Journal of Advances in Mathematics and Computer Science

31(2): 1-17, 2019; Article no.JAMCS.47812

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Reliability Analysis of a Commodity-Supply Multi-State System Using the Map Method

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Authors' contributions

This work was carried out in collaboration between the two authors. Author AMAR envisioned and designed the study, performed the symbolic analysis, constructed the map solution, managed the literature search and wrote the entire manuscript. Author ABA performed the computational task and drew the various figures. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2019/v31i230107

Editor(s):

(1) Dr. Paul Bracken, Professor, Department of Mathematics, The University of Texas-Pan American, USA.

Reviewers:

(1) J. Ladvánszky, Hungary.

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(3) Snehadri B. Ota, India.

Complete Peer review History: http://www.sdiarticle3.com/review-history/47812

Original Research Article

Received: 16 December 2018
Accepted: 10 March 2019
Published: 18 March 2019

Abstract

A multi-state k-out-of-n: G system is a multi-state system whose multi-valued success is greater than or equal to a certain value j (lying between 1 (the lowest non-zero output level) and M (the highest output level)) whenever at least k_m components are in state m or above for all m such that $1 \leq m \leq j$. This paper is devoted to the analysis of a commodity-supply system that serves as a standard gold example of a non-repairable multi-state k-out-of-n: G system with independent non-identical components. We express each instance of the multi-state system output as an explicit function of the multi-valued inputs of the system. The ultimate outcome of our analysis is a Multi-Valued Karnaugh Map (MVKM), which serves as a natural, unique, and complete representation of the multi-state system. To construct this MVKM, we use "binary" entities to relate each of the instances of the output to the multi-valued inputs. These binary entities are represented *via* an eight-variable Conventional Karnaugh Map (CKM) that is adapted to a map representing four variables that are four-valued each. Despite the relatively large size of the maps used, they are still very convenient, thanks to their regular structure. No attempt was made to draw loops on the maps or to seek minimal formulas. The maps just served as handy tools for combinatorial representation and for collectively implementing the operations of ANDing, ORing, and

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complementation. The MVKM obtained serves as a means for symbolic analysis yielding results that agree numerically with those obtained earlier. The map is a useful tool for visualizing many system properties, and is a valuable resource for computing a plethora of Importance Measures for the components of the system.

Keywords: System reliability; k-out-of-n system; Multi-state system; Multiple-valued logic; Eight-variable Karnaugh Map; Multi-Valued Karnaugh Map.

1 Introduction

A binary k-out-of-n: G system is *uniquely* defined as a dichotomous system that is successful if and only if at least k out of its n components are successful [1-23], By contrast, a multi-state k-out-of-n: G system does not possess a unique definition [24-43]. The definition adopted herein is that this system is a multi-state system (MSS) whose multi-valued success is greater than or equal to a certain value j (lying between 1 (the lowest non-zero output level) and M (the highest output level)) whenever at least k_m components are in state m or above for all m such that $1 \leq m \leq j$ [34,40-43].

In this paper, we a study a standard multi-state system, which was proposed and studied by Tian et al. [34], and further studied by Fadhel et al. [44], Mo et al. [40], Rushdi [41], Rushdi & Al-Amoudi [42,43]. The system (shown in Fig. 1) is a supply system of a certain commodity (e.g., oil, water, energy, transportation traffic, or communication traffic, *etc*.) that employs four pipelines to transport the given commodity from the given *source* to three *sink* nodes called *stations*. Both the system and each pipeline have four states, which are defined as shown in Table 1. The states of the system are defined according to whether the demands of *up to* a certain station can be met. We use $S\{k\}$ $\{0 \leq k \leq 3\}$ to denote a binary indicator that the system can meet the commodity demand *up to* the station number k, *i.e.*, for all stations m $(1 \leq m \leq k)$. The states of each pipeline are defined according to which station/stations can be reached by the commodity supply *via* this pipeline. Therefore, pipeline number i is represented by a multi-valued variable X_i, which has four values or instances $X_i\{j\}, (1 \leq i \leq 4,\ 0 \leq j \leq 3)$. The instance $X_i\{j\}$ is a binary indicator that the commodity can reach *up to* station j through pipeline i.

Table 1. Definition of the four-valued input variable X_i, which determines the status of pipeline i $(1 \leq i \leq 4)$, and the four-valued output variable S, which detemines the overall system status

Value of X_i	Meaning
0	Pipeline i cannot transmit the commodity to any station.
1	Pipeline i can transmit the commodity *up to* station 1.
2	Pipeline i can transmit the commodity *up to* station 2.
3	Pipeline i can transmit the commodity *up to* station 3.
Value of S	**Meaning**
0	The system cannot meet the commodity demand of any station.
1	The system can meet the commodity demand of *up to* station 1.
2	The system can meet the commodity demand of *up to* station 2.
3	The system can meet the commodity demand of *up to* station 3.

We have recently reported several solutions of the aforementioned problem, and our present paper offers yet another solution of this problem. In our earlier solutions, we employed *purely-algebraic methods* of multi-valued logic, in which we handled multi-valued variables either directly [41] or through some binary encoding [42,43], with various map versions used occasionally for verification. In this paper, however, we deliberately avoid the mathematically-demanding algebraic manipulations in [41-43] by employing the Karnaugh map [45-50] as the sole vehicle for our manipulations. There is a long history of utilization of the Karnaugh map as a probability map (or reliability map) in the binary case [51-59]. There are also some notable applications of the Karnaugh map as a multi-value map [60-61]. Our work herein combines the probability and multi-value notions by adapting the map to multi-valued reliability calculations. We modify a regular form of the binary eight-variable Karnaugh map (of $2^8 = 256$ cells) [62-64] for use as a map of $256 = 4^4$ cells representing four variables that are four-valued each.

The organization of the remainder of this paper is as follows. Section 2 retrieves from Rushdi [41] a mathematical description of the example multi-state k-out-of-n system. Section 3 implements a purely-map analysis of the system. Section 4 shows that our numerical results exactly agree with those obtained by earlier authors. Section 5 discusses certain advantages of using the map, while Section 6 concludes the paper.

2 Mathematical Description of the Example Multi-State *k*-out-of-*n* System

In this Section, we summarize from Rushdi [41] a mathematical description of the example multi-state k-out-of-n system. We use $S_m \{1 \leq m \leq 3\}$ to depict the success of station m (the indicator that the commodity demand of station m is met). The successes of the three stations are given by

$$\begin{aligned} S_1 &= Sy(4; \{4\};\ \bar{X}_1\{0\}, \bar{X}_2\{0\}, \bar{X}_3\{0\}, \bar{X}_4\{0\}) \\ &= \bar{X}_1\{0\}\ \bar{X}_2\{0\}\ \bar{X}_3\{0\}\ \bar{X}_4\{0\}, \end{aligned} \tag{1a}$$

$$\begin{aligned} S_2 &= Sy(4; \{2,3,4\};\ X_1\{2\} \vee X_1\{3\}, X_2\{2\} \vee X_2\{3\}, X_3\{2\} \vee X_3\{3\}, X_4\{2\} \vee X_4\{3\}) \\ &= (X_1\{2\} \vee X_1\{3\})(X_2\{2\} \vee X_2\{3\}) \vee (X_1\{2\} \vee X_1\{3\})(X_3\{2\} \vee X_3\{3\}) \vee (X_1\{2\} \vee X_1\{3\})(X_4\{2\} \vee X_4\{3\}) \\ &\vee (X_2\{2\} \vee X_2\{3\})(X_3\{2\} \vee X_3\{3\}) \\ &\vee (X_2\{2\} \vee X_2\{3\})(X_4\{2\} \vee X_4\{3\}) \vee (X_3\{2\} \vee X_3\{3\})(X_4\{2\} \vee X_4\{3\}), \end{aligned} \tag{1b}$$

$$\begin{aligned} S_3 &= Sy(4; \{3,4\};\ X_1\{3\}, X_2\{3\}, X_3\{3\}, X_4\{3\}) \\ &= X_1\{3\}\, X_2\{3\}\, X_3\{3\} \ \vee\ X_1\{3\}\, X_2\{3\}\, X_4\{3\} \ \vee\ X_1\{3\}\, X_3\{3\}\, X_4\{3\} \ \vee\ X_2\{3\}\, X_3\{3\}\, X_4\{3\}\,. \end{aligned} \tag{1c}$$

The notation $Sy(n; \boldsymbol{A}; \boldsymbol{X})$ denotes a symmetric switching function (*SSF*), which is defined as [1, 4, 20, 41-43, 46-48, 65-68]:

$$f = Sy(n; \boldsymbol{A}; \boldsymbol{X}) = Sy(n; \{a_1, a_2, \dots, a_m\};\ X_1, X_2, \dots, X_n), \tag{2}$$

and is specified *via* its number of inputs n, its characteristic set

$$\boldsymbol{A} = \{a_0, a_1, \dots, a_m\} \subseteq \boldsymbol{I}_{n+1} = \{0, 1, 2, \dots, n\},\ \{m \leq n\}, \tag{3}$$

and its inputs $\boldsymbol{X} = [X_1, X_2, \dots, X_n]^{\mathrm{T}}$. This function has the value 1 if and only if

$$\textstyle\sum_{i=1}^{n} X_i = a_i, \tag{4}$$

for all integers i such that $0 \leq i \leq m,$ and has the value 0, otherwise.

The four instances of the system output variable S are related to station successes by [41]

$$S\{0\} = \bar{S}_1, \tag{5a}$$

$$S\{1\} = S_1\, \bar{S}_2, \tag{5b}$$

$$S\{2\} = S_1\, S_2\, \bar{S}_3, \tag{5c}$$

$$S\{3\} = S_1\, S_2\, S_3\,. \tag{5d}$$

3 Karnaugh-map Construction and Analysis

This Section describes how the current problem is solved through the construction of a series of Karnaugh maps. Each of Figs. 2-11 is a Karnaugh map of four four-valued inputs X_1, X_2, X_3 and X_4. This map is considerably large as it has $4^4 = 256$ cells, and is simply an adaptation of a map of eight binary variables that has the same number of cells ($2^8 = 256$), introduced earlier in [62-64]. Each of the maps in Figs. 2-10 has binary outputs belonging to $\{0, 1\}$, while the map in Fig. 11 alone has four-valued entries belonging to $\{0, 1, 2, 3\}$. In Figs. 2-10, every 1-entry is written explicitly, while all 0-entered cells are left blank (as usual).

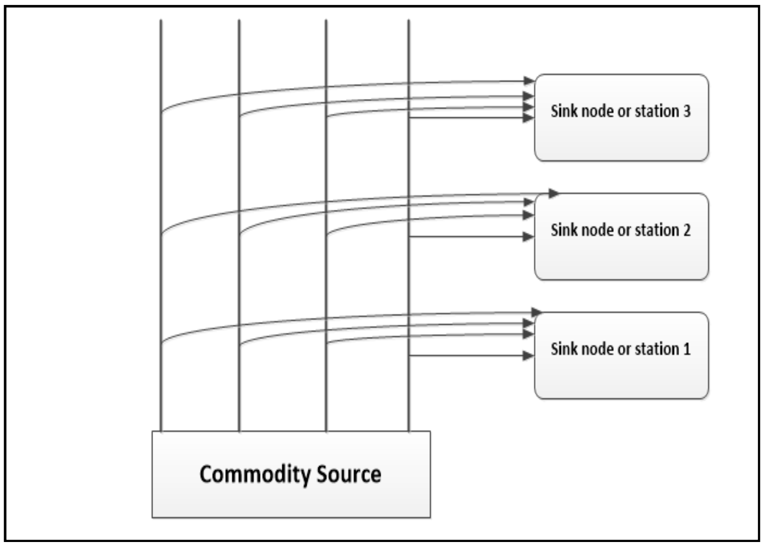

Fig. 1. A commodity-supply system that is modeled as a multi-state k-out-of-n: G system (Adapted from *Tian* et al. (2008))

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
																	0	0
																	1	
																	2	
																	3	
																	0	1
						1	1	1		1	1	1		1	1	1	1	
						1	1	1		1	1	1		1	1	1	2	
						1	1	1		1	1	1		1	1	1	3	
																	0	2
						1	1	1		1	1	1		1	1	1	1	
						1	1	1		1	1	1		1	1	1	2	
						1	1	1		1	1	1		1	1	1	3	
																	0	3
						1	1	1		1	1	1		1	1	1	1	
						1	1	1		1	1	1		1	1	1	2	
						1	1	1		1	1	1		1	1	1	3	
																	X_4	X_2

$$S_1$$

Fig. 2. A Karnaugh map (of four four-valued inputs) representing the success of station 1

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
											1	1			1	1	0	0
											1	1			1	1	1	
			1	1			1	1	1	1	1	1	1	1	1	1	2	
			1	1			1	1	1	1	1	1	1	1	1	1	3	
											1	1			1	1	0	1
											1	1			1	1	1	
			1	1			1	1	1	1	1	1	1	1	1	1	2	
			1	1			1	1	1	1	1	1	1	1	1	1	3	
			1	1			1	1	1	1	1	1	1	1	1	1	0	2
			1	1			1	1	1	1	1	1	1	1	1	1	1	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	3	
			1	1			1	1	1	1	1	1	1	1	1	1	0	3
			1	1			1	1	1	1	1	1	1	1	1	1	1	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	3	
																	X_4	X_2

$$S_2$$

Fig. 3. A Karnaugh map (of four four-valued inputs) representing the success of station 2

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
																	0	0
																	1	
																	2	
																1	3	
																	0	1
																	1	
																	2	
																1	3	
																	0	2
																	1	
																	2	
																1	3	
																1	0	3
																1	1	
																1	2	
				1				1					1	1	1	1	3	
																	X_4	X_2

$$S_3$$

Fig. 4. A Karnaugh map (of four four-valued inputs) representing the success of station 3.

The maps in Figs. 2-4 represent the two-valued station successes S_1, S_2 and S_3, as given by equations (1). These maps are filled-in *collectively* (and not in a cell-by-cell fashion), as we explain now. Equation (1a) sets to 1 (positively asserts) S_1 unless any of the four inputs X_1, X_2, X_3 or X_4 is negatively asserted (equated to 0). Excluding $\{X_1 = 0\}$ in Fig. 2 amounts to setting to 0 all cells in the first four columns of the map in Fig. 2, while avoiding $\{X_2 = 0\}$ assigns 0 to every cell in the first four rows of this map. Avoiding $\{X_3 = 0\}$ requires that 0 be entered in every cell in the first column of every group of four consecutive columns in Fig. 2, while rejecting $\{X_4 = 0\}$ does the same for every cell in the first row of every group of four consecutive rows in Fig. 2. For illustrative purposes, we highlight in yellow the blank (implicitly 0-entered) cells comprising $\{X_3 = 0\}$ in Fig. 2.

	X_1	0				1				2				3			
X_2	X_4 \ X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
0	0	1	1	1	1	1	1	1	1	1	1			1	1		
	1	1	1	1	1	1	1	1	1	1	1			1	1		
	2	1	1			1	1										
	3	1	1			1	1										
1	0	1	1	1	1	1	1	1	1	1	1			1	1		
	1	1	1	1	1	1	1	1	1	1	1			1	1		
	2	1	1			1	1										
	3	1	1			1	1										
2	0	1	1			1	1										
	1	1	1			1	1										
	2																
	3																
3	0	1	1			1	1										
	1	1	1			1	1										
	2																
	3																

$$\bar{S}_2$$

Fig. 5. A Karnaugh map (of four four-valued inputs) representing the failure of station 2, obtained by cell-wise complementation of the map in Fig. 3

	X_1	0				1				2				3			
X_2	X_4 \ X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
2	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
3	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
	3	1	1	1		1	1	1		1	1	1					

$$\bar{S}_3$$

Fig. 6. A Karnaugh map (of four four-valued inputs) representing the failure of station 3, obtained by cell-wise complementation of the map in Fig. 4

Equation (1b) sets to 1 (positively asserts) S_2 for six terms, the first of which is $(X_1\{2\} \vee X_1\{3\})(X_2\{2\} \vee X_2\{3\})$. The four columns covered by this term are highlighted in yellow in Fig. 3. Equation (1c) sets to 1 (positively asserts) S_3 for four terms, the first of which is $X_1\{3\}\, X_2\{3\}\, X_3\{3\}$. The four cells covered by this term are highlighted in yellow in Fig. 4. Figs. 5-7 are obtained by *collective* cell-wise complementation of the maps in Figs. 3, 4, and 2, respectively. Figures 7-10 express the four instances of the system output S *via* equations (5). Figures 8-10 use *collective* cell-wise ANDing of maps in the appropriate earlier figures. Figure 11 is a map of multi-valued entries, which represents the multi-valued output S. This map combines the results of the binary-entered maps in Figs. 7-10, which represent the four binary instances $S\{0\}, S\{1\}, S\{2\}$, and $S\{3\}$ of S. Either the four maps in Figs. 7-10, or (equivalently) the individual map in Fig. 11 can be read immediately to express the expectation of each instance (its probability of being equal to 1) as follows.

$$E\{S\{0\}\} = 1 - E\{\bar{X}_1\{0\}\}\, E\{\bar{X}_2\{0\}\}\, E\{\bar{X}_3\{0\}\}\, E\{\bar{X}_4\{0\}\}. \tag{6a}$$

$$\begin{aligned} E\{S\{1\}\} = {} & E\{X_1\{1\}\}\, E\{X_2\{1\}\}\, E\{X_3\{1\}\}\, (E\{X_4\{2\}\} + E\{X_4\{3\}\}) + \\ & E\{X_1\{1\}\}\, E\{X_2\{1\}\}\, (E\{X_3\{2\}\} + E\{X_3\{3\}\})\, E\{X_4\{1\}\} + \\ & E\{X_1\{1\}\}\, (E\{X_2\{2\}\} + E\{X_2\{3\}\})\, E\{X_3\{1\}\}\, E\{X_4\{1\}\} + \\ & (E\{X_1\{2\}\} + E\{X_1\{3\}\})\, E\{X_2\{1\}\}\, E\{X_3\{1\}\}\, E\{X_4\{1\}\} + \\ & E\{X_1\{1\}\}\, E\{X_2\{1\}\}\, E\{X_3\{1\}\}\, E\{X_4\{1\}\} \end{aligned} \tag{6b}$$

$$\begin{aligned} E\{S\{3\}\} = {} & E\{X_1\{3\}\}\, E\{X_2\{3\}\}\, E\{X_3\{3\}\}\, (E\{X_4\{2\}\} + E\{X_4\{1\}\}) + \\ & E\{X_1\{3\}\}\, E\{X_2\{3\}\}\, (E\{X_3\{2\}\} + E\{X_3\{1\}\})\, E\{X_4\{3\}\} + \\ & E\{X_1\{3\}\}\, (E\{X_2\{2\}\} + E\{X_2\{1\}\})\, E\{X_3\{3\}\}\, E\{X_4\{3\}\} + \\ & (E\{X_1\{2\}\} + E\{X_1\{1\}\})\, E\{X_2\{3\}\}\, E\{X_3\{3\}\}\, E\{X_4\{3\}\} + \\ & E\{X_1\{3\}\}\, E\{X_2\{3\}\}\, E\{X_3\{3\}\}\, E\{X_4\{3\}\}. \end{aligned} \tag{6c}$$

$$E\{S\{2\}\} = 1 - \left(E\{S\{0\}\} + E\{S\{1\}\} + E\{S\{3\}\}\right). \tag{6d}$$

X_2	X_4 \ X_1	0				1				2				3			
	X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1				1				1			
	2	1	1	1	1	1				1				1			
	3	1	1	1	1	1				1				1			
2	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1				1				1			
	2	1	1	1	1	1				1				1			
	3	1	1	1	1	1				1				1			
3	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1				1				1			
	2	1	1	1	1	1				1				1			
	3	1	1	1	1	1				1				1			

$$\boldsymbol{S\{0\} = \bar{S}_1}$$

Fig. 7. A Karnaugh map for the binary indicator of instant $S\{0\} = \bar{S}_1$ of system output, obtained by cell-wise complementation of the map in Fig. 2

4 Comparisons with Previous Work

The problem handled herein was solved *via* various techniques by Tian et al. [34], Mo. et al. [40], Rushdi [41], and Rushdi & Al-Amoudi [42,43]. In all cases, the results were tested by the following input matrix, in which the sum of entries in each row is 1, since such entries are the probabilities of mutually exclusive and exhaustive events.

$$\{E\{X_i\{j\}\}\} = \begin{bmatrix} .0500 & .0950 & .0684 & .7866 \\ .0500 & .0950 & .0684 & .7866 \\ .0300 & .0776 & .0446 & .8478 \\ .0300 & .0776 & .0446 & .8478 \end{bmatrix} \qquad (1 \le i \le 4\,,\ 0 \le j \le 3) \tag{7}$$

Table 2 compares our results for this specific input with the results of the earlier teams of authors. The six sets of results are essentially the same, despite the existence of minor differences in precision.

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
																	0	0
																	1	
																	2	
																	3	
																	0	1
						1	1	1		1				1			1	
						1											2	
						1											3	
																	0	2
						1											1	
																	2	
																	3	
																	0	3
						1											1	
																	2	
																	3	
																	X_4	X_2

$$S\{1\} = S_1\,\bar{S}_2$$

Fig. 8. A Karnaugh map for the binary indicator of instant $S\{1\} = S_1\,\bar{S}_2$ of system output, obtained by cell-wise ANDing of the maps in Figs. 2 and 5

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
																	0	0
																	1	
																	2	
																	3	
																	0	1
											1	1			1	1	1	
							1	1		1	1	1		1	1	1	2	
							1	1		1	1	1		1	1		3	
																	0	2
							1	1		1	1	1		1	1	1	1	
						1	1	1		1	1	1		1	1	1	2	
						1	1	1		1	1	1		1	1		3	
																	0	3
							1	1		1	1	1		1	1		1	
						1	1	1		1	1	1		1	1		2	
						1	1			1	1						3	
																	X_4	X_2

$$S\{2\} = S_1\,S_2\,\bar{S}_3$$

Fig. 9. A Karnaugh map for the binary indicator of instant $S\{2\} = S_1\,S_2\,\bar{S}_3$ of system output obtained by cell-wise ANDing of the maps in Figs. 2, 3 and 6

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
																	0	0
																	1	
																	2	
																	3	
																	0	1
																	1	
																	2	
																1	3	
																	0	2
																	1	
																	2	
																1	3	
																	0	3
																1	1	
																1	2	
								1				1		1	1	1	3	
																	X_4	X_2

$$S\{3\} = S_1 S_2 S_3$$

Fig. 10. A Karnaugh map for the binary indicator of instant $S\{3\} = S_1 S_2 S_3$ of system output, obtained by cell-wise ANDing of the maps in Figs. 2, 3 and 4

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	1	1	1	0	1	2	2	0	1	2	2	1	
	0	0	0	0	0	1	2	2	0	2	2	2	0	2	2	2	2	
	0	0	0	0	0	1	2	2	0	2	2	2	0	2	2	3	3	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
	0	0	0	0	0	1	2	2	0	2	2	2	0	2	2	2	1	
	0	0	0	0	0	2	2	2	0	2	2	2	0	2	2	2	2	
	0	0	0	0	0	2	2	2	0	2	2	2	0	2	2	3	3	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3
	0	0	0	0	0	1	2	2	0	2	2	2	0	2	2	3	1	
	0	0	0	0	0	2	2	2	0	2	2	2	0	2	2	3	2	
	0	0	0	0	0	2	2	3	0	2	2	3	0	3	3	3	3	
																	X_4	X_2

$$S$$

Fig. 11. A MVKM representing the multi-valued output S, obtained by combining information from the four maps in Figs. 7-10

Table 2. Comparison of the present results with those in earlier work

	Tian et al. [34]	**Mo et al. [40]**	**Rushdi [41]**	**Rushdi & Al-Amoudi [42,43]**	**Present results**
E{S(0)}	0.1508	0.150838	0.150837750000	0.150837750000000	0.150837750000
E{S(1)}	0.0023	0.002282	0.002282548128	0.002282548128000	0.002282548128
E{S(2)}	0.0892	0.089181	0.089180866436	0.089180866435691	0.089180866436
E{S(3)}	0.7577	0.757699	0.757698835436	0.757698835436309	0.757698835436
Total	1.0000	1.000000	1.000000000000	1.000000000000000	1.000000000000

5 Discussions

The ultimate outcome of our analysis is the Multi-Valued Karnaugh Map (MVKM) of Fig. 11, which serves as a natural, unique, and complete representation of the multi-state system. One can obtain many useful insights and deduce certain (not-so-obvious) facts from this map.

- The map reveals the nature of the four binary instances $S\{0\}, S\{1\}, S\{2\}$, and $S\{3\}$ of S, when these instances are viewed as individual binary reliability systems. The instance $S\{0\}$ acts like a coherent binary *failure* while the instance $S\{3\}$ behaves like a coherent binary *success*. Both $S\{1\}$ and $S\{2\}$ have a general *non-coherent* behavior, which somewhat mimics that of a k-to-l-out-of-n: G system [65, 66], or a double-threshold system [67, 68]. It is interesting to note that the instances $S\{0\}$, $S\{1\}$, and $S\{2\}$ are non-coherent in a binary sense, though each of the station successes S_1, S_2 and S_3 is coherent in the same sense. By contrast, the overall system output S is coherent in a multi-state sense.
- The map offers a convenient pictorial mechanism for *decomposing* its output function into various sub-functions, thereby constructing a multi-valued expansion tree or decision diagram for this function [1-4, 19-23, 41, 65-71].
- The map is a tool to visualize each of the properties of *causality*, *monotonicity*, and *relevancy*, which when combined together amount to labelling the present multi-state system as a *coherent* one [43].
- The map demonstrates *total symmetry* of the system function S with respect to its four arguments X_1, X_2, X_3 and X_4. Total symmetry means that the map entries are invariant to interchanging any two of the four arguments [46].
- The map in Fig. 11 is a valuable resource for computing a plethora of Importance Measures [72-96] for the current multi-state system. Importance Measures are used to assess the criticality of individual components within the system, identify system weaknesses, and rank components so as to prioritize potential reliability improvements A crucial map feature in this respect is the capability of the map to perform “Boolean differentiation” or “Boolean differencing” through appropriate map folding [87-100].
- Tedious algebraic manipulations were needed in [41-43] to prove that

  $$S_1 S_3 \leq S_1 S_2 , \tag{8}$$

  Equation (8) is a useful result, since it facilitates the derivation of an algebraic expression for $S\{3\}$. However, inspection of Figs. 2-4 reveals not only (8) but also the more powerful result

  $$S_3 \leq S_2 , \tag{9}$$

  Direct inspection of Figs. 2-4 also attests that S_1 is neither comparable to S_2 nor comparable to S_3. Figures 7-10 confirm that the four instances $S\{0\}, S\{1\}, S\{2\}$, and $S\{3\}$ of S form an orthonormal set, thereby allowing a consistent construction of the MVKM in Fig. 11.

6 Conclusions

This paper demonstrated how *MSS* reliability can be handled solely *via* Karnaugh maps of multi-valued inputs, and of binary or multi-valued entries. A classical *MSS* problem was manually analyzed by maps that resemble eight-variable Karnaugh maps. Despite the relatively large size of the maps used, they were very convenient, indeed. No attempt was made to draw loops on the maps or to seek minimal formulas. The maps just served as handy tools for combinatorial representation and for collective implementation of the operations of ANDing, ORing, and complementation. Results obtained are satisfactory as they exactly replicate earlier results obtained by various automated and manual means.

Acknowledgement

The first-named author (AMR) benefited greatly from (and is sincerely grateful for) his earlier collaboration and enlightening discussions with Engineer Mahmoud Ali Rushdi, Research Scientist at fortiss (Forschungsinstitut des Freistaats Bayern für softwareintensive Systeme und Services (Research Institute of the Free State of Bavaria for Software-Intensive Systems and Services)), Munich, Germany.

Competing Interests

Authors have declared that no competing interests exist.

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