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Relaxed sequencing games have a nonempty core[*]

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Abstract

We study sequencing situations with a fixed initial order and linear cost functions. In these sequencing situations cost savings can be obtained by rearranging jobs. When the jobs are owned by different players, next to the issue of finding an optimal order, the division of these cost savings forms an additional issue. Cooperative game theory studies this issue by taking into account that groups of players have possibilities to obtain cost-savings. For sequencing situations, a common assumption states that cooperation between players is restricted to groups that are connected according to the initial order. The value of disconnected groups is then defined as the sum of its connected components. In this paper we take a different approach. We allow for disconnected coalitions to switch places in any way they want as long as they don't hurt the players not in the coalition under consideration. The resulting games are called relaxed sequencing games and they have been studied before. No general results on stable profit divisions have been derived so far. In this paper we prove that relaxed sequencing games have a nonempty core, i.e., they all have stable profit divisions.

Keywords: COOPERATIVE GAME THEORY, SCHEDULING, BALANCEDNESS.

JEL classification: C71

1 Introduction

The analysis of sequencing situations from a game-theoretical point of view dates back to Curiel et al. (1989). They consider sequencing situations with linear cost functions for the jobs. Assuming that the jobs are placed according to some initial order and

owned by different customers, the problem of finding a cost-efficient processing order is supplemented with the issue of dividing the cost-savings among the different customers. The latter is one of the central issues in cooperative game theory and Curiel et al. (1989) use concepts from this theory to analyze sequencing situations.

Consider a machine and 3 companies who each have a broken tool that has to be repaired in order for the companies to restart production. Suppose that according to a first-come-first-serve principle, the tool of company 1 will be repaired first, followed by company 2, and, finally, company 3. Lost production in the different companies may have different values per time unit and therefore, taking repair times into account as well, reshuffling the order of the jobs can lead to cost savings. If we want to analyze the division of cost savings from a game-theoretical point of view we need to take the possibilities of subgroups of companies into account. Assume company 1 and 2 come together to discuss their possibilities to achieve cost savings without the help of company 3. Obviously, they could switch positions without changing the time span in which the job of company 3 is processed. Hence, the value that can be obtained by these two companies equals zero if switching attains no (or negative) cost savings and it equals the cost savings if they are nonnegative. A similar reasoning can be given for companies 2 and 3 together. The picture becomes less clear if we consider companies 1 and 3. Company 2 might have the opportunity to prevent switching between companies 1 and 3. Or, it might that he can only veto this switch if the processing time of company 3 is larger than the processing time of company 1. Different perspectives on possibilities of groups of companies that are not connected according to the initial order result in different cost savings that can be obtained by such a group and, therefore, might result in a different analysis from a game-theoretical point of view.

In the example just described, Curiel et al. (1989) take the perspective that companies 1 and 3 cannot switch positions without the cooperation of company 2. They follow standard game-theoretical literature, refer to what we called companies as players, and extend these switching possibilities to a general setting. The only switches they allow are switches within components that are connected according to the initial order. They study standard solution concepts from cooperative game theory in this setting, but they also define and characterize an allocation rule for sequencing situations. They show that this allocation rule is stable, which means that in any sequencing situation it ends up with payoffs to the players such that any set of players together receive at least as much as this group could obtain without the help of the other players. In cooperative game theory such a payoff vector is called a core-element.

The work of Curiel et al. (1989) has had several follow-ups, which extend the basic model by considering ready times, due dates, or multiple machines. A recent review

can be found in Curiel et al. (2002), who also consider more possible rearrangements by reviewing the work of van Velzen and Hamers (2003). They allow for one specific player who is allowed to switch with another player in any coalition (even if these two players have to jump over players outside the group under consideration), as long as the starting times of the players that are not considered are not delayed. They prove nonemptiness of the core for the associated sequencing games. This relaxation of the set of admissible rearrangements is different from the four relaxations that were introduced in Curiel et al. (1993). These relaxations are based on requirements on position in the queue (should stay the same/might change) and starting time (should stay the same/is not allowed to increase) for the players not under consideration. Results on nonemptiness of the core are available for two of the four relaxations only: for at most 4 players the associated relaxed sequencing games were proven to have a nonempty core (see Hamers (1988)).

In this work we focus on the strongest relaxation of Curiel et al. (1993). We will prove for any sequencing situation with any number of players that the core of the associated relaxed sequencing game is nonempty. This implies nonemptiness of the core for the other three relaxations as well. From a practical point of view this is an important result since in practice it might well be that players or companies that are not connected directly are allowed to change places even without the help of the players in between.

The setup of this paper is as follows. We continue in section 2 with preliminaries on sequencing situations and cooperative game theory. In section 3 we introduce the basic sequencing games and the four types of relaxed sequencing games of Curiel et al. (1993). In section 4 we prove our main result for situations with rational-valued processing times. We extend this result to real-valued processing times in section 5.

2   Preliminaries

In this section we will introduce some notation and standard results on cooperative game theory and sequencing situations.

A cooperative game with transferable utilities, *TU-game*, is a pair (N, v) with N a set of players and $v : 2^N \to \mathbb{R}$ the characteristic function, which assigns to every coalition $S \subseteq N$ its value $v(S)$ with $v(\emptyset) = 0$. The *core* Core(N, v) of a game (N, v) consists of the payoff vectors $x \in \mathbb{R}^N$ that satisfy condition $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N$ and $\sum_{i \in N} x_i = v(N)$. For a coalition $S \subseteq N$, $v_{|S}$ denotes the restriction of the characteristic function v to the player set S, i.e., $v_{|S}(T) = v(T)$ for each coalition $T \subseteq S$. The pair $(S, v_{|S})$ is a cooperative game with player set S, called a *subgame* of (N, v). A game is called *balanced* if it has a nonempty core and *totally balanced* if all its subgames are balanced. We will use balancedness and nonemptiness of the core

interchangeably. The terminology balanced is due to Bondareva (1963) and Shapley (1967). They independently identified the class of games that have nonempty cores as the class of balanced games. To describe this last class, we define for all $S \subseteq N$ the vector e^S by $e_i^S = 1$ for all $i \in S$ and $e_i^S = 0$ for all $i \in N \backslash S$. A map $\kappa : 2^N \backslash \{\emptyset\} \to [0, 1]$ is called a *balanced map* if $\sum_{S \in 2^N \backslash \{\emptyset\}} \kappa(S) e^S = e^N$. Now, a game (N, v) is called *balanced* if for every balanced map $\kappa : 2^N \backslash \{\emptyset\} \to [0, 1]$ it holds that $\sum_{S \in 2^N \backslash \{\emptyset\}} \kappa(S) v(S) \leq v(N)$. We will refer to this last condition as a balancedness condition and use that a game has a nonempty core if it satisfies all balancedness conditions.

A coalitional game is *convex* if a player's marginal contribution does not decrease if he joins a larger coalition. Formally, coalitional game (N, v) is convex if for each $i \in N$ and for all $S, T \subseteq N \backslash \{i\}$ with $S \subseteq T$ it holds that $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$. If a game is convex then it is totally balanced.

In proving our results we will encounter permutation games, introduced by Tijs et al. (1984). Let $A = [a_{ij}]_{i \in N, j \in N}$ be a square matrix and let $\Pi(S)$ be the set of permutations of coalition $S \subseteq N$. Then a permutation game (N, r) is defined by

$$r(S) = \max_{\pi \in \Pi(S)} \sum_{i \in S} [a_{ii} - a_{i\pi(i)}] \text{ for all } S \subseteq N.$$

Tijs et al. (1984) prove that permutation games are totally balanced.

Finally, we will introduce the sequencing situations that were considered from a game-theoretical point of view by Curiel et al. (1989). Without taking the game-theoretical point of view these models were studied by Smith (1956) already. In a sequencing situation there is a queue of agents, each with one job, in front of a machine. Each agent has to have his job processed on this machine. The finite set of agents is denoted by $N = \{1, \ldots, n\}$. The nonnegative processing time p_i of player i is the time the machine takes to handle the job. It is assumed that each player has a linear cost function $c_i : [0, \infty) \to \mathbb{R}$ defined by $c_i(t) = \alpha_i t$ with $\alpha_i > 0$. This cost function describes that player i has cost $c_i(t)$ if he stays in the system (i.e., has an unfinished job) for t time units. The position of the players in front of the machine can be described by a bijection $\sigma : N \to \{1, \ldots, n\}$. Here, $\sigma(i) = j$ means that player i is in position j. We will denote $\sigma^{-1} = (\sigma^{-1}(1), \ldots, \sigma^{-1}(n))$. For example, $\sigma^{-1} = (3, 1, 2)$ means that player 3 is first according to σ, followed by player 1 and, finally, player 2. It is assumed that initially there is some fixed order in front of the machine, which is denoted by σ_0. A sequencing situation is a tuple (N, σ_0, p, α) with N the set of players, σ_0 the initial order, $p = (p_i)_{i \in N}$ the processing times, and $\alpha = (\alpha_i)_{i \in N}$ the weights of the players, all as described above.

Let (N, σ_0, p, α) be a sequencing situation. We denote the set of all possible orders by Π_N and let $\sigma \in \Pi_N$. With an order we can associate a unique schedule by considering the associated semi-active schedule. A schedule is called semi-active if there is no idle

time between the jobs. Note that a schedule is completely determined by the starting times of the jobs, which will, for the semi-active schedule associated with σ, be denoted by t_σ. For $i \in N$ we have

$$
t_{\sigma,i} = \begin{cases} 0 & \text{if } \sigma(i) = 1; \\ t_{\sigma,j} + p_j & \text{if } \sigma(i) > 1, \end{cases} \tag{1}
$$

where, if $\sigma(i) > 1$, j is right in front of i, i.e., $\sigma(j) = \sigma(i) - 1$.

For order $\sigma \in \Pi_N$ the cost of player i are given by $C_i(\sigma) = c_i(t_{\sigma,i} + p_i) = \alpha_i(t_{\sigma,i} + p_i)$. Hence, the total cost for all players equal $C_N(\sigma) = \sum_{i \in N} C_i(\sigma)$. Starting from initial order σ_0 cost savings might be obtained by processing the jobs in a different order. The following theorem, due to Smith (1956) states that the order with lowest total cost (i.e., highest total cost savings) has its jobs ordered according to decreasing urgencies, where the urgency of player i is given by $u_i = \frac{\alpha_i}{p_i}$.

Theorem 2.1 Let (N, σ_0, p, α) be a sequencing situation. Then σ^* satisfies

$$
C_N(\sigma^*) - C_N(\sigma_0) = \max_{\sigma \in \Pi_N} \left(C_N(\sigma) - C_N(\sigma_0) \right)
$$

if and only if

$$
u_{(\sigma^*)^{-1}(1)} \geq u_{(\sigma^*)^{-1}(2)} \geq \ldots \geq u_{(\sigma^*)^{-1}(n)}.
$$

We remark that an order being optimal does not depend on the initial order.

Besides determining the optimal cost savings, we would like to redistribute these cost savings to the players. Here, we will use cooperative game theory. Thereby we need a formal description of the possibilities of any group of players to obtain cost savings. This description allows for different possibilities and will be the first subject of the next section.

3   Sequencing games

In this section we will define several different sequencing games associated with a sequencing situation. The differences between the games consists of different sets of possibilities for groups of players to obtain cost savings. We will first follow the line of Curiel et al. (1989) and then consider four different approaches as described by Curiel et al. (1993).

Let (N, σ_0, p, α) be a sequencing situation. The cost savings a coalition $S \subseteq N$ can obtain will in general depend on the possibilities of this coalition to make changes to the initial order and/or its associated time schedule. Curiel et al. (1989) take the approach that two players in S can switch positions if and only if all players in between these two

players belong to S as well. Formally, let $\Pi^0_S(\sigma_0)$ (or Π^0_S if there is no confusion on σ_0) be the subset of orders of Π_N such that $\sigma \in \Pi^0_S$ if and only if for all $i \in N \backslash S$ it holds that $P(\sigma, i) = P(\sigma_0, i)$, where $P(\sigma, i) = \{j \in N \mid \sigma(j) < \sigma(i)\}$ denotes the set of predecessors of i according to σ. Hence, no jumps over players outside S are allowed. Let (N, v_0) be the cooperative game that is constructed in this way, so $v_0(S)$ is the cost savings that can be obtained by coalition S if no jumping over jobs outside S is allowed. Curiel et al. (1989) show that (N, v_0) is convex and, consequently, that it is balanced.

Curiel et al. (1993) introduce four types of relaxed sequencing games associated with sequencing situation (N, σ_0, p, α). In the first relaxation, players in a coalition $S \subseteq N$ are allowed to jump over players outside S but both the position and the starting time of players outside S are not allowed to change. Hence, if the jobs have different processing times, then the schedule that is optimal for a coalition need not be semi-active.

In the second relaxation, coalition $S \subseteq N$ gets the additional possibility to decrease the starting time of players not in S but coalition S is not allowed to change the position in the queue of players not in S. It follows straightforwardly that for any S there exists an optimal schedule that is semi-active.

The third relaxation puts no restrictions on the place in the queue of jobs outside $S \subseteq N$, but requires that the starting time of jobs outside S remain unchanged only. As with relaxation 1 the schedule that is optimal for a coalition need not be semi-active.

Finally, we consider the fourth relaxation. As in relaxation 2 coalition $S \subseteq N$ gets the possibility to decrease the starting time of players not in S and, as in relaxation 3, there are no restrictions on the place in the queue of jobs outside S: for all $j \notin S$: $t_j \leq t_{0j}$. As with relaxation 2 it follows immediately that for any S there exists an optimal schedule that is semi-active.

Consider sequencing situation (N, σ_0, p, α) and let $S \subseteq N$. Furthermore, let $\sigma \in \Pi_N$. For any of the four relaxations we can then determine whether σ satisfies the requirements and, if so, whether there exists a time schedule associated with σ such that the requirements of the relaxation are satisfied.

To illustrate the four relaxations consider the following example, taken from Curiel et al. (1993).

Example 3.1 Consider sequencing situation (N, σ_0, p, α) with $N = \{1, 2, 3, 4\}$, $\sigma_0(i) = i$ for all $i \in N$, $p = (5, 3, 2, 1)$, and some unspecified weight vector. Consider $S = \{1, 3, 4\}$. The components of S are $\{1\}$ and $\{3, 4\}$. Hence, in the basic model a switch between job 3 and job 4 is the only possible switch.

According to relaxation 1, coalition $\{1, 3, 4\}$ can attain the schedule with $\sigma^{-1} = (4, 2, 1, 3)$ with respective starting times 0, 5, 8, 13.

For relaxation 2 the same order is admissible for $\{1, 3, 4\}$ but with respective starting times 0, 1, 4, 9.

If we consider relaxation 3 then we have an admissible schedule with $\sigma^{-1} = (4, 3, 2, 1)$ with associated starting times 0, 1, 5, 8.

For relaxation 4, these starting times change to 0, 1, 3, 6. $\diamondsuit$

For any $S \subseteq N$, denote the set of admissible orders, i.e., orders for which there exist an admissible time schedule, for the four relaxations by Π_S^1, Π_S^2, Π_S^3, and Π_S^4. Denote for any relaxation $k \in \{1, 2, 3, 4\}$, any $S \subseteq N$, any $j \in S$, and any admissible schedule $\sigma \in \Pi_S^k$

$$c_{S,j}^k(\sigma) = \alpha_j(t_{\sigma,j,S}^k + p_j),$$

where $t_{\sigma,j,S}^k$ is the starting time of job j according to schedule σ that is obtained according to relaxation k by coalition S.

Now, C_S^k and v^k are defined as in the basic model. So, the cooperative games associated with a sequencing situation are denoted by (N, v^1), (N, v^2), (N, v^3), and (N, v^4) for relaxations 1, 2, 3, and 4, respectively. The following result provides relations between these four games.

Theorem 3.1 Let (N, σ_0, p, α) be a sequencing situation. For the associated relaxed sequencing games we have:

$$v^1(N) = v^2(N) = v^3(N) = v^4(N);$$
$$v^1(S) \leq v^2(S) \leq v^4(S) \qquad\qquad \forall S \subseteq N;$$
$$v^1(S) \leq v^3(S) \leq v^4(S) \qquad\qquad \forall S \subseteq N.$$

Proof: Follows directly by definition of the admissible rearrangements/schedules of a coalition. Note that there is no $j \notin S$ if we consider $S = N$. $\square$

This theorem implies that any core-element of (N, v^4) is a core-element of (N, v^1), (N, v^2), and (N, v^3) as well. Consequently, if (N, v^4) has a nonempty core, then (N, v^1), (N, v^2), and (N, v^3) have a nonempty core as well. In this work we will prove that (N, v^4) has a nonempty core. For notational convenience, we will, from now on, denote (N, v^4) by (N, v) and refer to it as *the* relaxed sequencing game (associated with (N, σ_0, p, α)).

Note that for relaxation 4 it is easy to determine whether an order is admissible and, if so, its associated starting times. In sequencing situation (N, σ_0, p, α) coalition S can switch to order σ if and only if $t_{\sigma,i} \leq t_{\sigma_0,i}$ for all $i \in N\backslash S$. The associated (optimal) starting times are then given by $(t_{\sigma,i})_{i \in N}$.

Previous work on relaxed sequencing games have taken a much more restrictive perspective. Hamers (1988) proved balancedness of sequencing games (N, v^2) for $|N| \leq 4$. Van Velzen and Hamers (2003) study weak-relaxed sequencing games, in which only one specific player has the opportunity to switch with any other player in the coalition provided that the players outside this coalition do no suffer from this switch. They prove balancedness of weak-relaxed sequencing games.

4   Nonemptiness of the core

In this section we will prove nonemptiness of the core of relaxed sequencing games, but we restrict ourselves to situations with rational-valued processing times. We will extend our result to sequencing situations with real-valued processing times in the next section.

Let $\mathcal{S}$ denote the set of sequencing situations. The subset of sequencing situations with rational-valued processing times is denoted by $\mathcal{S}^{\mathbb{Q}}$. The subset of sequencing situations with each processing time equal to a natural number is denoted by $\mathcal{S}^{\mathbb{N}}$.

Let $(N, \sigma_0, p, \alpha) \in \mathcal{S}^{\mathbb{N}}$ be a sequencing situation. We define the associated sequencing situation $(\overline{N}, \overline{\sigma_0}, \overline{p}, \overline{\alpha})$ by

- $\overline{N} = N_1 \cup N_2 \cup \ldots \cup N_n$ with $N_i = \{j_i^1, \ldots, j_i^{p_i}\}$ $\forall i \in N$;

- $\overline{\sigma_0}(j_i^k) = k + \sum_{r=1}^{i-1} |N_{(\sigma_0)^{-1}(r)}|$ $\forall i \in N$ and $\forall k \in \{1, \ldots, p_i\}$;

- $\overline{p}_i^k = 1$ $\forall i \in N$ and $\forall k \in \{1, \ldots, p_i\}$;

- $\overline{\alpha}_i^k = \frac{\alpha_i}{p_i}$ $\forall i \in N$ and $\forall k \in \{1, \ldots, p_i\}$.

In this associated sequencing situation each original job has been split into a number of (sub-)jobs such that each subjob has unitary processing time. The weight of the original job is split over the subjobs as well. Finally, the new initial order respects the original order, the subjobs of one (original) job together are processed throughout exactly the same time span as the original job in the initial order. Note that all subjobs of a job are the same and, hence, their relative initial order is not important.

Above we associated an order of all subjobs with the initial order of the original jobs. We generalize this definition to introduce an order $\overline{\sigma}$ associated with any processing order $\sigma \in \Pi_N$:

$$\overline{\sigma}(j_i^k) = k + \sum_{r=1}^{\sigma(i)-1} |N_{\sigma^{-1}(r)}| \ \forall i \in N \text{ and } \forall k \in \{1, \ldots, p_i\}.$$

We illustrate the construction of the associated sequencing situation by means of an example.

Example 4.1 Let $(N, \sigma_0, p, \alpha) \in \mathcal{S}^{\mathbb{N}}$ be a sequencing situation with $N = \{1, 2, 3\}$, $\sigma_0(i) = i$ for all $i \in N$, $p_i = (1, 3, 2)$, and $\alpha = (2, 9, 2)$.

The associated sequencing situation $(\overline{N}, \overline{\sigma_0}, \overline{p}, \overline{\alpha})$ is then represented by $\overline{N} = \{j_1^1, j_2^1, j_2^2, j_2^3, j_3^1, j_3^2\}$, $\overline{\sigma_0}$ is such that $\overline{\sigma_0}^{-1}(1) = (j_1^1, j_2^1, j_2^2, j_2^3, j_3^1, j_3^2)$, $\overline{p}_i^k = 1$ for all $j_i^k \in \overline{N}$, and $(\overline{\alpha}_1^1, \overline{\alpha}_2^1, \overline{\alpha}_2^2, \overline{\alpha}_2^3, \overline{\alpha}_3^1, \overline{\alpha}_3^2) = (2, 3, 3, 3, 1, 1)$. $\diamond$

Since $(\overline{N}, \overline{\sigma_0}, \overline{p}, \overline{\alpha})$ is a sequencing situation we can consider its associated relaxed sequencing game. We will denote this relaxed sequencing game by $(\overline{N}, w)$ and refer to it as the *relaxed unit sequencing game* associated with the original sequencing situation (N, σ_0, p, α).

Since all processing times in $(\overline{N}, \overline{\sigma_0}, \overline{p}, \overline{\alpha})$ are equal (to one), the associated relaxed sequencing game coincides with a permutation game, which is balanced. This is captured in the following lemma.[1]

Lemma 4.1 Let $(N, \sigma_0, p, \alpha) \in \mathcal{S}^{\mathbb{N}}$ be a sequencing situation. The associated relaxed unit sequencing game $(\overline{N}, w)$ is balanced.

Proof: The relaxed unit sequencing game is the relaxed sequencing game associated with sequencing situation $(\overline{N}, \overline{\sigma_0}, \overline{p}, \overline{\alpha})$. Since all processing times are equal (to 1), this game coincides with a permutation game as already argued by Tijs et al. (1984). They prove that all permutation games are balanced. $\square$

In the following lemma we will derive some relations between the relaxed sequencing game and the relaxed unit sequencing game associated with a sequencing situation. For any $S \subseteq N$ will use $\overline{S}$ to denote $\cup_{i \in S} N_i$.

Lemma 4.2 Let $(N, \sigma_0, p, \alpha) \in \mathcal{S}^{\mathbb{N}}$ be a sequencing situation. Then $v(S) \leq w(\overline{S})$ for all $S \subseteq N$ with equality for $S = N$.

Proof: Let $i \in N$. Define

$$\Delta_i = p_i \alpha_i - \sum_{k=1}^{p_i} k \alpha_i^k.$$

Note that

$$\Delta_i = p_i \alpha_i - \frac{p_i(p_i + 1)}{2} \frac{\alpha_i}{p_i} = p_i \alpha_i - \frac{1}{2}(p_i + 1)\alpha_i = \frac{1}{2}(p_i - 1)\alpha_i.$$

---

[1]In fact, not only the relaxed sequencing game coincides with a permutation game, but it coincides with the cooperative game associated with any of the other relaxations as well.

Let $S \subseteq N$. We will prove that $v(S) \leq w(\overline{S})$. Let $\sigma \in \Pi_S^4$ be an admissible order for S. Then $\overline{\sigma}$ is obviously admissible for $\overline{S}$ since for all $j_i^k \notin \overline{S}$ we have that $t_{j_i^k} \leq t_{0j_i^k}$.[2]

Let $\tau \in \Pi^N$ be an order of the jobs and let $i \in N$. Then

$$
\begin{aligned}
C_i(\tau) &= \alpha_i \sum_{k:\tau(k) \leq \tau(i)} p_k \\
&= \alpha_i p_i + \alpha_i \sum_{k:\tau(k) < \tau(i)} p_k \\
&= \Delta_i + \sum_{k=1}^{p_i} k \frac{\alpha_i}{p_i} + \sum_{k=1}^{p_i} \left(\frac{\alpha_i}{p_i} \sum_{k:\tau(k) < \tau(i)} p_k \right) \\
&= \Delta_i + \sum_{k=1}^{p_i} C_{j_i^k}(\overline{\tau}).
\end{aligned}
$$

Using this, we have for all $S \subseteq N$

$$
C_S(\tau) = C_{\overline{S}}(\overline{\tau}) + \sum_{i \in S} \Delta_i.
$$

Hence, with σ_S the optimal order for S, $\overline{\sigma_S}$ its associated schedule which is admissible for $\overline{S}$, and $\sigma_{\overline{S}}$ the optimal schedule for $\overline{S}$ we have

$$
\begin{aligned}
w(\overline{S}) &= C_{\overline{S}}(\overline{\sigma}_0) - C_{\overline{S}}(\sigma_{\overline{S}}) \\
&\geq C_{\overline{S}}(\overline{\sigma}_0) - C_{\overline{S}}(\overline{\sigma}_S) \\
&= C_{\overline{S}}(\overline{\sigma}_0) + \sum_{i \in S} \Delta_i - \left(C_{\overline{S}}(\overline{\sigma}_S) + \sum_{i \in S} \Delta_i \right) \\
&= C_S(\sigma_0) - C_S(\sigma_S) = v(S).
\end{aligned}
$$

Finally, consider N. Let σ_N be an optimal rearrangement for N. Consider the associated order for $\overline{N}$, i.e., $\overline{\sigma_N}$. Let $i_1, i_2 \in N$, let $k_1 \in \{1, \ldots, p_{i_1}\}$ and $k_2 \in \{1, \ldots, p_{i_2}\}$ be such that $\overline{\sigma_N}(j_{i_1}^{k_1}) < \overline{\sigma_N}(j_{i_2}^{k_2})$. Then

$$
\frac{\alpha_{i_1}^{k_1}}{p_{i_1}^{k_1}} = \alpha_{i_1}^{k_1} = \frac{\alpha_{i_1}}{p_{i_1}} \geq \frac{\alpha_{i_2}}{p_{i_2}} = \alpha_{i_2}^{k_2} = \frac{\alpha_{i_2}^{k_2}}{p_{i_2}^{k_2}}, \tag{2}
$$

---

[2]Note that it need not be the case that $\overline{\sigma}$ can be obtained by a permutation of the jobs in $\overline{S}$, since jobs in $\overline{N}\backslash\overline{S}$ can have been moved forward. Hence, more cost savings can be obtained by $\overline{S}$ (not by S) if they move the jobs in $\overline{N}\backslash\overline{S}$ back to their original position and, while leaving the order within $\overline{S}$ unchanged, moving these jobs forward as much as possible. To prove that $v(S) \leq w(\overline{S})$ it suffices to consider $\overline{\sigma}$.

where the inequality follows by optimality of σ_N and theorem 2.1. Using (2) it follows by theorem 2.1 again that $\overline{\sigma}_N$ is optimal for $\overline{N}$. Hence,

$$
\begin{aligned}
w(\overline{N}) &= C_{\overline{N}}'(\overline{\sigma}_0) - C_{\overline{N}}'(\overline{\sigma}_N) \\
&= C_{\overline{N}}'(\overline{\sigma}_0) + \sum_{i \in N} \Delta_i - \left(C_{\overline{N}}'(\overline{\sigma}_N) + \sum_{i \in N} \Delta_i \right) \\
&= C_N(\sigma_0) - C_N(\sigma_N) = v(N).
\end{aligned}
$$

This completes the proof. $\qquad\square$

Using our results above, we can prove the first main result of this section.

Theorem 4.1 Let $(N, \sigma_0, p, \alpha) \in \mathcal{S}^{\mathbb{N}}$ be a sequencing situation. Then its associated relaxed sequencing game is balanced.

Proof: Let $(y_i^k)_{i \in N; 1 \leq k \leq p_i}$ be a core-element of $(\overline{N}, w)$, which exists by lemma 4.1. For all $i \in N$ define

$$
x_i = \sum_{k=1}^{p_i} y_i^k.
$$

Then

$$
\sum_{i \in N} x_i = w(\overline{N}) = v(N)
$$

and

$$
\sum_{i \in S} x_i \geq w(\overline{S}) \geq v(S),
$$

where we make use of $(y_i^k)_{i \in N; 1 \leq k \leq p_i} \in \mathrm{Core}(\overline{N}, w)$ and the (in-)equalities of lemma 4.2. We conclude that $(x_i)_{i \in N} \in \mathrm{Core}(N, v)$. $\qquad\square$

In the following theorem we extend the result on nonemptiness of the core from the class of sequencing games with natural-valued processing times to rational-valued processing times.

Theorem 4.2 Let $(N, \sigma_0, p, \alpha) \in \mathcal{S}^{\mathbb{Q}}$ be a sequencing situation. Then its associated relaxed sequencing game is balanced.

Proof: First, we prove that the theorem holds for all $(N, \sigma_0, p, \alpha) \in \mathcal{S}^{\mathbb{Q}}$ with $p_i > 0$ for all $i \in N$. Since $p_i \in \mathbb{Q}$ for all $i \in N$ we can write $p_i = a_i/b_i$ with $a_i, b_i \in \mathbb{N}$ for all $i \in N$. Let $\beta = \prod_{i \in N} b_i$. Then $\beta p_i \in \mathbb{N}$ for all $i \in N$. Hence, $(N, \sigma_0, \beta p, \alpha) \in \mathcal{S}^{\mathbb{N}}$. Denote

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the relaxed sequencing game associated with (N, σ_0, p, α) by (N, v) and the relaxed sequencing game associated with $(N, \sigma_0, \beta p, \alpha)$ by (N, z). Let $\sigma \in \Pi^N$ be an order of the jobs. Obviously, if the processing times are multiplied by a factor β then the cost of any player are multiplied by a factor β as well. Since this holds for any order and since the set of admissible orders for a coalition $S \subseteq N$ remains unchanged, we derive $z(S) = \beta v(S)$ for all $S \subseteq N$. By theorem 4.1 we know that (N, z) has a nonempty core, which implies that (N, v) has a nonempty core as well.

Finally, allowing for players with zero processing times just means including dummy players in the associated relaxed sequencing game. Hence, we can extend the balanced-result to $\mathcal{S}^{\mathbb{Q}}$. □

5   Real-valued processing times

In this section we will extend the core-nonemptiness theorem from the class with rational valued processing times to the class of real-valued processing times.

Taking into acccount the results of the previous section it seems obvious to approach a sequencing situation with real-valued processing times by means of a series of sequencing situations with rational-valued processing times. Taking an arbitrary series is not sufficient as is illustrated by the following example.

Example 5.1 Consider sequencing situation (N, σ_0, p, α) with $N = \{1, 2, 3\}$, $\sigma_0(i) = i$ for all $i \in N$, $p = (e, 1, e)$, and $\alpha = (1, 2, 3)$. Obviously, $v(1, 3) = 2 + 2e$. Define $p^m = ((1 + \frac{1}{m})^m, 1, (1 + \frac{1}{m})^m + \frac{1}{m})$ for any $m \in \mathbb{N}$. Then $\lim_{m \to \infty} p^m = (e, 1, e) = p$ but, with (N, v_m) the sequencing situation associated with (N, σ_0, p, α), we have that $v_m(1, 3) = 0$ for all $m \geq 1$ since $p_1^m < p_3^m$. Hence, $\lim_{m \to \infty} v_m(1, 3) = 0 \neq 2 + 2e = v(1, 3)$. We conclude that we cannot use this series in an obvious way to establish balancedness of (N, v). ◇

This example illustrates that an arbitrary series of processing time vectors will not be useful to prove balancedness of sequencing situations with real-valued processing times. However, a careful choice of series of processing time vectors will prove to be useful.

To describe this approach, we first need to introduce some additional notation. Denote the set of sequencing situations with positive processing times ($p_i > 0$ for all $i \in N$) that sum to 1 ($\sum_{i \in N} p_i = 1$) by $\mathcal{S}^{0,1}$. Let $(N, \sigma_0, p, \alpha) \in \mathcal{S}^{0,1}$ be such a sequencing situation and let $\delta \in \mathbb{Q}$ be such that $0 < \delta < \min_{i \in N} p_i$. Let $p_S = \sum_{i \in S} p_i$ for any $S \subseteq N$.

Furthermore, define

$$U(p, \delta) = \{q \in \mathbb{R}^N \mid \sum_{i \in N} q_i = 1; \forall_{i \in N} : q_i \geq \delta; \forall_{S,T \subseteq N} : p_S = p_T \Rightarrow q_S = q_T;$$

$$\forall_{S,T \subseteq N} : p_S > p_T \Rightarrow q_S > q_T\}.$$

The closure of this set, which we denote by $\overline{U(p, \delta)}$ is described by[3]

$$\overline{U(p, \delta)} = \{q \in \mathbb{R}^N \mid \sum_{i \in N} q_i = 1; \forall_{i \in N} : q_i \geq \delta; \forall_{S,T \subseteq N} : p_S \geq p_T \Rightarrow q_S \geq q_T\}. \quad (3)$$

Since $p \in U(p, \delta) \subseteq \overline{U(p, \delta)}$ and since $\overline{U(p, \delta)}$ is a subset of the N-dimensional simplex we have that $\overline{U(p, \delta)}$ is a bounded nonempty polyhedron. Hence, it is a polytope, and we will denote its vertices by $x_1, \ldots, x_t$.[4] Since all conditions in the description of $\overline{U(p, \delta)}$ have only rational coefficients and right-hand sides it follows that $x_i \in \mathbb{Q}^N$ for all $i \in \{1, \ldots, t\}$.

Since $p \in \overline{U(p, \delta)}$ we know that p is a convex combination of $x_1, \ldots, x_t$. Let $(\omega_i)_{i=1}^t$ be an arbitrary but fixed set of coefficients such that $p = \sum_{i=1}^t \omega_i x_i$. Note that some of the ω_i may be zero. For notational convenience, rename the vertices (and their associated coefficients) such that $\omega_i > 0$ for $i = 1, \ldots, r$ and $\omega_i = 0$ for $i = r + 1, \ldots, t$. So, $p = \sum_{i=1}^r \omega_i x_i$ with $\omega_i > 0$ for $i = 1, \ldots, r$.

We will construct a sequence of vectors of processing times in $U(p, \delta)$ that converges to p. For $i = 1, \ldots, r$ and any $n \geq 1$ let ω_i^m be the decimal representation of ω_i truncated after decimal position n, so $\omega_i^m = \frac{int(10^m \omega_i)}{10^m}$, where $int(a)$ is the integer part of a. Obviously, $\omega_i^m \in \mathbb{Q}$ for all $i = 1, \ldots, r$ and all $m \geq 1$. Let $m^* = \min\{m \mid \omega_i^m > 0$ for all $i \in \{1, \ldots, r\}\}$. Note that m^* is well-defined since $\omega_i > 0$ for all $i \in \{1, \ldots, r\}$. Furthermore, define for all $m \geq m^*$ and all $i \in \{1, \ldots, r\}$

$$\beta_i^m = \frac{\omega_i^m}{\sum_{j=1}^r \omega_j^m}.$$

Note that, since all $\omega_j^m \in \mathbb{Q}$ we have that $\beta_i^m \in \mathbb{Q}$ for all $i = 1, \ldots, r$ and all $m \geq m^*$ as well. Furthermore, note that for all $m \geq m^*$ it holds that $\sum_{i=1}^r \beta_i^m = 1$ and $\beta_i^m > 0$ for all $i \in \{1, \ldots, r\}$. Finally, define $p^m = \sum_{j=1}^r \beta_i^m x_i$. Since $\lim_{m \to \infty} \beta_i^m = \omega_i$ we have that $\lim_{m \to \infty} p^m = p$.

The following lemma shows that this sequence of processing vectors belongs to $U(p, \delta)$.

Lemma 5.1 For all $m \geq m^*$ it holds that $p^m \in U(p, \delta)$.

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[3]Let A be the right-hand side of equation (3). Then it follows immediately that $\overline{U(p, \delta)} \subseteq A$. The reverse relation holds since for any $q \in A$ and any $\alpha \in [0, 1)$ one easily verifies that $\alpha q + (1-\alpha)p \in U(p, \delta)$.

[4]We suppress that the vertices depend on $\overline{U(p, \delta)}$.

Proof: Let $m \geq m^*$. Since p^m is a convex combination of $x_1, \ldots, x_r$, which all belong to $\overline{U(p,\delta)}$ we have that $\sum_{i \in N} p_i^m = 1$ and $p_i^m \geq \delta$ for all $i \in N$. Furthermore, by the same reason it follows immediately for all $S, T \subseteq N$ that $p_S = p_T$ implies $p_S^m = p_T^m$. It remains to show that for all $S, T \subseteq N$ it holds that $p_S > p_T$ implies $p_S^m > p_T^m$. Let $S, T \subseteq N$ be such that $p_S > p_T$. Then, $(x_i)_S \geq (x_i)_T$ for all $i = 1, \ldots, r$ since all x_i belong to $\overline{U(p,\delta)}$. Since $p = \sum_{i=1}^{r} \omega_i x_i$ and $p_S > p_T$ there exists $i^* \in \{1, \ldots, r\}$ such that $(x_{i^*})_S > (x_{i^*})_T$. Using that $\beta_{i^*}^m > 0$ and $\beta_i^m \geq 0$ for all $i \in \{1, \ldots, r\}$ we derive that $p_S^m > p_T^m$. We conclude that $p^m \in U(p, \delta)$. $\qquad \square$

In the following lemma, we show that for any coalition it holds that two elements of $U(p, \delta)$ allow for the same rearrangements. Since the set of admissible orders of coalition S, Π_S^4, may depend on p we denote it by $\Pi_S^4(p)$.

Lemma 5.2 Let $(N, \sigma_0, p, \alpha) \in \mathcal{S}^{0,1}$ be a sequencing situation and let $\delta \in \mathbb{Q}$ be such that $0 < \delta < \min_{i \in N} p_i$. Let $p^* \in U(p, \delta)$ and $S \subseteq N$. Then $\Pi_S^4(p) = \Pi_S^4(p^*)$.

Proof: Let $\sigma \in \Pi^N$ and let $i \notin S$. Then $\sum_{j \in P(\sigma, i)} p_j \leq \sum_{j \in P(\sigma_0, i)} p_j$ if and only if $\sum_{j \in P(\sigma, i)} p_j^* \leq \sum_{j \in P(\sigma_0, i)} p_j^*$. Hence, $\sigma \in \Pi_S^4(p)$ if and only if $\sigma \in \Pi_S^4(p^*)$. We conclude that $\Pi_S^4(p) = \Pi_S^4(p^*)$. $\qquad \square$

As a last intermediate step, we prove that the sequence of games $(N, v^m)_{m \geq m^*}$, with (N, v^m) the relaxed sequencing game associated with $(N, \sigma_0, p^m, \alpha)$ converges to (N, v).

Lemma 5.3 Let $(N, \sigma_0, p, \alpha) \in \mathcal{S}^{0,1}$ be a sequencing situation and let $\delta \in \mathbb{Q}$ be such that $0 < \delta < \min_{i \in N} p_i$. Let $(N, v^m)_{m \geq m^*}$ be the sequence of relaxed sequencing games with (N, v^m) associated with $p^m = \sum_{j=1}^{r} \beta_i^m x_i$ Then $\lim_{m \to \infty} v^m = v$.

Proof: Let $S \subseteq N$. For any $\sigma \in \Pi_N$ and any $m \geq m^*$ we have by lemma 5.2 that $\sigma \in \Pi_S^4(p)$ if and only if $\sigma \in \Pi_S^4(p^m)$. For any $\sigma \in \Pi_S^4(p)(= \Pi_S^4(p^m))$ it follows immediately that $C_S(\sigma, p^m)$, defined by $C_S(\sigma)$ with p^m as its processing times, converges to $C_S(\sigma, p)$. Combining these two facts we derive that $v^m(S) = C_S(\sigma_0, p^m) - \min_{\sigma \in \Pi_S^4(p^m)} C_S(\sigma, p^m)$ converges to $C_S(\sigma_0, p) - \min_{\sigma \in \Pi_S^4(p)} C_S(\sigma, p) = v(S)$. This completes the proof. $\qquad \square$

Using the results so far, we can easily proof the main result of this paper, which states that all relaxed sequencing games have a nonempty core.

Theorem 5.1 Let $(N, \sigma_0, p, \alpha) \in \mathcal{S}$. The associated relaxed sequencing game (N, v) has a nonempty core.

Proof: First, let $(N, \sigma_0, p, \alpha) \in \mathcal{S}^{0,1}$. Let $(N, v^m)_{m \geq m^*}$ be the associated sequence of relaxed sequencing games as in lemma 5.3. By lemma 5.3 we know that (N, v^m) converges to (N, v). For all $m \geq m^*$, since $(N, \sigma_0, p^m, \alpha) \in \mathcal{S}^{\mathbb{Q}}$ we have that (N, v^m) is balanced by theorem 4.2. Hence, we derive that (N, v) satisfies all balancedness conditions as well.

Secondly, let $(N, \sigma_0, p, \alpha) \in \mathcal{S}$ with $p_i > 0$ for all $i \in N$. By the first part of this proof and a linearity-argument as in the proof of theorem 4.2 we derive balancedness of the relaxed sequencing game (N, v).

Finally, allowing for players with zero processing times just means including dummy players in the associated relaxed sequencing game. We conclude that the balancedness-result can be extended to $\mathcal{S}$. □

Earlier, in theorem 3.1, we stated that proving nonemptiness of the core for *the* relaxed sequencing games implies nonemptiness of the core for any of the four relaxed sequencing games that were considered in section 3. Hence, we have the following corollary to theorem 5.1.

Corollary 5.1 Let $(N, \sigma_0, p, \alpha) \in \mathcal{S}$. Any of the associated relaxed sequencing games (N, v^1), (N, v^2), (N, v^3), and (N, v^4) has a nonempty core.

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