gamma^2 = -(L^\dagger Q_1 L^\dagger QL^\dagger \delta q \mid Q_1 L^\dagger QL^\dagger \delta q)/(Q_1 L^\dagger QL^\dagger \delta q \mid Q_1 L^\dagger QL^\dagger \delta q) , \tag{2.16}$$

$$K_1 = (QL^\dagger \delta q \mid QL^\dagger \delta q)/(\delta q \mid \delta q) , \tag{2.17}$$

$$K_2 = (Q_1 L^\dagger QL^\dagger \delta q \mid Q_1 L^\dagger QL^\dagger \delta q)/(QL^\dagger \delta q \mid QL^\dagger \delta q) . \tag{2.18}$$

Truncating this expansion in a first approximation by setting $K_2 = 0$ we obtain a correlation function $C^0(s)$ given by a superposition of two exponentials:

$$C^0(t) = (1 - \delta)\exp(-\Gamma_1 t) + \delta \exp(-\Gamma_2 t) , \tag{2.19}$$

$$\Gamma_{1,2} = \tfrac{1}{2}\{\gamma^0 + \gamma^1 \pm [(\gamma^1 - \gamma^0)^2 + 4K_1]^{1/2}\} , \tag{2.20}$$

$$\delta = \frac{\gamma^1 - \Gamma_2}{\Gamma_1 - \Gamma_2} . \tag{2.21}$$

The parameter δ measures the importance of the memory effects. A value of δ close to 1 indicates that a truncation at the lowest order ($K_1 = 0$) is reliable.

From Eq. (2.14) we obtain a continued-fraction expansion for the relaxation time (1.9),

$$T = C^0(w=0) = \left[\gamma^0 - \cfrac{K_1}{\gamma^1 - \cfrac{K_2}{\gamma^2 - \cdots}} \right]^{-1} . \tag{2.22}$$

Before applying this calculational scheme to the model defined by (1.4) and (1.5) we wish to analyze the relation between (2.14) and an approximation proposed by Stratonovich.[17] This approximation will be used in Sec. III as a first step in the understanding of the non-white-noise problem. To the best of our knowledge, Stratonovich was the first author who tried to calculate the correlation function $C^0(s)$ for the model of (1.4) and (1.5). His calculation was based on a decoupling ansatz for the hierarchy of equations for the correlation function. The first of these equations is, for a general L,

$$\frac{d}{ds}\langle q(t+s)q(t)\rangle_{\rm st}=\langle [L^{\dagger}q(t+s)]q(t)\rangle_{\rm st}\ . \qquad (2.23)$$

The right-hand side (rhs) of (2.23) contains, in general, correlation functions of the form $\langle q^n(t+s)q(t)\rangle_{\rm st}$. Stratonovich made the assumption that

$$C_n^0(s)=(\langle q^n(t+s)q(t)\rangle_{\rm st}-\langle q^n\rangle_{\rm st}\langle q\rangle_{\rm st})/(\langle q^{n+1}\rangle_{\rm st}-\langle q^n\rangle_{\rm st}\langle q\rangle_{\rm st})$$

decays in the same way as $C^0(s)$. The ansatz consists, then, in replacing $C_n^0(s)$ by $C^0(s)$ in the rhs of (2.23). For a general case the decoupling ansatz can be written as

$$\frac{\langle [L^{\dagger}q(t+s)]q(t)\rangle-\langle L^{\dagger}q(t)\rangle\langle q(t)\rangle}{\langle [L^{\dagger}q(t)]q(t)\rangle-\langle L^{\dagger}q(t)\rangle\langle q(t)\rangle}=\frac{\langle q(t+s)q(t)\rangle-\langle q(t)\rangle^2}{\langle q^2(t)\rangle-\langle q(t)\rangle^2}\ . \qquad (2.24)$$

In the stationary state, in which we are interested, $\langle L^{\dagger}q(t)\rangle=\langle\dot{q}(t)\rangle=0$. Equation (2.24) is an identity for $s=0$ and leads to a linear dynamics in the sense that (2.23) with (2.24) becomes a linear equation for $\langle q(t+s)q(t)\rangle_{\rm st}$. The solution of this linear equation gives precisely (2.11). The zeroth-order approximation of (2.14) can then be interpreted as a linearization given by the ansatz (2.24). In the same way, the time constants $\gamma^1,\gamma^2,\ldots$ can be identified as inverse relaxation times of the correlation functions of $QL^{\dagger}\delta q, Q_1L^{\dagger}QL^{\dagger}\delta q,\ldots$ calculated by linearizing through a Stratonovich-like ansatz the equations they satisfy. This is equivalent to neglect memory effects at each step of the continued-fraction expansion calculation. As a consequence we can interpret a truncation of the continued-fraction expansion (2.14) at some step n as given by a decoupling ansatz on the corresponding Q_n subspace. This truncation is reliable whenever $\gamma^n \gg K_{n+1}/\gamma^{n+1}$. This will always occur if γ^{n+1} is sufficiently large, which means a small linearized relaxation time at the $n+1$ step; that is, a fast variable in the $n+1$ projection.

It is, finally, interesting to note that the ansatz (2.24) in the steady state corresponds to the following approximation for the stationary joint probability density of the process $P_{\rm st}(q,t+s;q',t)$:

$$P_{\rm st}(q,t+s;q',t)=P_{\rm st}(q)P_{\rm st}(q')\times\left[1+\frac{\langle\delta q(t+s)\delta q(t)\rangle_{\rm st}}{\langle(\delta q)^2\rangle_{\rm st}^2}\delta q(t+s)\delta q(t)\right]. \qquad (2.25)$$

We now consider the particular model of (1.4) and (1.5). The stationary moments needed to calculate (2.8) and (2.15)—(2.18) can be obtained from (1.6):

$$\langle q^n\rangle_{\rm st}=(2D)^{n/2}\Gamma^{-1}\left[\frac{1}{2D}\right]\Gamma\left[\frac{1}{2D}+\frac{n}{2}\right]. \qquad (2.26)$$

For this model, the eigenvalue spectrum of (2.2) is exactly known.[9] With the use of this spectrum the relaxation time can be explicitly calculated for small values of D. We have (see the Appendix)

$$T^{-1}=2-\tfrac{7}{4}D+O(D^2)\ . \qquad (2.27)$$

This result will be compared with (2.22).

The zeroth-order approximation of (2.22) gives, for (1.4) and (1.5),

$$T^{-1}=\gamma^0=\frac{D}{1-\langle q\rangle_{\rm st}^2}\ . \qquad (2.28)$$

This is shown in Fig. 1. It shows a monotonic increase with D in disagreement with the numerical simulation. Nevertheless, the limiting value for small D, $\lim_{D\to 0}T^{-1}=2$, is in agreement with (2.27) and with the simulation.

The first-order approximation of (2.22) gives

$$T^{-1}=\gamma^0-\frac{K_1}{\gamma^1}\ , \qquad (2.29)$$

where

$$K_1=\frac{2D+5D^2}{1-\langle q\rangle_{\rm st}^2}-(\gamma^0)^2\ , \qquad (2.30)$$

$$\gamma^1=\frac{1}{(2+5D-\gamma^0)}\times[4+38D+61D^2-\gamma^0(4+10D)+(\gamma^0)^2]\ . \qquad (2.31)$$

The value of T^{-1} [Eq. (2.29)] is also shown in Fig. 1. It is seen that for small D, T^{-1} decreases with D in agreement with the simulation. The expansion of (2.29) to first order in D agrees with the exact result (2.27). The correlation

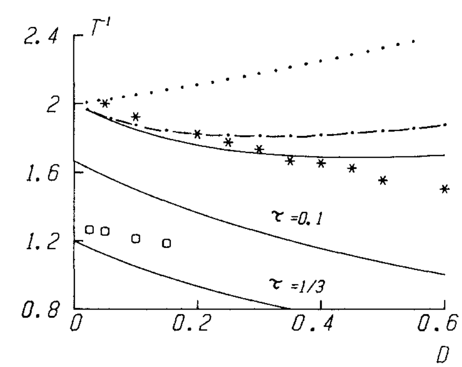

FIG. 1. Inverse of the relaxation time vs the noise intensity for different values of the correlation time of the noise. Asterisks and circles correspond to the numerical simulation of Ref. 1 for $\tau=0$ and $\frac{1}{3}$, respectively. Analytical results are shown as follows. $\cdots$, white noise ($\tau=0$) in zeroth-order approximation (2.28); —·—, white noise ($\tau=0$) in first-order approximation (2.29); and (——), white noise ($\tau=0$) in second-order approximation. The other two continuous lines correspond to (3.24) with $\tau=0,1;\frac{1}{3}$ as indicated.

function corresponding to (2.29) is given by (2.19). The dominant term in (2.19) is the second exponential $\delta\exp(-\Gamma_2 t)$.[7] Γ_2^{-1} gives a characteristic time of decay of the correlation function. Nevertheless, to obtain the correct relaxation time to order D [Eq. (2.27)] it is necessary to consider the contribution from the two exponentials in (2.19). If the term $(1-\delta)\exp(-\Gamma_1 t)$ is neglected, the inverse correlation time is identified with Γ_2, which for small D is

$$\Gamma_2=2-4D+O(D^2)\ . \tag{2.32}$$

This value of Γ_2 coincides with the smallest nonvanishing eigenvalue of the Fokker-Planck equation (2.1) (see the Appendix). Therefore, the identification of T^{-1} with the smallest eigenvalue is only correct in the limit $D=0$.

The inverse relaxation time T^{-1} obtained from (2.29) goes through a minimum as a function of D, for $D\simeq 0.3$, and grows with D for larger D (Fig. 1). This is in disagreement with the numerical simulation of Ref. 1. To elucidate the usefulness of the general scheme we have computed the second-order approximation of (2.22). The result is also shown in Fig. 1. It gives a fair agreement with the simulation up to $D\simeq 0.4$.[25] It gives a larger range of values of D for which T grows with D. We then conclude that the expansion (2.22) is useful to calculate T for small D but it becomes of limited practical use for larger values of D, probably due to the important effect of the continuous part of the eigenvalue spectrum of (2.1). The asymptotic value of T as $D\to\infty$ can also be calculated from the eigenvalue spectrum. In the Appendix we obtain that

$$T^{-1}\to\frac{2}{\pi}+O(D^{-1})\ \text{ as } D\to\infty\ . \tag{2.33}$$

It is, finally, interesting to discuss the differences of our multiplicative white-noise model of (1.4) and (1.5) with the corresponding additive model with the same $v(q)$ and $g(q)=1$. In this case the model represents the overdamped motion of a Brownian particle of position q in a symmetric double-well potential. In this bistable situation the relaxation of a fluctuation implies an equilibration process between the population of the two wells. Therefore, the relaxation time is related to the escape time of one of the wells.[26] For small noise intensity the escape time is given by the inverse of the lowest nonvanishing eigenvalue of the Fokker-Planck operator (2.2), $\lambda_1^{-1}\sim e^{1/D}$.[27] For small noise intensity the relaxation time T is also essentially given by λ_1^{-1}. This can be checked explicitly for a rectangular double-well potential for which the eigenfunctions and eigenvalues of L are exactly known.[28,29] Therefore, for additive fluctuations $T\to\infty$ as $D\to 0$ and for small D, T decreases with D. In the case of multiplicative fluctuations we have seen that T has a finite value as $D\to 0$ and that T increases with D. These two differences can be understood from a physical and intuitive point of view. The divergence of T as $D\to 0$ is due in the additive case to the impossibility of escaping from one of the wells in the absence of fluctuations. For multiplicative fluctuations there is a boundary at $q=0$ which cannot be crossed by the stochastic trajectories. As a consequence, only one of the two wells is accessible and the relaxation time has an asymptotic finite value as $D\to 0$. In the additive noise case a large fluctuation, in which the process overcomes the energy barrier at $q=0$, becomes more probable when D increases and therefore T decreases. A heuristic argument to understand the slowing down for the multiplicative case can be given rewriting (1.6) in terms of a noise modified potential $V(q,D)$:

$$P_{\rm st}^0(q)=2\left[\frac{1}{2D}\right]^{1/2D}\Gamma^{-1}\left[\frac{1}{2D}\right]\exp\left[-\frac{V(q,D)}{D}\right]. \tag{2.34}$$

In the additive case (besides modifications of the normalization factor) V would be a symmetric double-well potential independent of D. In the multiplicative case $V(q,D)$ has a single minimum which becomes less pronounced as D increases. This implies a slower relaxation analogous to the slowing down which exists for additive noise and fixed value of D in a single-well potential when the pump parameter goes to zero [$\alpha<0$ and going to zero in (1.4)]. At $D=1$ the minimum of $V(q,D)$ reaches the boundary at $q=0$ and the slowing down phenomenon becomes less important, with T going to an asymptotic constant value independent of D.

III. ORNSTEIN-UHLENBECK NOISE

In the case in which $\xi(t)$ is the Ornstein-Uhlenbeck process (1.2), the problem becomes non-Markovian. This non-Markovian behavior is due to the fact that $\xi(t)$ has a slower time evolution than in the white-noise case and the random values of $\xi(t)$ at different times are correlated. As a consequence $\xi(t)$ evolves in less-random trajectories and there are memory effects in the evolution of the process $q(t)$. The memory effects clearly cause the slowing down phenomenon observed in the numerical simulation of Ref. 1 (see Fig. 1) when τ increases from its zero value in the white-noise limit. It is important to note that an increase of the relaxation time with respect to its white-noise limit also exists for simpler non-Markovian processes as for example a linear process [$v(q)=-aq$] with additive noise.[19] Here we want to discuss by means of an explicit calculation of T how this effect is modified in the presence of nonlinearities and multiplicative noise. The calculational scheme of Sec. II cannot be directly applied to a non-Markovian problem. We first propose a generalization of that scheme for the general model (1.1). The exact equations satisfied by the probability density and the correlation function are not known. For a small value of the correlation time of the noise τ, approximate equations can be obtained by a systematic expansion in powers of τ. For the probability density and to first order in τ the equation satisfied by the probability density has been derived in Ref. 1. This equation is

$$\frac{\partial}{\partial t}P(q,t)=L(\tau)P(q,t)\ , \tag{3.1}$$

where

$$L(\tau)=-\frac{\partial}{\partial q}v(q)+D\frac{\partial}{\partial q}g(q)\frac{\partial}{\partial q}h(q)\ , \tag{3.2}$$

$$h(q)=g(q)\left[1+\tau g(q)\left[\frac{v(q)}{g(q)}\right]'\right]. \tag{3.3}$$

Due to the fact that the process is non-Markovian, the joint probability distribution $P_2(q,t;q't')$ does not satisfy (3.1) and $L(\tau)$ does not determine the evolution of the correlation function.[19,30] To first order in τ the equation satisfied by the correlation function has been derived in Refs. 19 and 30 in different contexts:

$$\begin{aligned}\frac{d}{ds}\langle q(t+s)q(t)\rangle_{st} &= \langle [L^{\dagger}(\tau)q(t+s)]q(t)\rangle_{st} \\ &\quad + D\exp(-s/\tau)\langle g(q(t+s))h(q(t))\rangle_{st}\,.\end{aligned} \tag{3.4}$$

The correlation function cannot be expressed in the form (2.3) due to the existence of the second term in the rhs of (3.4). It will be shown below that the slowing down phenomenon as a function of τ is mainly due to the existence of this term which disappears in the white-noise limit ($\tau \to 0$).

As a first step toward understanding the problem we consider the limit as $D\to 0$. From the discussion in Sec. II we expect that an extension of Stratonovich's ansatz should give a correct result: We use (2.24) with $L^{\dagger}$ replaced by $L^{\dagger}(\tau)$ to linearize the first term in the rhs of (3.4). The resulting equation can be formally integrated to give

$$C(s)=\exp\left[\frac{\langle [L^{\dagger}(\tau)\delta q]\delta q\rangle_{st}}{\langle(\delta q)^2\rangle_{st}}s\right]\langle(\delta q)^2\rangle_{st}+\int_0^s ds'\exp\left[\frac{\langle [L^{\dagger}(\tau)\delta q]\delta q\rangle_{st}}{\langle(\delta q)^2\rangle_{st}}(s-s')\right]D\exp\left[-\frac{s'}{\tau}\right]\langle g(q(t+s'))h(q(t))\rangle_{st}\,. \tag{3.5}$$

By successive partial integration in the second term in the rhs of (3.5) we obtain

$$\begin{aligned}C(s)=&\exp\left[\frac{\langle [L^{\dagger}(\tau)\delta q]\delta q\rangle_{st}}{\langle(\delta q)^2\rangle_{st}}s\right]\langle(\delta q)^2\rangle_{st}+D\tau\exp\left[\frac{\langle [L^{\dagger}(\tau)\delta q]\delta q\rangle_{st}}{\langle(\delta q)^2\rangle_{st}}s\right]\langle g^2(q)\rangle_{st} \\ &-D\tau\exp(-s/\tau)\langle g(q(t+s))g(q(t))\rangle_{st}+O(\tau^2)\,,\end{aligned} \tag{3.6}$$

where to lowest order in τ we have replaced $h(q)$ by $g(q)$. The relaxation time T to order τ is obtained from (3.6) as

$$\begin{aligned}T&=-\frac{\langle(\delta q)^2\rangle_{st}}{\langle [L^{\dagger}(\tau)\delta q]\delta q\rangle_{st}}\left[1+\tau D\frac{\langle g^2(q)\rangle_{st}}{\langle(\delta q)^2\rangle_{st}}\right]+O(\tau^2) \\ &=[\gamma^0(\tau)]^{-1}\left[1+\tau D\frac{\langle g^2(q)\rangle_{st}}{\langle(\delta q)^2\rangle_{st}}\right]+O(\tau^2)\,,\end{aligned} \tag{3.7}$$

where $[\gamma^0(\tau)]^{-1}$ is the time constant defined in (2.8) for the white-noise case but now calculated with $L^{\dagger}$ replaced by $L^{\dagger}(\tau)$. In this limit of $D\to 0$ we observe that the effect of having a correlation time $\tau\neq 0$ is twofold. First, the change in the evolution operator for the probability density produces a replacement of $L^{\dagger}$ by $L^{\dagger}(\tau)$. Second, there is a new contribution proportional to τ which has its origin in the second term of the rhs of (3.4). We note that the last term in the rhs of (3.6) does not contribute to T in order τ. Because of this fact, (3.7) is also obtained if a Stratonovich-like ansatz is made for the last term in (3.4) or if in lowest order τ we directly set

$$\exp(-s/\tau)\langle g(q(t+s))h(q(t))\rangle_{st}\simeq\exp(-s/\tau)\langle g^2(q)\rangle_{st}\,. \tag{3.8}$$

This approximation is valid because, for small τ, and due to the exponential factor, the main contribution of this term is for $s\simeq 0$.

The expression for the relaxation time T, Eq. (3.7), can be further simplified: The evolution equation for $\langle(\delta q)^2\rangle$ can be written from (3.1) as

$$\begin{aligned}\frac{d}{dt}\langle(\delta q)^2\rangle &= 2\langle [L^{\dagger}(\tau)\delta q(t)]\delta q(t)\rangle \\ &\quad +2D\langle g(q(t))h(q(t))\rangle\,.\end{aligned} \tag{3.9}$$

Therefore, in the stationary state $\langle [L^{\dagger}(\tau)\delta q]\delta q\rangle_{st} = -D\langle g(q)h(q)\rangle_{st}$ and

$$[\gamma^0(\tau)]^{-1}=\frac{\langle(\delta q)^2\rangle_{st}}{D\langle g(q)h(q)\rangle_{st}}\,. \tag{3.10}$$

Substituting (3.10) in (3.7) we obtain

$$T=[\gamma^0(\tau)]^{-1}+\tau+O(\tau^2)\,. \tag{3.11}$$

As $D\to 0$, $[\gamma^0(\tau)]^{-1}$ is expected to be independent of τ. In this asymptotic limit (3.11) implies a simple shift of order τ of the value of T with respect to the white-noise limit. This shift comes entirely from the last term in the rhs of (3.4). It implies a slowing down for $D\to 0$ which has the same origin as that for a simple linear non-Markovian process: It is a pure non-Markovian effect.

For the particular model of (1.4) and (1.5), the stationary moments are calculated from the stationary solution of (3.1):[1]

$$\begin{aligned}\langle q^n\rangle_{st}=&\langle q^n\rangle^0_{st}\left[1-\tau\frac{1+2D}{2D}\right] \\ &+\langle q^{n+2}\rangle^0_{st}\tau\frac{2D+1}{D}-\langle q^{n+4}\rangle^0_{st}\frac{\tau}{2D}\,,\end{aligned} \tag{3.12}$$

where $\langle q^n\rangle^0_{st}$ are the moments in the white-noise case, given in (2.26). With these values

$$\gamma^0(\tau)=2+D(\tfrac{1}{2}-9\tau)+O(\tau^2,\,D^2) \tag{3.13}$$

and from (3.11)

$$T(D=0)=\tfrac{1}{2}+\tau+O(\tau^2)\,. \tag{3.14}$$

The shift of the value of T at $D=0$ is explicitly seen here and it is in agreement with the numerical simulation of Ref. 1.

We now extend the general result (3.7) for values of $D\neq0$ going beyond the approximation of Stratonovich. We will see that the basic structure of (3.7) regarding the effect of $\tau\neq0$ remains valid for $D\neq0$. With the same argument as that in (3.8) we make in (3.4) the approximation

$$\exp(-s/\tau)\langle g(q(t+s))h(q(t))\rangle_{st} \simeq \exp(-s/\tau)\frac{\langle g^2(q)\rangle_{st}}{\langle(\delta q)^2\rangle_{st}}\langle \delta q(t+s)\delta q(t)\rangle_{st} \quad (3.15)$$

with (3.15) Eq. (3.4) can be rewritten as

$$\frac{d}{ds}C(s)=\left\langle \left[\left[L^{\dagger}(\tau)+D\exp(-s/\tau)\frac{\langle g^2(q)\rangle_{st}}{\langle(\delta q)^2\rangle_{st}}\right]\delta q(t+s)\right]\delta q(t)\right\rangle_{st} . \quad (3.16)$$

Defining the operator

$$\mathscr{L}(s)=L(\tau)+D\exp(-s/\tau)\frac{\langle g^2(q)\rangle_{st}}{\langle(\delta q)^2\rangle_{st}} \quad (3.17)$$

and integrating (3.16) we have

$$\begin{aligned} C(s)&=\int dq\, P_{st}(q)\delta q \exp\left[\int_0^s ds'\mathscr{L}^{\dagger}(s')\right]\delta q\\ &=\exp\left[-\tau\frac{D\langle g^2(q)\rangle_{st}}{\langle(\delta q)^2\rangle_{st}}[\exp(-s/\tau)-1]\right]\\ &\quad\times\int dq\, P_{st}(q)\delta q\exp[L^{\dagger}(\tau)s]\delta q . \end{aligned} \quad (3.18)$$

Expanding the exponential prefactor to order τ and defining

$$\bar{C}(s)=\int dq\, P_{st}(q)\delta q\exp[L^{\dagger}(\tau)s]\delta q \quad (3.19)$$

we obtain

$$C(s)=\left[1+\tau D\frac{\langle g^2(q)\rangle_{st}}{\langle(\delta q)^2\rangle_{st}}[1-\exp(-s/\tau)]\right]\bar{C}(s)+O(\tau^2) . \quad (3.20)$$

Comparing (3.19) with (2.3) we see that $\bar{C}(s)$ is the correlation function of a Markovian process with Fokker-Planck operator $L(\tau)$. Therefore, the relaxation time $\bar{T}$ of $\bar{C}(s)$ is given by (2.22) using $L^{\dagger}(\tau)$ instead of $L^{\dagger}$. Then from (3.20) we obtain

$$\begin{aligned} T&=\bar{T}\left[1+\tau D\frac{\langle g^2(q)\rangle_{st}}{\langle(\delta q)^2\rangle_{st}}\right]+O(\tau^2)\\ &=\left[\gamma^0(\tau)-\frac{K_1(\tau)}{\gamma^1(\tau)-\cdots}\right]^{-1}\\ &\quad\times\left[1+\tau D\frac{\langle g^2(q)\rangle_{st}}{\langle(\delta q)^2\rangle_{st}}\right]+O(\tau^2) , \end{aligned} \quad (3.21)$$

where $\gamma^1(\tau),K_1(\tau),\ldots$ are defined as $\gamma^0(\tau)$ in (3.7), that is, by (2.8) and (2.15)–(2.18) with $L^{\dagger}$ replaced by $L^{\dagger}(\tau)$. The stationary moments in (3.21) are calculated from (3.12). As we anticipated, (3.21) has the same structure as (3.7). The two different effects of τ discussed in (3.7) are the ones that appear in (3.21). The term explicitly proportional to τ is the same as that in (3.7) but $[\gamma^0(\tau)]^{-1}$ is now replaced by $\bar{T}$. The contribution $\tau D\langle g^2(q)\rangle_{st}/\langle(\delta q)^2\rangle_{st}$ appears disentangled from the other contributions. It comes from the last term in the rhs of (3.4) and it is the only τ-dependent contribution that exists for a linear process. $\bar{T}$ is calculated as in the white-noise case, now using $L^{\dagger}(\tau)$ instead of $L^{\dagger}(\tau=0)$.

We have applied the formula (3.21) to the model of (1.4) and (1.5) in the first-order approximation ($K_2=0$). From the discussion of the white-noise case we expect good results of this approximation for small enough values of D. We obtain

$$K_1(\tau)=9D(1-2\tau)+O(\tau^2,\, D^2) , \quad (3.22)$$

$$\frac{K_1(\tau)}{\gamma^1(\tau)}=\frac{9D}{4}(1-2\tau)+O(\tau^2,\, D^2) . \quad (3.23)$$

The expansion of (3.21) to first order in D gives

$$T=\tfrac{1}{2}+\tau+D(\tfrac{7}{16}+\tfrac{9}{4}\tau)+O(\tau^2,\, D^2) . \quad (3.24)$$

For $\tau=0$, (3.24) reduces to (2.27). The term proportional to $D\tau$ originates from the τ dependence of $L^{\dagger}(\tau)$ in Eq. (3.4) and also from the second term in the rhs of (3.4). This result, in addition to the τ dependence of (3.14), identifies the last term in (3.4) as crucial for the understanding of T when $\tau\neq0$. In Fig. 1 we have plotted (3.24) for $\tau=\frac{1}{10};\frac{1}{3}$. Our result accounts qualitatively for the slowing down observed in the numerical simulation of Ref. 1 when increasing D or τ. We have also included in Fig. 1 the available numerical data for $\tau=\frac{1}{3}$. A detailed comparison does not seem to be completely justified for this value of τ. Nevertheless, we note that from (3.24) we obtain $T^{-1}(D=0,\tau=\frac{1}{3})=1.2$ which is in good agreement with the simulation.

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APPENDIX

For the model of (1.4) and (1.5) in the white-noise case the correlation function can be exactly written as[9,14]

$$\langle q(s)q(0)\rangle_{st}=\sum_{n=0}^{n_0}\alpha_n\exp(-\lambda_n s) + \int_0^{\infty}dx\, 2x\alpha(x)\exp[-\lambda(x)s] , \quad (A1)$$

where the eigenvalue spectrum is[9]

$$\lambda_n=2n-4n^2D,\quad n\le\frac{1}{4D}=n_0 \quad (A2)$$

$$\lambda(x)=\frac{1}{4D}+4Dx^2 \quad (A3)$$

and

$$\alpha_n=\frac{4D}{\pi}\left[\frac{1}{4D}-n\right]\frac{\Gamma^2(-n+\frac{1}{2}+1/2D)\Gamma^2(n+1/2D)}{n!\Gamma(1/2D)\Gamma(-n+1+1/2D)}\,, \tag{A4}$$

$$\alpha(x)=\frac{D\sinh(2\pi x)}{\pi^3}\frac{|\Gamma(ix-1/4D)\Gamma(ix+\frac{1}{2}+1/4D)\Gamma(ix+\frac{1}{2}+1/4D)|^2}{\Gamma(1/2D)}\,. \tag{A5}$$

For very small D, $n_0\to\infty$ and only the discrete part of the spectrum will contribute, so that

$$T=\frac{\sum_{n=1}^{\infty}\alpha_n/\lambda_n}{\sum_{n=1}^{\infty}\alpha_n}\,. \tag{A6}$$

From (A4) it is easy to see that for small D, $\alpha_n\sim D^n$. Neglecting terms of order D^2, (A6) reduces to

$$T=\frac{\alpha_1/\lambda_1+\alpha_2/\lambda_2}{\alpha_1+\alpha_2}+O(D^2)\,. \tag{A7}$$

The explicit values of α_1 and α_2 necessary to evaluate T in first order in D are, from (A4),

$$\alpha_1=\frac{D}{2}+O(D^2)\,, \tag{A8}$$

$$\alpha_2=\tfrac{9}{8}D^2+O(D^3)\,. \tag{A9}$$

Equation (2.27) is obtained from (A7)—(A9) and (A2).

In the limit, $D\to\infty$, $n_0\to 0$, and the continuous part of the spectrum contributes in the calculation of T.

The integral in (A1) is evaluated in this limit using the following approximations:

$$\Gamma\left[ix+\frac{1}{2}+\frac{1}{2D}\right]\Bigg|_{D\to\infty}\to\Gamma(ix+\tfrac{1}{2}) \tag{A10}$$

and[31]

$$|\Gamma(ix+\tfrac{1}{2})|^2=\frac{\pi}{\cosh\pi x}\,, \tag{A11}$$

$$\left|\Gamma\left[ix-\frac{1}{4D}\right]\right|^2\Bigg|_{D\to\infty}\simeq\Gamma\left[-\frac{1}{4D}\right]^2\times\frac{1}{(1+x^2 4^2 D^2)}\frac{\pi x}{\sinh(\pi x)}\,. \tag{A12}$$

With (A11) and (A12) the correlation function (A1) is expressed as

$$C(s)=\frac{4D\Gamma^2(-1/4D)}{\Gamma(1/2D)}\int_0^\infty dx\frac{x^2\exp[-\lambda(x)s]}{\cosh\pi x(1+4^2x^2D^2)}\,. \tag{A13}$$

After lengthy but straightforward algebra and taking the limit $D\to\infty$, we arrive at

$$\int_0^\infty ds\,C(s)=2\int_0^\infty dy\frac{y^2}{(1+y^2)^2}+O\left[\frac{1}{D}\right]=\frac{\pi}{2}+O\left[\frac{1}{D}\right]\,. \tag{A14}$$

The value of $\langle(\delta q)^2\rangle_{st}$ is obtained from (2.26):

$$\langle(\delta q)^2\rangle_{st}=1+O\left[\frac{1}{D}\right]\,. \tag{A15}$$

Finally, substituting (A14) and (A15) in (1.9) and (1.10) we obtain the asymptotic value (2.33) for T.

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