Relaxed Abduction: Robust Information Interpretation for Incomplete Models

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Abstract. This paper introduces relaxed abduction, a novel non-standard reasoning task for description logics. Although abductive reasoning over description logic knowledge bases has been applied successfully to various information interpretation tasks, it typically fails to provide adequate (or even any) results when confronted with spurious information or incomplete models. Relaxed abduction addresses this flaw by ignoring such pieces of information automatically based on a joint optimization of the sets of explained observations and required assumptions. We present a method to solve relaxed abduction over $\mathcal{EL}^+$ TBoxes based on the notion of multi-criterion shortest hyperpaths.

Keywords: abduction, interpretation, non-standard reasoning

1 Introduction

Abduction was introduced in the late 19th century by Charles Sanders Pierce as an inference scheme aimed at deriving potential explanations for some observation [7]. It is conveniently expressed by the derivation rule

$$\frac{\phi \supset \omega \qquad \omega}{\phi}$$

which can be understood as an inversion of the modus ponens rule that permits to derive ϕ as a hypothetical explanation for the occurrence of ω, given that the presence of ϕ in some sense justifies ω. Note that this general formulation does not presuppose any causality between ϕ and ω; various notions of how ϕ sanctions the presence of ω give rise to different notions of abductive inference such as the set-cover-based approach, logic-based approaches, and the knowledge-level approach (see [12] for a survey). This paper focuses on logic-based abduction over $\mathcal{EL}^+$ TBoxes, however all results except the algorithm presented in Sect. 3 carry over to other logic-based representation schemes straightforwardly.

Due to its hypothetical nature, an abduction problem typically does not have a single solution but a collection of alternative answers $A_1, A_2, \ldots, A_k$ among which optimal solutions are selected by means of a preference order $\preceq$. We denote

A_i being not worse than A_j by $A_i \preceq A_j$, indifference ($A_i \preceq A_j \wedge A_j \preceq A_i$) is abbreviated by $A_i \simeq A_j$, and strict preference ($A_i \preceq A_j \wedge A_i \not\simeq A_j$) by $A_i \prec A_j$. Then a (normal) preferential abduction problem can be defined as follows:

Definition 1 (Preferential abduction problem $\mathcal{PAP} = (\mathcal{T}, \mathcal{A}, \mathcal{O}, \preceq_{\mathcal{A}})$). *Given a set of axioms $\mathcal{T}$ called the theory, a set of abducible axioms $\mathcal{A}$, a set $\mathcal{O}$ of axioms representing observations such that $\mathcal{T} \not\models \mathcal{O}$, and a (not necessarily total) order relation $\preceq_{\mathcal{A}} \subseteq \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A})$, determine all $\preceq_{\mathcal{A}}$-minimal sets $A \subseteq \mathcal{A}$ such that $\mathcal{T} \cup A$ is consistent and $\mathcal{T} \cup A \models \mathcal{O}$.*

Typical preference orders over sets include subset-minimality ($A_i \preceq^{\mathrm{s}} A_j \leftrightarrow A_i \subseteq A_j$), minimum cardinality ($A_i \preceq^{\mathrm{c}} A_j \leftrightarrow |A_i| \leq |A_j|$), and weighting-based orders defined by a function w that assigns numerical weights to subsets of $\mathcal{A}$ ($A_i \preceq^{\mathrm{w}} A_j \leftrightarrow w(A_i) \leq w(A_j)$). The first two orders prefer a set A over any of its supersets, this monotonicity property is formalized in Def. 2.

Definition 2 (Monotone and anti-monotone order). *An order $\preceq$ ($\prec$) over sets is monotone (strictly monotone) for set inclusion if and only if $S' \subseteq S$ implies $S' \preceq S$ ($S' \subset S$ implies $S' \prec S$). Conversely, $\preceq$ ($\prec$) is anti-monotone (strictly anti-monotone) for set inclusion if and only if $S' \supseteq S$ implies $S' \preceq S$ ($S' \supset S$ implies $S' \prec S$).*

Applications of abductive information interpretation using a formal domain model include media interpretation [4] and diagnostics for complex technical systems such as production machinery [9]. These domains are characterized by an abundance of low-level observations due to a large number of sensors whereas the model is often unelaborate or incomplete. The next example illustrates how the classical definition of abduction may fail to handle such situations adequately.

Example 1 (Sensitivity to spurious information). Consider the diagnostic unit of a production system whose model states that a fluctuating power supply manifests by intermittent outages of the main control unit while the communication links remain functional and the mechanical gripper of the production system is unaffected (the observations entailed by the diagnosis). Assume a new vibration sensor additionally observes low-frequency vibrations of the system. If the diagnostic model has not been extended yet to encompass these observations, the additional data will in fact distract the diagnostic process and invalidate the diagnosis concerning the power supply, although it might be completely unrelated.

This flaw rests on the requirement that every single observation $o_i \in \mathcal{O}$ be entailed by an admissible solution. It severely restricts the practical applicability of logic-based abduction to real-world industrial applications where an ever-growing amount of sensor data almost inevitably generates pieces of information that the model cannot account for. We therefore extend logic-based abduction in Sect. 2 to handle such cases in a more flexible yet formally sound way, and propose a method to solve such extended abduction problems expressed in the description logic $\mathcal{EL}^+$ in Sect. 3. Section 4 contrasts our proposal with relevant related work on logics and abduction, and we conclude in Sect. 5.

2 Relaxed Abduction

While for very simple models it is possible to identify and remove spurious information in a preprocessing step, this is not feasible for reasonably complex models since the (ir-)relevance of a piece of information depends on the interpretation and is thusly not known beforehand. We therefore propose a general approach based on the intuition that spurious and missing information are two complementary facets of information imperfection and should thus be treated similarly: In addition to assuming information as needed based on the set of abducibles $\mathcal{A}$, relaxed abduction ignores observations from $\mathcal{O}$ during hypotheses generation if required. This intuition is formalized in the next definition.

Definition 3 (Relaxed abduction problem $\mathcal{RAP} = (\mathcal{T}, \mathcal{A}, \mathcal{O}, \preceq_{\mathcal{A}}, \preceq_{\mathcal{O}})$). *Given a set of axioms $\mathcal{T}$ called the theory, a set of abducible axioms $\mathcal{A}$, a set $\mathcal{O}$ of axioms representing observations such that $\mathcal{T} \not\models \mathcal{O}$, and two (not necessarily total) order relations $\preceq_{\mathcal{A}} \subseteq \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A})$ and $\preceq_{\mathcal{O}} \subseteq \mathcal{P}(\mathcal{O}) \times \mathcal{P}(\mathcal{O})$, determine all $\preceq$-minimal tuples $(A, O) \in \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{O})$ such that $\mathcal{T} \cup A$ is consistent and $\mathcal{T} \cup A \models O$. The order $\preceq$ is defined based on $\preceq_{\mathcal{A}}$ and $\preceq_{\mathcal{O}}$ as follows:*

- $(A, O) \simeq (A', O') \leftrightarrow A \simeq_{\mathcal{A}} A' \wedge O \simeq_{\mathcal{O}} O'$
- $(A, O) \prec (A', O') \leftrightarrow (A \preceq_{\mathcal{A}} A' \wedge O \prec_{\mathcal{O}} O') \vee (A \prec_{\mathcal{A}} A' \wedge O \preceq_{\mathcal{O}} O')$
- $(A, O) \preceq (A', O') \leftrightarrow ((A, O) \prec (A', O')) \vee ((A, O) \simeq (A', O'))$

Intuitively, a good solution will have high expressive power regarding the observations while being as non-assumptive as possible, which suggests to chose $\preceq_{\mathcal{A}}$ monotone and $\preceq_{\mathcal{O}}$ anti-monotone for set inclusion, respectively. The following example uses one such combination to solve the problem presented in Ex. 1.

Example 2 (Sensitivity to irrelevant data (cont.)). Using inclusion as order criterion over sets, we let $A \preceq_{\mathcal{A}} A' \leftrightarrow A \subseteq A'$ and $O \preceq_{\mathcal{O}} O' \leftrightarrow O \supseteq O'$. As intended, the resulting order $\preceq$ gives rise to the minimal solution which explains all observations but the vibrations and only requires to assume the diagnosis, namely a fluctuating power supply.

Proposition 1 (Conservativeness). *$A \subseteq \mathcal{A}$ is a solution to the preferential abduction problem $\mathcal{PAP} = (\mathcal{T}, \mathcal{A}, \mathcal{O}, \preceq_{\mathcal{A}})$ if and only if $(A, \mathcal{O})$ is a solution to the relaxed abduction problem $\mathcal{RAP} = (\mathcal{T}, \mathcal{A}, \mathcal{O}, \preceq_{\mathcal{A}}, \preceq_{\mathcal{O}})$ for an arbitrary order $\preceq_{\mathcal{O}}$ that is anti-monotone for set inclusion.*

Proof. Assume A solves $\mathcal{PAP}$. Then $\mathcal{T} \cup A$ is consistent, $\mathcal{T} \cup A \models \mathcal{O}$, and A is $\preceq_{\mathcal{A}}$-minimal. As $\preceq_{\mathcal{O}}$ is anti-monotone for set inclusion $\mathcal{O}$ is naturally $\preceq_{\mathcal{O}}$-minimal; $(A, \mathcal{O})$ is therefore $\preceq$-minimal and thus solves $\mathcal{RAP}$.
Conversely if $(A, \mathcal{O})$ solves $\mathcal{RAP}$ then $\mathcal{T} \cup A$ is consistent, $\mathcal{T} \cup A \models \mathcal{O}$, and $(A, \mathcal{O})$ is $\preceq$-minimal. Assume $A' \prec_{\mathcal{A}} A$ s. t. $A' \subseteq \mathcal{A}$, $\mathcal{T} \cup A'$ is consistent, $\mathcal{T} \cup A' \models \mathcal{O}$. Then $(A', \mathcal{O}) \prec (A, \mathcal{O})$, contradicting $\preceq$-minimality of $(A, \mathcal{O})$. □

Conservativeness states that, under natural conditions, relaxed abduction is guaranteed to reproduce all (if any) solutions of the corresponding standard abduction problem. Since $\preceq_{\mathcal{A}}$ and $\preceq_{\mathcal{O}}$ will typically represent competing optimization objectives, it is convenient to treat relaxed abduction as a bi-criterion optimization problem. $\preceq$-minimal solutions then correspond to Pareto-optimal points in the space of all combinations (A, O) meeting the logical requirements of a solution (consistency and entailment) as shown next.

Proposition 2 (Pareto-optimality of $\mathcal{RAP}$). *Let $\mathcal{RAP} = (\mathcal{T}, \mathcal{A}, \mathcal{O}, \preceq_{\mathcal{A}}, \preceq_{\mathcal{O}})$ be a relaxed abduction problem. (A^*, O^*) is a solution to $\mathcal{RAP}$ if and only if it is a Pareto-optimal element (subject to $\preceq_{\mathcal{A}}$ and $\preceq_{\mathcal{O}}$) of the candidate space $\{(A, O) \in \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{O}) \mid \mathcal{T} \cup A \models O \wedge \mathcal{T} \cup A \not\models \bot\}$.*

Proof. If (A^*, O^*) solves $\mathcal{RAP}$, then $\mathcal{T} \cup A^*$ is consistent and $\mathcal{T} \cup A^* \models O^*$ holds. (A^*, O^*) is thus an element of the explanation space (ES), furthermore (A^*, O^*) must be $\preceq$-minimal. Now assume (A^*, O^*) is not Pareto-optimal for ES, and let $(A', O') \in ES$ such that (w. l. o. g.) $A' \prec_{\mathcal{A}} A^*$ and $O' \preceq_{\mathcal{O}} O^*$. Then $(A', O') \prec (A^*, O^*)$, contradicting $\preceq$-minimality of (A^*, O^*). Thus, (A^*, O^*) is a Pareto-optimal element of the explanation space.
Analogously, let (A', O') be a Pareto-optimal element of ES. To show that the tuple is $\preceq$-minimal, let (A^*, O^*) be a solution to $\mathcal{RAP}$ such that $(A^*, O^*) \prec (A', O')$. Then w. l. o. g. $A^* \prec_{\mathcal{A}} A'$ and $O^* \preceq_{\mathcal{O}} O'$, contradicting Pareto-optimality of (A', O'). Conclusively, (A', O') must be $\preceq$-minimal and therefore solves $\mathcal{RAP}$. □

The next section presents an approach to solving relaxed abduction for $\mathcal{EL}^+$ that explicitly addresses the bi-criterial nature of the problem.

3 Solving Relaxed Abduction for $\mathcal{EL}^+$

The description logic $\mathcal{EL}^+$ is a member of the $\mathcal{EL}$ family of lightweight DLs for which subsumption can be tested in PTime [1]. $\mathcal{EL}^+$ concept descriptions are defined by $C ::= \top \mid A \mid C \sqcap C \mid \exists r.C$ (for $A \in N_{\mathrm{C}}$, $r \in N_{\mathrm{R}}$ a basic concept / role name); $\mathcal{EL}^+$ axioms are either concept inclusion axioms $C \sqcap D$ or role inclusion axioms $r_1 \circ \cdots \circ r_k \sqsubseteq r$ (C, D concept descriptions, $r, r_1, \ldots, r_k \in N_{\mathrm{R}}, k \geq 1$). Since any $\mathcal{EL}^+$ TBox can be normalized with only a linear increase in size, we can assume w. l. o. g. that all axioms are of one of the following forms **(NF1)** $A_1 \sqsubseteq B$, **(NF2)** $A_1 \sqcap A_2 \sqsubseteq B$, **(NF3)** $A_1 \sqsubseteq \exists r.B$, **(NF4)** $\exists r.A_2 \sqsubseteq B$, **(NF5)** $r_1 \sqsubseteq s$, and **(NF6)** $r_1 \circ r_2 \sqsubseteq s$ (for $A_1, A_2, B \in N_{\mathrm{C}}^{\top} = N_{\mathrm{C}} \cup \{\top\}$ and $r_1, r_2, s \in N_{\mathrm{R}}$). In addition to standard refutation-based tableau reasoning, the $\mathcal{EL}$ family allows for a completion-based reasoning scheme that explicitly derives valid subsumptions using a set of rules in the style of Gentzen's sequent calculus. The rules are depicted in Fig. 1, the graph-structure created by applying them to derive subsumptions provides the basis for our approach as shown in the next subsection.

In contrast to other work such as [3, 5] where observations and abducibles are represented by means of named concepts, we assume that both $\mathcal{A}$ and $\mathcal{O}$ are

$$\textbf{(CR1)}\ \frac{A \sqsubseteq A_1}{A \sqsubseteq B}\ [A_1 \sqsubseteq B \in \mathcal{T}]$$

$$\textbf{(CR2)}\ \frac{A \sqsubseteq A_1 \qquad A \sqsubseteq A_2}{A \sqsubseteq B}\ [A_1 \sqcap A_2 \sqsubseteq B \in \mathcal{T}]$$

$$\textbf{(CR3)}\ \frac{A \sqsubseteq A_1}{A \sqsubseteq \exists r.B}\ [A_1 \sqsubseteq \exists r.B \in \mathcal{T}]$$

$$\textbf{(CR4)}\ \frac{A \sqsubseteq \exists r.A_1 \qquad A_1 \sqsubseteq A_2}{A \sqsubseteq B}\ [\exists r.A_2 \sqsubseteq B \in \mathcal{T}]$$

$$\textbf{(CR5)}\ \frac{A \sqsubseteq \exists r_1.B}{A \sqsubseteq \exists s.B}\ [r_1 \sqsubseteq s \in \mathcal{T}]$$

$$\textbf{(CR6)}\ \frac{A \sqsubseteq \exists r_1.A_1 \qquad A_1 \sqsubseteq \exists r_2.B}{A \sqsubseteq \exists s.B}\ [r_1 \circ r_2 \sqsubseteq s \in \mathcal{T}]$$

$$\textbf{(IR1)}\ \frac{}{A \sqsubseteq A} \qquad \textbf{(IR2)}\ \frac{}{A \sqsubseteq \top}$$

Fig. 1. Completion rules for $\mathcal{EL}^+$

sets of DL axioms just like $\mathcal{T}$. In our experience the axiom-oriented representation provides greater flexibility and information reuse as well as being easier to understand for non-expert users; we furthermore conjecture without formal proof that the concept-based definition is subsumed by the axiom-based one.[3]

3.1 From Completion Rules to Hypergraphs

Since the rules shown in Fig. 1 constitute a sound and complete proof system for $\mathcal{EL}^+$, any normalized axiom set can be represented equivalently as a hypergraph whose vertices are all axioms of type **(NF1)** and **(NF3)** over the concept and role names used in the axiom set (corresponding to all statements admissible as premise or conclusion in a derivation step). The hyperedges are induced by instantiations of the rules **(CR1)-(CR6)**; for example an instantiation of **(CR4)** that derives $C \sqsubseteq F$ from $C \sqsubseteq \exists r.D$ and $D \sqsubseteq E$ using the axiom $\exists r.E \sqsubseteq F$ induces a hyperedge $e = (T(e), h(e), w(e))$ with $T(e) = \{C \sqsubseteq \exists r.D, D \sqsubseteq E\}$, $h(e) = C \sqsubseteq F$, and $w(e) = \exists r.E \sqsubseteq F$.

This correspondence can be extended to relaxed abduction problems as follows: Both $\mathcal{T}$ and $\mathcal{A}$ contain arbitrary $\mathcal{EL}^+$ normal form axioms that can justify

[3] First observe that $\mathcal{T} \models A_1 \sqcap \cdots \sqcap A_n \sqsubseteq O$ as required in [3] straightforwardly implies $\{\top \sqsubseteq A_1, \ldots, \top \sqsubseteq A_k\} \cup \mathcal{T} \models \top \sqsubseteq O$, i. e. a special case of our definition. Concept abduction and contraction introduced in [5] can conceptually be seen as abduction problems in the line of [3] with additional limitations on the solution A (namely $A = \{C, H\}$ in the former and $A = \{K, D\}$ in the latter case).

single derivation steps represented by a hyperedge (to simplify presentation we assume w. l. o. g. that $\mathcal{A} \cap \mathcal{O} = \emptyset$). Elements from $\mathcal{O}$ on the other hand represent information to be justified (i. e. derived), they therefore correspond to vertices of the hypergraph. This leads to the requirement that axioms in $\mathcal{O}$ may be of type **(NF1)** and **(NF3)** only – this restriction is however negligible in practice since **(NF2)**- and **(NF4)**-axioms can be translated into a **(NF1)**-axiom by introducing a new concept name, and role inclusion axioms are not required for expressing observations about domain objects. To keep track of required assumptions and explained observations, the hyperedges are labelled according to these criteria. This intuition is formalized in the next definition.

Definition 4 (Induced hypergraph $H_{\mathcal{RAP}}$). *Let $\mathcal{RAP} = (\mathcal{T}, \mathcal{A}, \mathcal{O}, \preceq_{\mathcal{A}}, \preceq_{\mathcal{O}})$ be a relaxed abduction problem. The weighted hypergraph $H_{\mathcal{RAP}} = (V, E)$ induced by $\mathcal{RAP}$ is defined by $V = \{(A \sqsubseteq B), (A \sqsubseteq \exists r.B) \mid A, B \in N_C^\top, r \in N_R\}$ where $V_\top = \{(A, A), (A, \top) \mid A \in N_C^\top\} \subseteq V$ denotes the set of terminal states, and E the set of all hyperedges $e = (T(e), h(e), w(e))$ s. t. there is an axiom $ax \in \mathcal{T} \cup \mathcal{A}$ justifying the derivation of $h(e) \in V$ from $T(e) \subseteq V$ due to one of* **(CR1)**-**(CR6)**. *The edge weight $w(e) = (A, O)$ is defined by*

$$A = \begin{cases} \{ax\} & \text{if } ax \in \mathcal{A}, \\ \emptyset & \text{otherwise} \end{cases}, \quad O = \begin{cases} \{h(e)\} & \text{if } h(e) \in \mathcal{O}, \\ \emptyset & \text{otherwise} \end{cases}.$$

Note that the size of $H_{\mathcal{RAP}}$ is bounded polynomially in $|N_C|$ and $|N_R|$. Checking whether a concept inclusion $D \sqsubseteq E$ ($C \sqsubseteq \exists r.D$) is derivable corresponds to checking if in the graph there exists a hyperpath from $V_\top$ to the vertex $D \sqsubseteq E$ ($C \sqsubseteq \exists r.D$). Intuitively, there is a hyperpath from X to t if there is a hyperedge connecting some set of nodes Y to t, and each $y_i \in Y$ is reachable from X via a hyperpath; Def. 5 formalizes this intuitive picture.

Definition 5 (Hyperpath). *$p_{X,t} = (V_{X,t}, E_{X,t})$ is a hyperpath in $H = (V, E)$ from X to t if and only if (i) $t \in X$ and $p_{X,t} = (\{t\}, \emptyset)$, or (ii) there is an edge $e \in E$ such that $h(e) = t, T(e) = \{y_1, \ldots, y_k\}$, p_{X,y_i} are hyperpaths from X to y_i, $V \supseteq V_{X,t} = \{t\} \cup \bigcup_{y_i \in T(e)} V_{X,y_i}$, and $E \supseteq E_{X,t} = \{e\} \cup \bigcup_{y_i \in T(e)} E_{X,y_i}$.*

3.2 Hyperpath Search for Relaxed Abduction

This section presents an algorithm for solving a relaxed abduction problem $\mathcal{RAP}$ by determining bi-criterion shortest hyperpaths. The graph algorithm extends a label-correcting algorithm for finding bi-criterion shortest paths in graphs, which is one of the most efficient algorithms known for this problem [14]. It compactly represents the graph using two lists S and R as proposed in [1], the entries are however extended with labels encoding the Pareto-optimal paths to the vertex found so far, and changes are propagated along the weighted edges using two operators called `meet` ($\otimes$) and `join` ($\oplus$). When saturation has terminated, the labels of all $\preceq$-minimal paths in $H_{\mathcal{RAP}}$ are collected in the set $MP(H_{\mathcal{RAP}}) := \bigcup_{v \in V} label(v)$. Algorithm 1 depicts the label propagation algorithm restricted to rule **(CR4)** only due to space limitations. Note that while the order of propagations is irrelevant for correctness, it may have a significant effect on the number of candidates generated: Finding near-optimal solutions early leads to many suboptimal solutions being dominated and therefore not propagated further. As a heuristic to improve performance, we therefore suggest to exhaustively apply $\mathcal{T}$-propagations first, and introduce assumptions only if no other propagation is possible.

Algorithm 1: Label correcting construction of $H_{\mathcal{RAP}}$

Data : $\mathcal{RAP} = (\mathcal{T}, \mathcal{A}, \mathcal{O}, \preceq_{\mathcal{A}}, \preceq_{\mathcal{O}})$, a relaxed abduction problem over $N_C^\top$ and N_R.
Result : $H_{\mathcal{RAP}}$, the induced hypergraph.

// initialization
foreach $r \in N_R$ **do**
  $\mathsf{R}\,(r) \leftarrow \emptyset$;
foreach $C \in N_C^\top$ **do**
  $\mathsf{S}\,(C) \leftarrow \{\top : \{(\emptyset,\emptyset)\}, C : \{(\emptyset,\emptyset)\}\}$;
// propagation
repeat
  changed $\leftarrow$ false;
  foreach $ax \in \mathcal{T} \cup \mathcal{A}$ **do**
    else if $ax = \exists r.A_2 \sqsubseteq B$ **then** // CR4
      foreach $A_1 \in N_C^\top$ *s. t.* $\mathsf{S}(A_1) \ni A_2 : L_{A_1,A_2}$ **do**
        foreach $A \in N_C^\top$ *s. t.* $\mathsf{R}(r) \ni (A, A_1) : L_{A,r,A_1}$ **do**
          $L \leftarrow \emptyset$;
          if $\mathsf{S}(A) \ni B : L_{A,B}$ **then** $L \leftarrow L_{A,B}$;
          $L^* \leftarrow$ join(L, meet(L_{A_1,A_2}, L_{A,r,A_1}, ax, $A \sqsubseteq B$));
          if $L^* \neq L$ **then**
            $\mathsf{S}(A) \leftarrow (\mathsf{S}(A) \setminus \{B : L_{A,B}\}) \cup \{B : L^*\}$;
            changed $\leftarrow$ true;
until changed = false;

Proposition 3 (Correctness). *The set of all solutions to a relaxed abduction problem $\mathcal{RAP} = (\mathcal{T}, \mathcal{A}, \mathcal{O}, \preceq_{\mathcal{A}}, \preceq_{\mathcal{O}})$ is given by the $\preceq$-minimal closure of MP($H_{\mathcal{RAP}}$) under component-wise union $(A, O) \uplus (A', O') := (A \cup A', O \cup O')$.*

Proof. Due to space limitations we can only present an outline of the proof here. Following the argumentation in [13, 8], it is clear that hyperpaths in $H_{\mathcal{RAP}}$ starting in $V_\top$ do indeed represent derivations, and that labels constructed from the hyperpaths can be used to encode relevant pieces of information used during that derivation. By Prop. 2, it then suffices to show that the proposed algorithm correctly determines the labels of all Pareto-optimal paths in $H_{\mathcal{RAP}}$ starting in

Function meet(L_1, L_2, *just*, *concl*)

Input : L_1, L_2, two label sets; *just*, *concl*, two normal form axioms.
Output : The label set produced by the meet-operator $\otimes$.

```
result ← {(A1 ∪ A2, O1 ∪ O2) | (A1, O1) ∈ L1, (A2, O2) ∈ L2};
if just ∈ 𝒜 then result ← {(A ∪ {just}, O) | (A, O) ∈ result};
if concl ∈ 𝒪 then result ← {(A, O ∪ {concl}) | (A, O) ∈ result};
return result;
```

Function join(L_1, L_2)

Input : L_1, L_2, two label sets.
Output : The label set produced by the join-operator $\oplus$.

```
result ← L1 ∪ L2;
result ← remove-dominated(result, ⪯𝒜, ⪯𝒪);
return result;
```

$V_\top$. This can be proven inductively based on the correctness of the operators $\oplus$ and $\otimes$, which can easily be established in a case-by-case analysis. The terminal closure of $\bigcup_{v \in V} label(v)$ under component-wise union is based on the intuition that, having proved two statements a and b, we can obviously prove $a \wedge b$ by joining the two proofs (corresponding to the $\otimes$ operator). Graphically, this can be seen as adding a dedicated vertex $\top$ such that any other $v \in V$ is connected to $\top$ by a hyperedge $(\{v\}, \top, \{\emptyset, \emptyset\})$, and determining the label of this node that intuitively represents anything that can be derived at all. □

Since the node labels may grow exponentially in the size of $\mathcal{A}$ and $\mathcal{O}$ for general preference orders such as set inclusion, it is worthwhile investigating the benefit of our method as compared to the following simple brute-force approach: Iterating over all pairs $(A, O) \in \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{O})$, collect all (A, O) such that $\mathcal{T} \cup A \models O$ holds and finally drop all $\preceq$-dominated tuples among them. This approach obviously requires $2^{|\mathcal{A}|+|\mathcal{O}|}$ entailment tests, each set passing this test is consequently tested for $\preceq$-minimality. We argue that the our approach is superior to the brute-force method due to three aspects:

1. In contrast to the uninformed search outlined above, the approach proposed in this paper realizes an informed search as it does not generate all possible (A, O)-pairs haphazardly but only those for which the property $\mathcal{T} \cup A \models O$ actually holds, without requiring any additional entailment tests. The net effect of this property depends on the model $\mathcal{T}$ as well as on $\mathcal{A}$ and $\mathcal{O}$; problems having only few solutions at all will obviously benefit most.
2. Dropping $\preceq$-dominated labels for $\preceq_{\mathcal{O}}$ and $\preceq_{\mathcal{A}}$ being (anti-)monotone for set inclusion reduces the worst-case size of node labels from by at least a factor of $O(\sqrt{|\mathcal{A}| \cdot |\mathcal{O}|})$. This can be justified as follows: Fixing a set $A^* \subseteq \mathcal{A}$, the sets $O_i \subseteq \mathcal{O}$ that constitute the (non-dominated) label entries (A^*, O_i) must form an antichain w. r. t. set inclusion. The maximum size of such an antichain is

given by $\binom{|\mathcal{O}|}{\lfloor|\mathcal{O}|/2\rfloor}$ according to Sperner's theorem [15], and can be bounded by $2^{|\mathcal{O}|}/\sqrt{\pi/2\cdot|\mathcal{O}|}$ using Stirling's approximation.[4] An analogous argument holds for fixed O^*; the size of the cross product can therefore be bounded by $O((2^{|\mathcal{A}|}/\sqrt{|\mathcal{A}|})\cdot(2^{|\mathcal{O}|}/\sqrt{|\mathcal{O}|}))$, resulting in the factor stated above.

3. In addition to the strict upper bound to the size of labels provided by the preceding line of argumentation, we can also determine the expected number of non-dominated paths to a state as follows: We assume two arbitrary orders over the elements of $\mathcal{A}$ and $\mathcal{O}$ such that any subset can be encoded straightforwardly as a binary vector of length $|\mathcal{A}|$ (resp. $|\mathcal{O}|$). Fixing $A^* \subseteq \mathcal{A}$, an estimate of the expected number of label entries (A^*, O_i) is given by the expected number $A(n,l)$ of maximal $(0,1)$-vectors of length $l = |\mathcal{O}|$ among a set of k distinct such vectors chosen uniformly at random. For our estimation, we let $k := 2^{|\mathcal{O}|}$ to get an upper bound though the actual number is expected to be less (c. f. aspect 1). Adapting the technique used in [2], $A(n,l)$ can be expressed by the recurrence $A(n,l) \leq \lceil\frac{n}{2}\rceil \cdot \frac{A(n,l-1)}{n} + \lceil\frac{n}{2}\rceil \cdot \frac{A(\lceil n/2\rceil,l-1)}{\lceil n/2\rceil} \approx \frac{1}{2}\cdot A(n,l-1) + A(n/2,l-1)$.[5] Assuming $n \geq 2^{l-1}$, the recursion is limited only by l and terminates with the terms $A(n,1) = A(n^{1/(l-1)},1) = 1$ at depth $l-1$. An upper bound is thus given by $A(n,l) \leq A'(l) = A'(l-1) + \frac{1}{2}\cdot A(l-1) = \frac{3}{2}\cdot A(l-1) = (\frac{3}{2})^{l-1}$; the expected label size is thus $O(1.5^{|\mathcal{A}|+|\mathcal{O}|})$.

Other choices for $\preceq_{\mathcal{A}}$ and $\preceq_{\mathcal{O}}$ can lead to more substantial savings; since the preference orders are used as a pruning criterion during solution generation this may however turn the approach into an approximate one. For instance if the assumption and observation sets are not compared by set inclusion but by cardinality, the maximum label size is reduced to $|\mathcal{A}|\cdot|\mathcal{O}|$ – dependent on the order of rule application the algorithm may however fail to find the optimal solutions. In a more complex setting, assigning numerical weights to observations and abducibles allows to drop only solutions that are significantly worse than others, or to compute bounds on the maximum score a partial solution may still achieve, and use this value as a pruning criterion.

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[4] For $m \to \infty$ it holds that $\binom{2m}{m} \sim \frac{4^m}{\sqrt{\pi\cdot m}}$. Letting $m := \lfloor\frac{n}{2}\rfloor$, this yields the estimate $\binom{n}{\lfloor\frac{n}{2}\rfloor} \sim \frac{4^{\lfloor\frac{n}{2}\rfloor}}{\sqrt{\pi\cdot\lfloor\frac{n}{2}\rfloor}} \approx \frac{4^{\frac{n}{2}}}{\sqrt{\pi\cdot\frac{n}{2}}} = \frac{2^n}{\sqrt{\frac{\pi}{2}\cdot n}}$.

[5] This recurrence can be understood as follows: Assume the vectors are arranged in a $(n\times l)$-matrix, sorted by the first component. A randomly chosen vector $\boldsymbol{v}$ starts with 1 or 0 with probability 0.5 each. In the former case, $\boldsymbol{v}$ cannot be dominated by any vector starting with a 0, i. e. the "lower half" of the table is ruled out instantly, and its probability of being dominated by another vector starting with 1 is given by the expected number of maxima among the remaining $\lceil n/2\rceil$ vectors divided by their number, taken together $\boldsymbol{v}$ is maximal with probability $A(\lceil n/2\rceil, l-1)/\lceil n/2\rceil$. If $\boldsymbol{v}$ starts with 0, we can similarly determine its probability of being maximal to be $A(n,l-1)/n$. Summing up these probabilities and and multiplying the result by the number n of original vectors yields the expected number of maxima given above.

4 Related Work

While abductive reasoning naturally addresses the problem of missing observations, there are to the authors' best knowledge no other approaches providing a formally sound solution to logic-based abduction with incomplete models.

The idea of considering abduction as a multi-criteria optimization problem is also central to [10], where multi-criteria decision making techniques are employed to red-cell antibody identification in blood samples. The task is solved using domain-specific operators for combining entries in tables representing the hypotheses. Being an instance of the set-cover approach to abduction, the proposed method does however not address the problem of hypotheses generation, and requires a simple tabular mapping from hypotheses to effects. In the context of abductive (or diagnostic) inference in Bayesian networks, [11] distinguishes between most informative and most simple explanations which correspond to the $\preceq_{\mathcal{O}}$-minimal and the $\preceq_{\mathcal{A}}$-minimal solution in our approach, respectively. However, intermediary Pareto-optimal combinations are not considered in their approach which is furthermore limited to propositional Bayes nets. The algorithm presented in [4] for ABox abduction resembles our approach as it determines alternative explanation sets with varying expressive power, keeping track of the assumptions required for each of them. Unlike the approach presented in this paper, the work by Castano et al. requires special handcrafted models combining forward- and backward-chaining rules, and uses an iterative approach to handle models expressed in the more expressive description logic $\mathcal{ALCQ}$.

[13, 8] use an automaton which is structurally similar to the hypergraph $H_{\mathcal{RAP}}$ introduced in Def. 4 to generate a formula encoding all solutions to a pinpointing respectively a (standard) abduction problem. In contrast to our approach these works guarantee polynomial runtime for solution generation, they do however impose strong restrictions on the combination function, and are inherently limited to uni-criterion problems. Assumption-based Truth Maintenance Systems (ATMSs) [6] impose fewer restrictions on edge weights as compared to the previously mentioned approaches, and similarly to our approach labels containing information on required assumptions are propagated between vertices in a hypergraph structure. We are however not aware of any extension to ATMSs allowing for a tradeoff between assumptions and explanatory power, nor do ATMSs consider any order over labels other than implication.

5 Conclusions and Outlook

We have introduced relaxed abduction, a novel non-standard reasoning task for description logics. Relaxed abduction extends logic-based abduction to a general and formally sound framework for interpreting spurious information w. r. t. incomplete models. We have presented an algorithm for relaxed abduction over $\mathcal{EL}^+$ knowledge bases based on the notion of Pareto-optimal hyperpaths in the derivation graph, and motivated its superiority to a straightforward enumeration approach despite the inherent exponential growth of node labels. The proposed algorithm is straightforwardly extensible to other DLs for which subsumption can be decided by completion such as $\mathcal{EL}^{++}$ which supports nominals and thus ABox abduction. The very general notion of relaxed abduction allows for several interesting specializations resulting from different choices for $\preceq_{\mathcal{A}}$ and $\preceq_{\mathcal{O}}$: Approximate solutions can for example be generated very efficiently (i. e. with linear label size) if we use set cardinality as a dominance criterion. More elaborate schemes based on weights assigned to the axioms allow for early and even lossless pruning of suboptimal partial solutions while also reducing label sizes.

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