Relaxed stability and performance LMI conditions for Takagi-Sugeno fuzzy systems with polynomial constraints on membership function shapes

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***Abstract*—Most LMI fuzzy control results in literature are valid for any membership function, i.e., independent of the actual membership shape. Hence, they are conservative (with respect to other nonlinear control approaches) when specific knowledge on the shapes is available. This paper presents relaxed LMI conditions for fuzzy control which incorporate such shape information, in the form of polynomial constraints, generalizing previous works by the authors. Interesting particular cases are overlap (product) bounds and ellipsoidal regions. Numerical examples illustrate the achieved improvements, as well as the possibilities of solving some multi-objective problems. The results also apply to polynomial-in-membership TS fuzzy systems.**

***Index Terms*—Takagi-Sugeno fuzzy control, LMI, relaxed condition, quadratic stability, parallel distributed compensation, polynomial fuzzy systems**

I. Introduction

LINEAR Matrix Inequality (LMI) techniques, introduced by [1] in the fuzzy community, have become the tool of choice [2], [3] in order to design controllers for Takagi-Sugeno (TS) models [4]. Most LMI control design techniques are based on proving positiveness (or negativeness) of a so-called double fuzzy summation [1], in expressions such as $\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j x^T Q_{ij} x > 0$, where μ_i are denoted as membership functions. Other settings may require higher summation dimension, see Section II-C.

Early sufficient conditions for positivity of the above double fuzzy summation were $Q_{ii} > 0$, $Q_{ij} + Q_{ji} > 0$ [1]; widely-used less conservative conditions are those in [5], improved by [6]. The conditions can be further generalized, until asymptotically necessary and sufficient conditions are obtained [7].

The above cited conditions are independent of the "shape" of the membership functions: sufficient conditions are stated via properties of Q_{ij} (disregarding any property that μ_i might have other than $\mu_i \geq 0$, $\sum_i \mu_i = 1$). Shape-independence is a source of conservativeness: for instance, the system $\dot{x} = \mu_1(z)\cdot x + (1-\mu_1(z))\cdot(-x)$ cannot be proved stable for an arbitrary μ_1, as it is unstable for $\mu_1(z) = 1$. However, it is an exponentially stable system for, say, $\mu_1 = 0.25 + 0.2sin(x)$. As a conclusion from the above, "pure nonlinear" strategies [8] on an original nonlinear model may find better solutions than fuzzy ones on an equivalent fuzzy TS model.

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In summary, there is a "gap" between fuzzy and nonlinear control caused by shape-independence. Indeed, in an actual application, when explicit expressions of the memberships as a function of some variables are known, some zones of the possible membership space can be excluded (only *for that particular application*, of course). If the set where the membership functions take values is reduced, the family of models described by the TS expressions gets smaller; in this way, less conservative conditions are possible.

The problem of shape-dependent information in fuzzy control LMIs has only been partially addressed in [9], [10] (for non-PDC control) and [11] for PDC control with known bounds on the product of two membership functions.

This contribution focuses on the PDC case, and presents results generalizing [11]. Indeed, [11] discusses constraints such as $\mu_i \leq \beta_i$, with β_i known, in order to prove shape-dependent stability of open-loop fuzzy systems and constraints in the form $\mu_i\mu_j \leq \beta_{ij}$, with β_{ij} known, in order to prove closed-loop shape-dependent stability. In this work the above cited procedures will be generalized and recent results from [7] will be incorporated. In this way, simple methodologies will allow for any polynomial constraint on the membership shapes, such as $\mu_1^2 - \mu_2^3\mu_3 - 4\mu_2\mu_4 - 0.1 \geq 0$, to be incorporated in the LMIs in order to relax conservativeness based on shape information.

In summary, this paper presents general polynomial shape-dependent relaxations of [5]–[7]. Interestingly, the results apply seamlessly to polynomial-in-membership fuzzy systems, *i.e.*, those described by, say, $\dot{x} = p_1(\mu)(A_1x + B_1u) + \cdots + p_s(\mu)(A_sx + B_su)$, where p_i are polynomials in the membership functions.

Furthermore, the ideas in this work allow a new type of fuzzy *multi-objective* designs (complementary to those in, for instance, [12], [13]) to be accomplished, so different control performance criteria may be specified in different regions of the operation space, as outlined via an example in Section V.

The structure of the paper is as follows: next section will discuss previous literature results, polynomial-in-membership fuzzy systems, and it will set the notation for fuzzy summations. Section III will discuss how to incorporate membership-shape information (in the form of polynomials of degree 2) to obtain shape-dependent relaxations for TS and polynomial-in-membership fuzzy systems. Section IV will generalize the results to fuzzy summations and polynomials of arbitrary degree. Numerical examples will illustrate the achieved improvements in Section V; some possibly interesting multi-objective problems will also be discussed there. A conclusion section closes this paper.

II. Preliminaries and notation

A. Double fuzzy summations

In many situations, Lyapunov-based conditions for stability or performance of a fuzzy control system may be expressed in the form

$$\Xi(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(z(t))\mu_j(z(t))x(t)^TQ_{ij}x(t) > 0 \quad \forall\ x \neq 0 \tag{1}$$

where $z(t)$ are denoted as premise variables (usually measurable) and r denotes the number of fuzzy "rules" or "local models". Symmetry of Q_{ij} and a fuzzy partition condition

$$\sum_{i=1}^{r}\mu_i(z(t)) = 1 \quad \mu_i(z(t)) \geq 0 \tag{2}$$

are assumed to hold. Notation μ_i will be used as shorthand for $\mu_i(z(t))$. Also, in most cases, "positive" in the text below should be understood as shorthand for "positive for $x \neq 0$".

Widely-used sufficient conditions for (1) are:

[5, Theorem 2]. Expression (1) under fuzzy partition condition holds if there exist matrices $X_{ij} = X_{ji}^T$ such that:

$$X_{ii} \leq Q_{ii} \tag{3}$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} \quad i \neq j \tag{4}$$

$$\mathbf{X} = \begin{pmatrix} X_{11} & \dots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \dots & X_{rr} \end{pmatrix} > 0 \tag{5}$$

The above conditions have been recently relaxed in [6, Theorem 5], and in [7]; see Section IV for further discussion on shape-dependent versions of them (shape-independent formulation omitted for brevity).

One of the simplest, well-known, examples in which the above arises is quadratic decay-rate performance of a Takagi-Sugeno fuzzy system

$$\dot{x} = \sum_{i=1}^{r}\mu_i(A_ix + B_iu) \tag{6}$$

Indeed, following [1], we have:

$$Q_{ij} = -(A_iX + XA_i{}^T - B_iM_j - M_j^TB_i^T + 2\alpha X) \tag{7}$$

where $X > 0$ and M_i are LMI decision variables. There are many other situations (see [1]–[3], [14], [15] and references therein for details) giving rise to different Q_{ij}, contemplating $\mathcal{H}_2$, $\mathcal{H}_\infty$, etc. performance measures, as well as uncertainty and delay, for both continuous-time and discrete fuzzy systems.

B. Previous shape-dependent conditions in literature

As discussed in the introduction, the problem of shape-dependent conditions has been addressed in literature only recently.

PDC case (1): To the authors' knowledge, the shape-dependent PDC case (1) has been considered only in [11], assuming that a bound on the overlap between fuzzy sets defined by μ_k and μ_l, for $k \neq l$, is expressed as:

$$\mu_k\mu_l \leq \gamma_{kl} \tag{8}$$

[11, Theorem 3] Consider an antecedent fuzzy partition fulfilling the overlap bounds (8). Expression (1) holds if there exist matrices $X_{ij} = X_{ji}^T$ and symmetric R_{ij}, $i \leq j$, such that:

$$X_{ii} \leq Q_{ii} + R_{ii} - \Lambda \tag{9}$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} + R_{ij} - 2\Lambda \tag{10}$$

$$\mathbf{X} = \begin{pmatrix} X_{11} & \dots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \dots & X_{rr} \end{pmatrix} > 0, \qquad R_{ij} \geq 0 \tag{11}$$

where

$$\Lambda = \sum_{k=1}^{r}\sum_{k \leq l \leq r}\beta_{kl}R_{kl}$$

Also, the conditions in [16] may be considered to be shape-dependent, exploiting the particular product structure present in many TS obtained from the sector-nonlinearity modelling technique [1]; however, albeit slightly less powerful, the relaxations in [11] are significantly less computationally demanding than [16] when applied to the same problem.

Non-PDC case: The non-PDC case, with $u = \sum_{i=1}^{r}\eta_iF_ix$, a model (6), and shape information in, for instance, the form $\rho_i^m \leq \eta_i/\mu_i \leq \rho_i^M$ is considered in [9], [10]. As the objective of this paper is the PDC case (1) and its multi-dimensional generalizations (see below), the reader is referred to the cited works for details on shape-dependent issues for non-PDC controllers. Anyway, the conditions in those works are always more conservative than the PDC ones, so they are not useful to accomplish the objectives of this paper.

C. Multi-dimensional fuzzy summations

As a generalisation of (1), other fuzzy control results require positiveness of a p-dimensional fuzzy summation, *i.e.*, checking

$$\Xi = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_p=1}^{r}\mu_{i_1}(z)\mu_{i_2}(z)\dots\mu_{i_p}(z)x^TQ_{i_1i_2\dots i_p}x \geq 0 \tag{12}$$

The case $p = 2$ reduces to (1).

1) Takagi-Sugeno systems with output equations: Conditions with $p = 3$ for TS systems are, for instance, the fuzzy dynamic controllers in [1], [12], using $Q_{ijk} = E_{ijk} + E_{ijk}^T$, where, for suitable $\mathcal{A}$,$\mathcal{B}$,$\mathcal{C}$,$\mathcal{D}$,

$$E_{ijk} = \begin{pmatrix} A_iQ_{11} + B_i\mathcal{C}_{jk} & A_i + B_i\mathcal{D}_jC_k \\ \mathcal{A}_{ijk} & A_iP_{11} + \mathcal{B}_{ij}C_k \end{pmatrix} < 0 \tag{13}$$

Triple sums appear also in output-feedback settings [6], [17].

2) Polynomial-in-membership fuzzy systems: Multi-dimensional summations in fuzzy control problems may also be obtained when *polynomial-in-membership* processes or controllers are used (see [18] for the latter case). A continuous-time polynomial-in-membership TS system is defined as:

$$\dot{x} = \sum_{i=1}^{s} \pi_i(\mu)(A_i x + B_i u) \tag{14}$$

being μ the vector of r membership functions, and π_i a polynomial in μ. The definition of a discrete system is analogous. Repeatedly multiplying by $\sum_{i=1}^{r} \mu_i$, all polynomials can be made homogeneous (see Section IV).

For instance, let $\sin z$ be fuzzified as $\sin z = \mu_1 * (-1) + \mu_2 * (+1)$, with $\mu_1 = 0.5(1 - \sin z)$, $\mu_2 = 1 - \mu_1$; consider the system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} (\sin z)x_1 + u \\ (\sin z)^2 x_1 - x_2 \end{pmatrix} \tag{15}$$

and replace $\sin z = (\mu_2 - \mu_1) \cdot (\mu_2 + \mu_1) = \mu_2^2 - \mu_1^2$ and $(\sin z)^2 = (\mu_2 - \mu_1)^2 = \mu_2^2 + \mu_1^2 - 2\mu_1\mu_2$. Then, a double-sum TS model $\sum_{i=1}^{2} \sum_{j=1}^{2} \mu_i \mu_j (A_{ij} x + B_{ij} u)$ is readily obtained, with $B_{ij} = (1\ 0)^T$ and

$$A_{11} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, A_{12} = A_{21} = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, A_{22} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

Considering also a quadratic-in-membership (QPDC) controller $u = -\sum_{k=1}^{2} \sum_{l=1}^{2} \mu_k \mu_l F_{kl} x$, decay rate performance would be proved if a 4-dimensional sum (12) were positive with $M_{kl} = F_{kl} X$ and

$$Q_{ijkl} = -(A_{ij} X + X {A_{ij}}^T - B_{ij} M_{kl} - M_{kl}^T B_{ij}^T + 2\alpha X) \tag{16}$$

for which the shape-independent results in [7] readily apply, as well as the shape-dependent ones in Section IV in this paper.

D. Multi-dimensional index notation

In order to streamline notation in multi-dimensional summations (12), the following notation, from [7], will be used to handle p-dimensional vectors of natural numbers (denoted by boldfaced variables), and its associated p-dimensional summations:

$$\begin{aligned} \mathbb{I}_p &= \{(i_1, i_2, \ldots, i_p) \mid 1 \le i_j \le r, j = 1, 2, \ldots, p\} \\ \sum_{\mathbf{i} \in \mathbb{I}_p} \gamma_{\mathbf{i}} &= \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \cdots \sum_{i_p=1}^{r} \gamma_{i_1 i_2 \ldots i_p} \end{aligned} \tag{17}$$

for some suitably defined multidimensional $\gamma_{\mathbf{i}} = \gamma_{i_1 i_2 \ldots i_p}$ ($\gamma_{i_1 i_2 \ldots i_p}$ will be either a number or a matrix), *i.e.*, boldface symbol $\mathbf{i}$ will denote a multi-index in a p-dimensional index set $\mathbb{I}_p$ ($\mathbb{I}_p$ has r^p elements). For instance, triple-summations in a four-rule fuzzy system will be spanned by a multi-index $\mathbf{i} \in \mathbb{I}_3$, where $\mathbb{I}_3$ has 64 elements: $\{1,1,1\}, \{1,1,2\}, \ldots, \{1,1,4\}, \{1,2,1\}, \ldots, \{4,4,4\}$. The following subset of $\mathbb{I}_p$ will also be later used:

$$\mathbb{I}_p^+ = \{\mathbf{i} \in \mathbb{I}_p \mid i_k \le i_{k+1}, \quad k = 1 \ldots, p-1\}$$

For instance, the elements of $\mathbb{I}_3^+$ are 20 $(6!/(3!3!))$: $\{1,1,1\}, \{1,1,2\}, \{1,1,3\}, \{1,1,4\}, \{1,2,2\}, \ldots, \{4,4,4\}$.

By convention, the cartesian product of several multi-indices, resulting in a higher-dimensional one, will be symbolized by parentheses:

$$\mathbf{i} \in \mathbb{I}_p,\ \mathbf{j} \in \mathbb{I}_q, \ldots, \mathbf{m} \in \mathbb{I}_t \ \Rightarrow\ (\mathbf{i}, \mathbf{j}, \ldots, \mathbf{m}) \in \mathbb{I}_{p+q+\cdots+t} \tag{18}$$

and sometimes by mere juxtaposition, such as in $\gamma_{i_1 \ldots i_p}$ in (17). One-dimensional indices, say $\mathbf{j} \in \mathbb{I}_1$ are ordinary integer index variables: they will be typed in italic typeface as j, $1 \le j \le r$ when its one-dimensionality should be emphasized.

The purpose of multi-index notation is to compactly represent multi-dimensional fuzzy summations, as follows.

First, let us define the following notation, specific for membership functions as a shorthand for a product:

$$\mu_{\mathbf{i}} = \prod_{l=1}^{p} \mu_{i_l} = \mu_{i_1} \mu_{i_2} \ldots \mu_{i_p} \quad \mathbf{i} \in \mathbb{I}_p \tag{19}$$

For instance $\mu_{(3,4,1,1)} = \mu_3 \mu_4 \mu_1^2$ will be the membership associated to the term Q_{3411} in a 4-dimensional fuzzy summation. Note that if $\mathbf{t} = (\mathbf{i}, \mathbf{k})$, $\mu_{\mathbf{t}} = \mu_{\mathbf{i}} \mu_{\mathbf{k}}$. With the above notation, p-dimensional fuzzy summations (12) may be written as follows:

$$\Xi(t) = \sum_{\mathbf{i} \in \mathbb{I}_p} \mu_{\mathbf{i}} x^T Q_{\mathbf{i}} x \tag{20}$$

where the basic memberships $\mu = \{\mu_1, \ldots, \mu_r\}$ from which $\mu_{\mathbf{i}}$ stem fulfill the add-1 partition condition.

1) Permutations: Given a multi-index $\mathbf{i} \in \mathbb{I}_p$, let us denote by $\mathcal{P}(\mathbf{i}) \subset \mathbb{I}_p$ the set of permutations (with, possibly, repeated elements) of the multi-index $\mathbf{i}$. For instance, $\mathbf{i} = (3,3,1,1)$ has $4!/(2!2!) = 6$ elements in its set of permutations. Of course if $\mathbf{i} \in \mathcal{P}(\mathbf{j})$, then $\mathbf{j} \in \mathcal{P}(\mathbf{i})$. The permutations will be used to group elements in multiple fuzzy summations which share the same antecedent: it's an evident fact that $\mathbf{j} \in \mathcal{P}(\mathbf{i}) \ \Rightarrow\ \mu_{\mathbf{j}} = \mu_{\mathbf{i}}$.

III. Improved shape-dependent positivity conditions for double fuzzy summations

Denote by $\mu(z)$ the column vector of membership functions $\mu(z) = (\mu_1(z), \mu_2(z), \ldots, \mu_r(z))^T$ in a fuzzy model. On the sequel, the shorthand notation μ will be used instead of $\mu(z)$, as previously introduced for the individual membership components.

Assume that knowledge of:

- the specific shape of the membership functions,
- the set of values Ω taken by premise variables z,

allows to set up a bound in the form:

$$\mu^T S \mu + w\mu + v \le 0 \quad \forall\ z \in \Omega \tag{21}$$

where S, w and v are, respectively, a matrix (of dimensions $r \times r$, with elements s_{ij}), a row vector ($1 \times r$, with elements w_i) and a scalar. All S, w and v are assumed known.

The left-hand side term in (21) is a second-order polynomial in the membership functions. Particular examples are, for instance, knowledge on degrees of membership function overlap (say, $\mu_1 \mu_2 < 0.15$, $\mu_1 \mu_3 = 0$, $(\mu_1 + \mu_2) * \mu_4 \le 0.4$), ellipsoidal sets (such as $(\mu_1 - 0.9)^2 + 2(\mu_2 - 0.1)^2 \le 0.05^2$) or drilling ellipsoidal "holes" (such as $(\mu_1 - 0.9)^2 + 2(\mu_2 - 0.1)^2 \ge 0.05^2$); see section III-A for further discussion.

Proposition III.1 *If the membership functions conform a fuzzy partition, then the matrix $\mathcal{B}$ whose elements, denoted by β_{ij}, are defined as*

$$\beta_{ij} = (s_{ij} + w_i + v) \tag{22}$$

fulfills:

$$\mu^T \mathcal{B} \mu \leq 0 \tag{23}$$

Proof: Indeed, using the partition condition (2),

$$0 \geq \mu^T S \mu + w\mu + v = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j s_{ij} + \sum_{i=1}^{r} w_i \mu_i + v =$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j s_{ij} + \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j w_i + \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j v =$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j (s_{ij} + w_i + v) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j \beta_{ij} \tag{24}$$

Hence, any second-order polynomial restriction on the membership shape can be written as an homogeneous form. ■

Theorem III.1 *Assume knowledge about the particular membership function shape is available via a constraint matrix $\mathcal{B}$, with elements β_{ij}, fulfilling* (23)*. Then, expression* (1) *is proved if there exists a symmetric matrix $R \geq 0$ so that the condition:*

$$\Xi'(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j x^T Q'_{ij} x > 0 \quad \forall\ x \neq 0 \tag{25}$$

holds, where $Q'_{ij} = Q_{ij} + \beta_{ij} R$, i.e., *$Q_{ij}$ is replaced in* (1) *by Q'_{ij} involving an additional matrix decision variable.*

Proof: consider an arbitrary symmetric positive semi-definite R. Then, the term

$$H = x^T R x \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j \beta_{ij} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j x^T \beta_{ij} R x \tag{26}$$

verifies $H \leq 0$, because it is the product of a positive number, $x^T R x$, and a non-positive one, $\mu^T \mathcal{B} \mu$. Then, H may be added to Ξ in (1) and, if the resulting sum is positive, Ξ will evidently be positive, *i.e.*, $\Xi \geq \Xi + H = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j x^T (Q_{ij} + \beta_{ij} R) x = \Xi'$. ■

The numerical examples in Section V will prove that the above result actually improves achievable performance over previous literature ones.

Multiple restrictions can be incorporated by repeated application of Theorem III.1, *i.e.*:

Corollary III.1 *Assume that knowledge about the particular membership function shape is available via a set of matrices $\mathcal{B}^{[k]}$, $k = 1, \ldots, n_c$, with elements $\beta_{ij}^{[k]}$, where n_c denotes the number of constraints. Expression* (1) *is proved if there exist positive semi-definite matrices R_k, $k = 1, \ldots, n_c$, so that the condition* (25) *holds, with*

$$Q'_{ij} = Q_{ij} + \sum_{k=1}^{n_c} \beta_{ij}^{[k]} R_k$$

Note that the above results adapt the basic idea behind the well-known S-procedure [19] (sufficient conditions for positivity of quadratic forms under quadratic constraints) to the fuzzy restrictions in consideration here, by considering a relaxation matrix instead of a scalar.

In order to check condition (25), usual expressions can be used, such as [5] or the relaxations in [7]. For instance, straightforward application of [5] and Corollary III.1 yields:

Theorem III.2 *Consider an antecedent fuzzy partition fulfilling a set of bounds in the form* (21) *or, equivalently,* (23)*. Expression* (1)*, under shape constraints $\mathcal{B}^{[k]}$, $k = 1, \ldots, t$, holds if there exist matrices $X_{ij} = X_{ji}^T$ and symmetric $R_k \geq 0$ such that* (5) *holds and*

$$X_{ii} \leq Q_{ii} + \sum_{k=1}^{n_c} \beta_{ii}^{[k]} R_k \tag{27}$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} + \sum_{k=1}^{n_c} (\beta_{ij}^{[k]} + \beta_{ji}^{[k]}) R_k \tag{28}$$

also do.

A. Particular cases

1) Membership function overlap [11]: Assume that a bound on the overlap between fuzzy sets is available in the form (8). Such bound is a particular case of (21) with $w = 0$ and

$$s_{kl} = s_{lk} = \frac{1}{2}, \quad v = -\gamma_{kl} \tag{29}$$

$$s_{ij} = 0 \quad \forall\ (i,j) \neq (k,l) \tag{30}$$

resulting in $\beta_{kl} = \beta_{lk} = \frac{1}{2} - \gamma_{kl}$ and, for $(i,j) \neq (k,l)$, $\beta_{ij} = -\gamma_{kl}$. The conditions of Theorem III.2 result in (5), $R \geq 0$ and

$$\begin{aligned}
X_{ii} &\leq Q_{ii} - \gamma_{kl} R && (31)\\
X_{ij} + X_{ji} &\leq Q_{ij} + Q_{ji} - 2\gamma_{kl} R \quad (i,j) \neq (k,l) && (32)\\
X_{kl} + X_{lk} &\leq Q_{ij} + Q_{ji} + (1 - 2\gamma_{kl}) R && (33)
\end{aligned}$$

By adding the restrictions for all possible pairs (k,l), the results in Section II-B from [11] are obtained.

2) Ellipsoidal hole: The interior (or exterior) of any ellipse, parabola or hyperbola can be considered via a suitable $\mathcal{B}$. As a simple example, let us assume that the condition

$$\sum_{i=1}^{r} (\mu_i - c_i)^2 \geq \delta^2 \tag{34}$$

is known to hold, *i.e.*, the membership functions are known to lie outside of a particular hyper-sphere.

Then, $\sum_{i=1}^{r} (\mu_i^2 + c_i^2 - 2c_i\mu_i) \geq \delta^2$, *i.e.*,

$$\sum_{i=1}^{r} \mu_i^2 + \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j (-\delta^2 + \sum_{k=1}^{r} c_k^2) - 2\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j c_i \geq 0$$

so we have, denoting by $\phi = -\delta^2 + \sum_{k=1}^{r} c_k^2$,

$$\sum_{i=1}^{r} \mu_i^2 (1 - 2c_i + \phi) + \sum_{i=1}^{r}\sum_{j=i+1}^{r} \mu_i \mu_j (2\phi - 2c_i - 2c_j) \geq 0$$

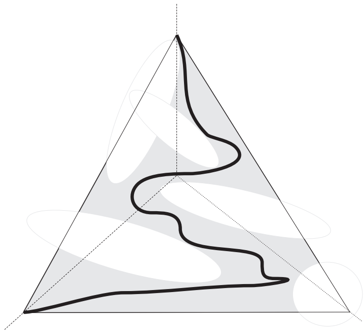

Fig. 1. Ellipsoidal holes on the add-1 simplex in $\mathbb{R}^3$.

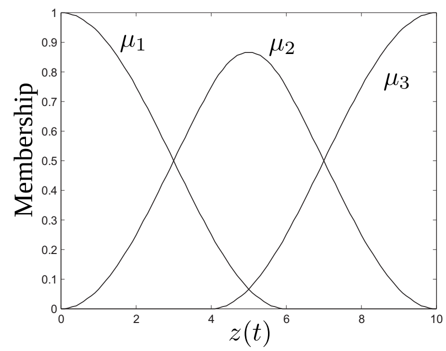

Fig. 2. Fuzzy partition with limited overlap.

Hence, the previous results apply with

$$\beta_{ii} = -(1-2c_i) - \phi \tag{35}$$

$$\beta_{ij} = -\phi + 2c_i \quad i \neq j \tag{36}$$

In general, following a similar methodology, one may specify that the membership functions are inside or outside of an arbitrary ellipsoidal quadratic form with expressions in the form:

$$\mu^T S \mu + v \leq 0 \tag{37}$$

Details are left to the reader. A collection of such ellipsoids may be used to exclude any zones which are known to lie out of the range of the membership function vector (Figure 1).

B. Obtaining bounds in practice

As the shape of the membership functions is known in PDC fuzzy control, obtaining the bounds for any second-order polynomial in the form (21) may be cast as an optimization problem. Indeed, choosing arbitrary S and w, maximising $J(z) = \mu(z)^T S \mu(z) + w\mu(z)$ over the expected range of values of the premise variables z (previously denoted as Ω), yields a value $J_{max} = \max_{z\in\Omega} J(z)$. Once $J_{\max}$ is available[1], an expression in the form (21) can be obtained from $J(z) - J_{max} \leq 0$, *i.e.*, the polynomial $\mu^T S\mu + w\mu - J_{\max} \leq 0$.

As an example, consider the fuzzy partition in Figure 2. In this case, the bounds $\mu_2 - 0.86 \leq 0$ and $\mu_1\mu_3 - 0.0045 \leq 0$ may be easily computed by line-search on the one-dimensional set where the premise variable takes values. In fact, following a similar optimization approach, higher order polynomial bounds may be obtained; for instance, maximizing $\mu_1\mu_2\mu_3$: This idea inspires next section.

In other common cases, membership functions are the cartesian tensor product of simpler ones, describing either fuzzy partitions on individual variables or basic nonlinearities in the system equations, following the modeling methodology in [1]. In that case, it can be shown that certain products of memberships can be bounded by a power of 0.25 (because $\mu(1-\mu) \leq 0.25$). For instance, consider "high" and "low" to be contrary concepts defined on pressure and temperature universes. If a membership μ_1 is "temperature is low and pressure is low", and other membership μ_2 is "temperature is $high$ and pressure is $high$", then $\mu_1\mu_4 \leq 0.25^2$. Details of these situations appear in [11], [16].

[1]The optimization on Ω can be carried out by using any optimization technique; even a brute-force approach evaluating the memberships on a dense-enough grid on z may suffice. Note, however, that the approach may find difficulties with highly nonlinear memberships or a large number of premise variables in z.

IV. Generalisation to higher dimensions.

Consider also the possibility that a restriction on the shape of the membership functions is given by a multivariate polynomial of degree d, for instance

$$\mu_1^4 - 2\mu_3 + \mu_1\mu_2 - 0.3 < 0 \tag{38}$$

with $d = 4$, where monomials of degree four, one, two and zero appear. As another example, the memberships in Figure 2 fulfill $\mu_1\mu_2\mu_3 - 0.0039 \leq 0$.

Now, choose any arbitrary integer q fulfilling $q \geq d$. By multiplying each of the atomic monomials of degree d_i ($0 \leq d_i \leq d$) by $(\sum_{i=1}^r \mu_i)^{q-d_i}$ (which is identically equal to one), any polynomial of degree d can be converted to an homogeneous polynomial of degree q. For instance, the above (38) gets converted in

$$\mu_1^4 - 2\mu_3(\sum_{i=1}^4 \mu_i)^3 + \mu_1\mu_2(\sum_{i=1}^4 \mu_i)^2 - 0.3(\sum_{i=1}^4 \mu_i)^4 < 0 \tag{39}$$

So, let us restate a generalized version of Theorem III.1 for higher-degree homogeneous polynomial constraints, using the notation in Section II-D.

Consider a degree-q homogeneous polynomial constraint in memberships:

$$\sum_{\mathbf{i}\in\mathbb{I}_q} \mu_{\mathbf{i}}\beta_{\mathbf{i}} \leq 0 \tag{40}$$

The coefficients $\beta_{\mathbf{i}}$ may be considered elements of a multi-dimensional array (*i.e.*, a tensor [20]) $\mathcal{B}$, generalising the matrix appearing in (23). Note that any polynomial of degree lower or equal to q may be expressed as an homogeneous q-dimensional summation (40), by following the methodology used to obtain (39).

Theorem IV.1 *Consider a p-dimensional fuzzy summation condition* (12)*, jointly with shape-dependent knowledge expressed as constraints of degree q* (40)*. Choose any arbitrary integer n so that $n \geq \max(p,q)$. The positivity condition* (12) *(in the region determined by the constraints* (40)*) is fulfilled if* *there exists a positive definite matrix R so that the condition $\sum_{\mathbf{i}\in\mathbb{I}_n}\mu_{\mathbf{i}}x^TQ'_{\mathbf{i}}x\geq 0$, $\mathbf{i}=(i_1,i_2,\ldots,i_n)$, holds with:*

$$Q'_{\mathbf{i}}=Q_{i_1i_2\ldots i_p}+\beta_{i_1i_2\ldots i_q}R \tag{41}$$

The integer n will be denoted as the complexity parameter.

Proof: As $\sum_{k=1}^r\mu_k=1$, it's straightforward that:

$$\begin{aligned}\sum_{\mathbf{i}\in\mathbb{I}_p}\mu_{\mathbf{i}}x^TQ_{\mathbf{i}}x\geq\sum_{\mathbf{i}\in\mathbb{I}_p}\mu_{\mathbf{i}}x^TQ_{\mathbf{i}}x+(x^TRx)\cdot(\sum_{\mathbf{i}\in\mathbb{I}_q}\mu_{\mathbf{i}}\beta_{\mathbf{i}})=\\=(\sum_{k=1}^r\mu_k)^{n-p}\sum_{\mathbf{i}\in\mathbb{I}_p}\mu_{\mathbf{i}}x^TQ_{\mathbf{i}}x+(\sum_{k=1}^r\mu_k)^{n-q}\sum_{\mathbf{i}\in\mathbb{I}_q}\mu_{\mathbf{i}}x^T\beta_{\mathbf{i}}Rx=\\=\sum_{\mathbf{i}\in\mathbb{I}_n}\mu_{\mathbf{i}}x^TQ'_{\mathbf{i}}x\end{aligned} \tag{42}$$

so positivity of the last term implies positivity of the first one. ∎

Note that an extension to n_c constraints of degree q_k, $k=1,\ldots,n_c$, is straightforward, by using:

$$Q'_{\mathbf{i}}=Q_{i_1i_2\ldots i_p}+\sum_{k=1}^{n_c}\beta^{[k]}_{i_1i_2\ldots i_{q_k}}R_k \tag{43}$$

with $\mathbf{i}\in\mathbb{I}_n$, $n\geq\max(p,q_1,\ldots,q_{n_c})$, for some to-be-determined positive-definite R_k associated to each constraint tensor $\mathcal{B}^{[k]}$ (details omitted for brevity). As there is no loss of generality in assuming $q=\max_k q_k$ (by multiplying by a suitable power of $\sum_{i=1}^r\mu_i$), only a unique q will be considered in the sequel.

Once the new $Q'_{\mathbf{i}}$ are defined in (43), any sufficient condition to prove positivity of an n-dimensional fuzzy summation may be used. For instance, $n=3$ may be handled by adapting the conditions in [6, Theorem 5].

Corollary IV.1 *Given a set of n_c degree-3 polynomial restrictions, expressed as n_c homogeneous forms* (40)*, and any Q_{ijk} expressing some fuzzy control requirements, the 3-dimensional summation $\sum_{i=1}^r\sum_{j=1}^r\sum_{k=1}^r\mu_i\mu_j\mu_kx^TQ_{ijk}x$ is positive (in the region determined by the constraints) if there exist X_{ijk}, R_l so that, being $Q'_{ijk}=Q_{ijk}+\sum_{l=1}^{n_c}\beta^{[l]}_{ijk}R_l$:*

$$R\geq 0\quad X_{ijk}=X_{ikj}^T \tag{44}$$

$$Q'_{iii}\leq X_{iii} \tag{45}$$

$$Q'_{iij}+Q'_{iji}+Q'_{jii}\leq X_{iij}+X_{iji}+X_{jii}\quad i<j \tag{46}$$

$$\begin{aligned}Q'_{ijk}+Q'_{ikj}+Q'_{jik}+Q'_{jki}+Q'_{kij}+Q'_{kji}\leq\\X_{ijk}+X_{ikj}+X_{jik}+X_{jki}+X_{kij}+X_{kji}\quad i<j<k\end{aligned} \tag{47}$$

$$\begin{pmatrix}X_{i11}\ldots X_{i1r}\\\vdots\ddots\vdots\\X_{ir1}\ldots X_{irr}\end{pmatrix}>0\quad\forall i \tag{48}$$

And any complexity parameter $n\geq 2$ may be handled by using Theorems 4 and 5 in [7], replacing the original $Q_{\mathbf{i}}$ by Q' from (43):

Corollary IV.2 *A sufficient condition for positivity of Ξ in* (12)*, under as set of n_c constraints in the form* (40)*, is the positivity condition below, for $n\geq\max(p,q)$:*

$$\sum_{\mathbf{k}\in\mathbb{I}_{n-2}}\mu_{\mathbf{k}}\xi^T\begin{pmatrix}X_{(\mathbf{k},1,1)}&\ldots&X_{(\mathbf{k},1,r)}\\\vdots&\ddots&\vdots\\X_{(\mathbf{k},r,1)}&\ldots&X_{(\mathbf{k},r,r)}\end{pmatrix}\xi>0 \tag{49}$$

if there exist matrices $X_{\mathbf{j}}$, $\mathbf{j}\in\mathbb{I}_n$, and positive-definite R_l, $l=1,\ldots,t$, so that, for all $\mathbf{i}\in\mathbb{I}_n^+$

$$\sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})}(Q_{j_1j_2\ldots j_p}+\sum_{l=1}^{n_c}\beta^{[l]}_{j_1j_2\ldots j_q}R_l)\geq\sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})}\frac{1}{2}(X_{\mathbf{j}}+X_{\mathbf{j}}^T) \tag{50}$$

As the above corollary is the most general statement in this work, its proof is included for readability, even if it is a trivial adaptation of that in [7].

Proof:

$$\Xi=\sum_{\mathbf{i}\in\mathbb{I}_n}\mu_{\mathbf{i}}x^TQ'_{i_1i_2\ldots i_p}x=\sum_{\mathbf{i}\in\mathbb{I}_n^+}\mu_{\mathbf{i}}\sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})}x^TQ'_{j_1j_2\ldots j_p}x \tag{51}$$

Hence, if (50) holds, as $\mu_{\mathbf{i}}\geq 0$,

$$\begin{aligned}\sum_{\mathbf{i}\in\mathbb{I}_n^+}\mu_{\mathbf{i}}\sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})}x^TQ'_{j_1\ldots j_p}x\geq\sum_{\mathbf{i}\in\mathbb{I}_n^+}\mu_{\mathbf{i}}\sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})}x^T\frac{1}{2}(X_{\mathbf{j}}+X_{\mathbf{j}}^T)x=\\=\sum_{\mathbf{i}\in\mathbb{I}_n^+}\mu_{\mathbf{i}}\sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})}x^TX_{\mathbf{j}}x=\sum_{\mathbf{i}\in\mathbb{I}_n}\mu_{\mathbf{i}}x^TX_{\mathbf{i}}x=\\=\sum_{\mathbf{k}\in\mathbb{I}_{n-2}}\sum_{i=1}^r\sum_{j=1}^r\mu_{\mathbf{k}}\mu_i\mu_jx^TX_{(\mathbf{k},i,j)}x=\\=\sum_{\mathbf{k}\in\mathbb{I}_{n-2}}\mu_{\mathbf{k}}\xi^T\begin{pmatrix}X_{(\mathbf{k},1,1)}&\ldots&X_{(\mathbf{k},1,r)}\\\vdots&\ddots&\vdots\\X_{(\mathbf{k},r,1)}&\ldots&X_{(\mathbf{k},r,r)}\end{pmatrix}\xi\end{aligned} \tag{52}$$

where $\xi=(\mu_1x^T\ \mu_2x^T\ \ldots\ \mu_rx^T)^T$. Hence, if (49) and (50) hold, (12) also does. ∎

The number of decision variables can be reduced by assuming $X_{(\mathbf{k},i,j)}=X_{(\mathbf{k},j,i)}^T$ with no loss of generality, and practical application for $n>3$ requires setting up a recursive procedure: given a starting value of n, it provides sufficient conditions for the positivity of the n-dimensional sum expressed as an $(n-2)$-dimensional one. Hence, repeated application of the theorem is needed until: (a) 2-dimensional fuzzy summations are obtained (using Theorem 2 in [5] as a last step) when starting from an even n; (b) in the odd-n case, one-dimensional fuzzy summations are obtained, stating then the condition that each of the elements in the sum must be positive. The reader is referred to [7] for details.

V. Examples

Examples showing the usefulness of the particular case of known bounds on products of two memberships (Section III-A), appear in [11]. This section will present examples which allow to take into account some spherical shapes and higher-order products of memberships.

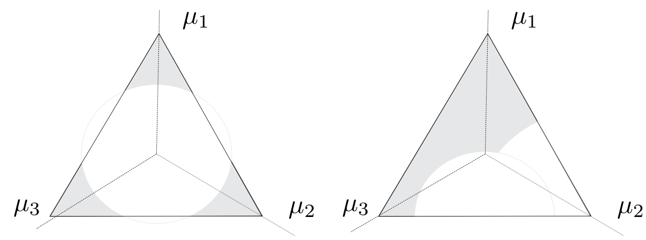

Fig. 3. Example 1: (left) case 1, (right) case 2. Shaded area denotes possible values of membership functions inside the triangle $\{\mu_1 + \mu_2 + \mu_3 = 1, \mu_i \geq 0\}$. Isometric projection.

A. Example 1 (quadratic constraints)

Consider the system (6), with $r = 3$ and

$$A_1 = \begin{pmatrix} -0.74 & 0.61 & 0.87 \\ 0.39 & -0.26 & 0.56 \\ 0.99 & 0.05 & -0.16 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0.99 & 0.65 \\ 0.2 & 0.87 \\ 0.76 & 0.12 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0.69 & 0.07 & 0.35 \\ 0.48 & 0.86 & 0.37 \\ 0.1 & 0.31 & 0.3 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0.96 & 0.36 \\ 0.76 & 0.17 \\ 0.04 & 0.20 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0.28 & 0.39 & 0.58 \\ 0.32 & 0.13 & 0.20 \\ 0.68 & 0.34 & 0.19 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0.32 & 0.02 \\ 0.72 & 0.15 \\ 0.29 & 0.06 \end{pmatrix}$$

The fastest decay rate[2] with a state feedback PDC law provable by Theorem 2 in [5], using the Q_{ij} in (7), is $\alpha = 0.51$. The procedures in this paper will be applied to achieve improved decay rates, when some knowledge about the membership function shape is available. Let us consider below three illustrative cases:

1) Case 1: Assume that, for a particular system, the membership vector *does not* lie inside a sphere centered at the origin ($c_i = 0$ in (34)) with radius δ (Figure 3-left), and that fact is known to the designer to take advantage of it.

Then, conditions (35) and (36) in Theorem III.2 result in the shape-dependent LMI conditions:

$$X_{ii} \leq Q_{ii} + (\delta^2 - 1)R \tag{53}$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} + 2\delta^2 R \quad i < j \tag{54}$$

Note that, for $0 \leq \delta \leq 1$, the conditions are progressively less conservative as δ increases, because the right-hand-side terms above add the positive terms $\delta^2 R$ and $2\delta^2 R$.

Let us check different cases for δ with the above system. In particular, values of $\delta > 1/\sqrt{2}$ produce faster decay rates than the case with arbitrary memberships: for $\delta = 0.72$ the fastest decay rate for which the LMI solver found a feasible solution is $\alpha = 0.96$; for $\delta = 1$ (only vertices) it is $\alpha = 3.4$, almost 7 times faster than the conventional shape-independent conditions[3] from [5].

[2]Note that the chosen performance measure has been decay rate, for simplicity; other features such as robustness margins or $\mathcal{H}_\infty$ bounds may be tested by selecting a different Q_{ij}, as discussed in Section II.

[3]The case $\delta = 1$ indicates that only the canonical vertices $(\mu_1, \mu_2, \mu_3) \in \{(1,0,0), (0,1,0), (0,0,1)\}$ are possible values of memberships (corresponding, for instance, to a switching linear system). In this case, conditions (53) and (54) result in: $\{X_{ii} \leq Q_{ii}, X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} + 2R\}$. Hence, with a large enough R, $X_{ij} = 0$ for $i \neq j$ can be made to be a solution, i.e., (1) holds if $Q_{ii} > 0$, which is a well-known result in "crisp" switching linear systems.

2) Case 2: Using expressions (35)-(36) as before, the knowledge that the membership vector lies outside of a sphere with center (0, 0.5, 0.5) and radius 0.4 (see Figure 3-right) is cast into the LMIs via (35)–(36). The result is a decay $\alpha = 0.62$. If a sphere with center (0,0,1) and radius 0.49 is used, the achievable decay is $\alpha = 0.56$. With both circles and two relaxation variables R_1, R_2, the achievable decay rate is $\alpha = 0.66$. All of the results are, as expected, better than the decay 0.51 with the standard shape-independent conditions.

3) Case 3: Multi-objective design (dual specifications): Consider the problem of achieving different levels of specifications in different regions of the working space[4]. For instance, only stability may suffice for certain infrequent cases whereas a faster decay may be desired at particular operation region, given by a particular polynomial constraint on the membership functions.

Under these assumptions, the LMI conditions for mere stability (stated for an unrestricted membership shape), may be adjoined with the decay rate ones incorporating membership shape information which would apply to a particular region.

Consider designing a regulator which achieves stability for the TS model under consideration, and simultaneously tries to achieve the fastest decay rate when when the distance of the membership vector to the point $(1, 0, 0)$ is smaller than 0.75. Then, conditions in [5, Theorem 2] with Q_{ij} in (7) with $\alpha = 0$ must be set up (with additional decision variables X_{ij}), as well as those with Q'_{ij} given by suitable β_{ij} from (35)–(36), using another set of decision variables X'_{ij}, evidently.

The result is that $\alpha = 1.11$ is the fastest "local" decay rate which can be proved feasible with the proposed setup, which more than doubles the one obtainable with a PDC on the whole operating space. Note, however, that faster fuzzy controllers might exist: on one hand, a source of conservativeness (well-known in multi-criteria LMI synthesis) is the shared Lyapunov function; on the other hand, the conservativeness inherent in the use of [5, Theorem 2] may be relaxed by using the results in [7].

B. Example 2 (cubic constraints)

Consider a 3-rule 2nd-order TS system with:

$$A_1 = \begin{pmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{pmatrix} \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{pmatrix} \quad B_2 = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -a & -4.33 \\ 0 & 0.05 \end{pmatrix} \quad B_3 = \begin{pmatrix} -b + 6 \\ -1 \end{pmatrix}$$

[4]The above multi-objective design is different, and complementary, to the usual approach in literature of achieving different sorts of performance bounds on *all* the state space (mixed $\mathcal{H}_2/\mathcal{H}_\infty$, $\mathcal{H}_\infty$ plus decay rate, etc. [12], [21]); as an alternative approach, the *same* performance type but with different level in different *regions* is suggested here. Note also that some definitions are needed in order to rigorously define the meaning of "local" decay rate in terms of basins of attraction and Lyapunov level sets. The definition of, say, a "local" $\mathcal{H}_\infty$ norm would also be cumbersome. These issues are, however, omitted for brevity as they are not the main objective of the paper. In the example in consideration, in a set of premise variables $\Omega_l \subset \Omega$, a "local" decay rate α will be said to have been proved when a Lyapunov function is found fulfilling $\dot{V}(x) \leq -2\alpha V(x)$ for all $z \in \Omega_l$ (indeed, z may include some or all of the components of x, as usual in TS modeling [1]).

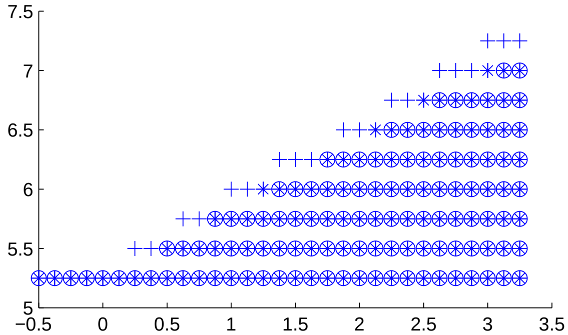

Fig. 4. Feasible points (example 2) under different restrictions with Theorem IV.2, $n = 3$. '⊗': $\beta_{ijk} = 0$; '*': $\mu_1\mu_2\mu_3 \leq 0.0004$; '+' $\mu_1\mu_2\mu_3 = 0$. Horizontal axis denotes parameter a, vertical axis denotes parameter b.

A stabilizing PDC controller is to be designed. The ranges of values for a and b yielding a feasible LMI are tested for several methods, in the grid $a \in [-0.5, 3.25]$, $b \in [5.25, 7.25]$.

The procedure in [5, Theorem 2] does not yield any feasible value in this parameter range. Even with no restriction ($\beta_{\mathbf{i}} = 0$), Corollary IV.1 (*i.e.*, [6]), produces feasible stabilising regulators for the values of a and b indicated by the "cart wheel" symbol in Figure 4.

When the restriction $\mu_1\mu_2\mu_3 \leq 0.0004$ is enforced, a few more feasible points appear (indicated by a star).

The restriction $\mu_1\mu_2\mu_3 = 0$ produces a larger set of feasible points; the $+$ sign on the figure pinpoints those combinations of parameter values yielding feasible controllers only under the last restriction. In all cases, the LMI solver was Matlab LMI Toolbox with default options.

VI. Conclusions

This paper presents results which relax stability and performance conditions for (possibly polynomial-in-membership) fuzzy control systems if knowledge of membership function shape is available. Such knowledge must be in the form of polynomial constraints. Particular cases are membership function overlap bounds (product of two or more membership functions) and ellipsoidal constraints (*i.e.*, incorporating knowledge about the membership functions lying either inside or outside of a particular ellipsoid). However, any polynomial in the membership functions, with arbitrary degree, is allowed.

Furthermore, multi-objective designs can be accomplished so that different control performance criteria may be specified in different regions of the operation space. This idea may be interesting in practical gain-scheduling applications.

As a result, more freedom in guaranteeing control requirements is available in any particular application with explicitly known membership functions; such situation is indeed the case when actually implementing PDC controllers.

Acknowledgment

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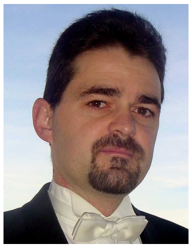

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