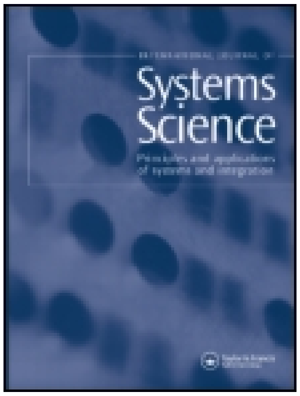

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Reliability assessment for fuzzy multi-state systems

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Fuzzy multi-state system (FMSS) is defined as a multi-state system (MSS) consisting of multi-state elements (MSE) whose performance rates and transition intensities are presented as fuzzy values. Due to the lack, inaccuracy or fluctuation of data, it is oftentimes impossible to evaluate the performance rates and transition intensities of MSE with precise values. This is true especially in continuously degrading elements that are usually simplified to MSE for computation convenience. To overcome these challenges in evaluating the behaviour of MSS, fuzzy theory is employed to facilitate MSS reliability assessment. Given the fuzzy transition intensities and performance rates, the state probabilities of MSE and MSS are also fuzzy values. A fuzzy continuous-time Markov model with finite discrete states is proposed to assess the fuzzy state probability of MSE at any time instant. The universal generating function with fuzzy state probability function and performance rate is applied to evaluate fuzzy state probability of MSS in accordance with the system structure. A modified FMSS availability assessment approach is introduced to compute the system availability under the fuzzy user demand. In order to obtain the membership functions of the indices of interest, parametric programming technique is employed according to Zadeh's extension principle. The effectiveness of the proposed method is illustrated and verified via reliability assessment of a multi-state power generation system.

Keywords: fuzzy multi-state system (FMSS); fuzzy multi-state element (FMSE); fuzzy reliability assessment; fuzzy Markov model; fuzzy universal generating function (FUGF); parametric programming

1. Introduction

In the real world, many systems perform their task with degraded performance levels (performance rates). This phenomenon is mainly caused by the degradation of components and parts in the system or/and the failure of some elements which deteriorates the system performance. This type of system is called multi-state system (MSS) and was first introduced in the mid-1970s by Murchland (1975).

The MSS widely exists in industrial engineering (Lisnianski and Levitin 2003), e.g. power generation systems, computing systems, transportation systems, and radio relay station, etc. Many novel methods were developed to facilitate the MSS reliability assessment, e.g. the extended decision diagram-based method (Shrestha and Xing 2008), the stochastic process (Li and Pham 2005), the universal generating function (UGF) (Ushakov 1986; Levitin 2005), and the Monte Carlo simulation (Zio, Podofillini and Levitin 2004; Zio, Marella and Podofillini 2007), etc. Some specific MSS existing in particular fields were also studied in recent years, e.g. the dependent MSS (Levitin 2004), the multi-state weighted system (Li and Zuo 2008), the generalised multi-state k-out-of-n:F system (Zuo and Tian 2006) and acyclic multi-state-node networks (Yeh 2006), etc.

However, conventional MSS reliability assessment methods are based on the following two assumptions (Ding and Lisnianski 2008):

(1) The state probabilistic distributions of multi-state element (MSE) in the MSS are precisely known and measurable;
(2) The performance rate of MSE is precisely determined.

Actually, these assumptions do not always hold when precisely evaluating the probability distribution and performance rate in each state is difficult. There are two main reasons for this lack of precise information:

(1) Getting accurate and sufficient data is impossible in some systems and environments. Therefore, the evaluation of element/system characteristics can only be expressed in terms like 'a unit would fail in about 1 year' and 'system performance degrades nearly 200 per unit time'. Thus, crisp values used to represent the probabilistic distributions and performance rates sometimes make no sense.

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(2) Many elements/systems deteriorate continuously or nearly continuously with time, and they are usually considered as having many discrete states. To avoid the 'dimension damnation' (Lisnianski 2007), the model is simplified via state combination to reduce the computational burden. Therefore, the continuously degrading element/system is finally simplified to one with several states separated by the distinguishable performance rates, and the number of discrete states is usually not large to make the computation tractable (Lisnianski 2001).

Because of these two reasons, the conventional approach for representing the performance and state distribution of MSE/MSS in crisp values fails to capture the actual element/system behaviour.

Fuzzy reliability theory, which employs the fuzzy theory introduced by Zadeh (1965, 1978), is becoming a new methodology to study the imprecision and uncertainty phenomena in reliability engineering (Cai 1991), and it has since received increasing attention. For example, Cai, Wen and Zhang (1991) introduced the fuzzy success/failure state and the reliability model to study a gradually degrading computing system. Huang (1995) assessed the reliability a system in the presence of fuzziness in operating time. Huang, Tong and Zuo (2004) proposed to evaluate the failure possibility via posbist fault tree analysis when statistical data is scarce or failure probability is extremely small. A novel fuzzy Bayesian approach was developed by Wu (2004), to create the fuzzy Bayes point estimator of reliability. Huang, Zuo and Sun (2006) introduced a Bayesian method to assess system reliability when lifetime data is presented as a fuzzy value. Fuzzy dynamic reliability evaluation for a deteriorating system under imperfect repair action was addressed by Verma, Srividya and Gaonkar (2004). Ke, Huang and Lin (2006) developed a procedure to construct the fuzzy steady-state availability when obtained data are subjective. Two-unit repairable systems suffering common-cause failure was discussed by Huang, Lin and Ke (2008), where the time to failure follows fuzzified exponential distribution. Pandey and Tyagi (2007) proposed a new method to assess the profust reliability indices. However, most of the reported works mainly focus on binary-state system issues. As stated in Lisnianski and Levitin (2003), the MSS is already very popular in industry, so the fuzzy reliability under MSS context remains an emerging research paradigm. The concept of fuzzy multi-state system (FMSS) was first used by Ding and Lisnianski (2008) in a modelling study of the state probabilities and performances of a component presented as fuzzy values. In their work, fuzzy UGF (FUGF) method was proposed to assess reliability and availability of FMSS under the fuzzy demand. Afterwards, some general definitions involving relevancy, coherency, dominance and equivalence in FMSS were provided by Ding, Zuo, Lisnianski and Tian (2008), to extend the basic properties of MSS in crisp case to the fuzzy context. However, they just gave the steady fuzzy state probabilities and performance rate for each MSE. To facilitate engineers in understanding the system behaviour and risk with respect to working time, a dynamic fuzzy reliability assessment method for the FMSS is investigated in this work.

In this article, FMSS is introduced to overcome the deficiencies of conventional MSS theory. The state transition intensity and the performance rates of MSE are presented as fuzzy values. Fuzzy Markov model are developed in accordance with the fuzzy transition intensity matrix. Dynamic fuzzy reliability with respect to working time can be calculated through the fuzzy Kolmogorov's equation of fuzzy Markov model. With the assistance of FUGF, the fuzzy state probability and performance rate of FMSS can be evaluated through aggregating the fuzzy behaviour of FMSE. The membership functions of indices of interest are computed by parametric programming technique. In addition, a modified approach is introduced to assess the fuzzy availability when the membership functions of performances rate and user demand are overlapping, and the membership function of the fuzzy availability is examined by regarding the α-cut level as 'fuzzy risk' which can be tolerated.

The remainder of this article is organised as follows. Section 2 introduces the definition of MSS. Fuzzy set, fuzzy number and extension principle are briefly reviewed in Section 3. The definitions of FMSE and FMSS are given in Section 4. The fuzzy Markov model and FUGF, as well as fuzzy availability, are also discussed in this section. The proposed model and approach are illustrated in Section 5 via a power generation system. A brief conclusion is given in Section 6.

Nomenclature

N:	number of independent elements in the MSS
k_l:	number of states for element l
$\mathbf{g}_l$:	set of possible crisp performance rates for element l
$g_{(l,i)}$:	crisp performance rate of element l in state i
$\mathbf{p}_l(t)$:	set of crisp state probabilities for element l at time t
$p_{(l,i)}(t)$:	crisp probability of element l staying in state i at time t

$G_l(t)$: random variable representing the crisp performance rate of element l at time t
K: number of states for the MSS
$\mathbf{G_s}$: set of possible crisp performance rates for the MSS
g_{si}: crisp performance rate of the MSS staying in state i
$\mathbf{p}(t)$: set of crisp state probabilities for the MSS at time t
$p_{si}(t)$: crisp probability of the MSS staying in state i at time t
$G_s(t)$: random variable representing the crisp performance rate of the MSS at time t
$\phi(\cdot)$: MSS structure function
$E(t)$: crisp instantaneous expected performance rate of the MSS at time t
w: crisp user demand
$A(t, w)$: crisp instantaneous availability of the MSS when user demand is w at time t
$\tilde{\mathbf{g}}_l$: set of possible fuzzy performance rates for element l
$\tilde{g}_{(l,i)}$: fuzzy performance rate of element l in state i
$\tilde{g}^L_{(l,i)\alpha}$ lower bound of the α-cut level set of fuzzy performance rate when the element l in state i
$\tilde{g}^U_{(l,i)\alpha}$ upper bound of the α-cut level set of fuzzy performance rate when the element l in state i
$\tilde{p}_{(l,i)}(t)$: fuzzy probability of element l staying in state i at time t
$\tilde{p}_{si}(t)$: fuzzy probability of the FMSS staying in state i at time t
$\lambda^l_{(i,j)}$: crisp intensity of element l transiting from state i to state j
$\tilde{\lambda}^l_{(i,j)}$: fuzzy intensity of element l transiting from state i to state j
$\tilde{p}^L_{(l,i)\alpha}(t)$: lower bound of the α-cut level set of the fuzzy probability that element l is staying in state i at time t
$\tilde{p}^U_{(l,i)\alpha}(t)$: upper bound of the α-cut level set of the fuzzy probability that element l is staying in state i at time t
$\tilde{\lambda}^{l,L}_{(i,j)\alpha}$: lower bound of the α-cut level set of the fuzzy intensity that element l transits from state i to state j
$\tilde{\lambda}^{l,U}_{(i,j)\alpha}$: upper bound of the α-cut level set of the fuzzy intensity that element l transits from state i to state j
$\tilde{p}^L_{si\alpha}(t)$: lower bound of the α-cut level set of the fuzzy probability that the FMSS is staying in state i at time t
$\tilde{p}^U_{si\alpha}(t)$: upper bound of the α-cut level set of the fuzzy probability that the FMSS is staying in state i at time t
$\tilde{\Omega}_\phi$: fuzzy performance rate composition operator in FUGF
$\tilde{g}_{si}$: fuzzy performance rate of the FMSS in state i
$\tilde{g}^L_{(si\alpha)}$ lower bound of the α-cut level set of fuzzy performance rate when the FMSS in state i
$\tilde{g}^U_{(si\alpha)}$ upper bound of the α-cut level set of fuzzy performance rate when the FMSS in state i
$\tilde{E}(t)$: fuzzy instantaneous expected performance rate of the FMSS at time t
$\tilde{A}^L_\alpha(t, \tilde{w})$: lower bound of the α-cut level set of the FMSS instantaneous availability under the fuzzy user demand $\tilde{w}$
$\tilde{A}^U_\alpha(t, \tilde{w})$: upper bound of the α-cut level set of the FMSS instantaneous availability under the fuzzy user demand $\tilde{w}$

2. Multi-state system

According to Lisnianski and Levitin (2003), a system that can have a finite number of performance rates is called an MSS. There are many different situations in which a system should be considered to be an MSS:

(1) Any system consisting of different units that have a cumulative performance effect on the entire system.
(2) The system consisting of elements with performances that can vary as a result of their deterioration (fatigue, partial failures, etc.) and repairs.

In order to analyse MSS behaviour under crisp value context, one has to know the characteristics of its elements. Any system element l can have k_l different states corresponding to the performance rates, which is represented by the set:

$$\mathbf{g}_l = \{g_{(l,1)}, g_{(l,2)}, \ldots, g_{(l,k_l)}\}, \tag{1}$$

where $g_{(l,i)}$ $(g_{(l,i)} \geq 0)$ is the performance rate of element l in its state $i, i \in \{1, 2, \ldots, k_l\}$, and if k_l is greater than two, the element is called an MSE.

Performance rate $G_l(t)$ of element l at any instant $t \geq 0$ is a random variable, taking value from $\mathbf{g}_l : G_l(t) \in \mathbf{g}_l$. Therefore, for any time interval $[0, T]$, the performance rate of element l is defined as a stochastic process. The probabilities associated with different states of the element l at any instant t can be represented by the set:

$$\mathbf{p}_l(t) = \{p_{(l,1)}(t), p_{(l,2)}(t), \ldots, p_{(l,k_l)}(t)\}, \tag{2}$$

where $p_{(l,i)}(t)$ represents the probability that $G_l(t) = g_{(l,i)}$. The state probabilities should satisfy the condition $\sum_{i=1}^{k_l} p_{(l,i)}(t) = 1$, for the elements' states at any time instant t compose the complete group of mutually exclusive events.

Suppose an MSS is consisting of N independent elements: its states are separated through its performance rate, which is unambiguously determined by the system configuration and performance rates of elements. Without loss of generality, it assumes that the entire MSS has K different states according to its possible performance rates, and g_{si} denotes the system performance rate in state $i(i \in \{1, \ldots, K\})$. Thus, the MSS performance rate at time t is also a random variable $G_s(t)$ that takes values from the set $\mathbf{G_s} = \{g_{s1}, \ldots, g_{sK}\}$, and is given by:

$$G_s(t) = \phi(G_1(t), \ldots, G_l(t), \ldots, G_N(t)), \tag{3}$$

where $G_l(t), (1 \le l \le N)$ is the performance stochastic process of the l-th element, and $\phi(\cdot)$ is system structure function. Thus, the instantaneous probabilities associated with the individual system state can be denoted by the set:

$$\mathbf{p}(t) = \{p_{s1}(t), p_{s2}(t), \ldots, p_{sK}(t)\}, \tag{4}$$

where $p_{si}(t)$ represents the probability that $G_s(t) = g_{si}$, and g_{si} is corresponding performance rate at the i-th system state. Thus, the instantaneous expected system performance rate is formulated as:

$$E(t) = \sum_{i=1}^{K} p_{si}(t) \cdot g_{si}. \tag{5}$$

The instantaneous availability of the MSS is defined as the probability that system performance rate is not less than the user demand w at any instant t, and is written as:

$$A(t, w) = \Pr(G_s(t) \ge w) = \sum_{i=1}^{K} p_{si}(t) 1(F(g_{si}, w) \ge 0), \tag{6}$$

where $1(x)$ is unity function: $1(TRUE) = 1$, $1(FALSE) = 0$, and $F(g_{si}, w) = g_{si} - w$.

3. Fuzzy set theory

3.1. *Fuzzy set and fuzzy number*

A fuzzy subset $\tilde{X}$ of a universal set U is defined by its membership (or characteristic) function $\mu_{\tilde{X}} : U \to [0, 1]$. The values of $\mu_{\tilde{X}}(x)$ extend from zero to one which can be interpreted as the membership degree at which x belongs to $\tilde{X}$.

Let $\Re$ be a universal set of real numbers and $\tilde{X}$ be a fuzzy subset of $\Re$. $\tilde{X}_\alpha = \{x | \mu_{\tilde{X}}(x) \ge \alpha\}$ denotes the α-cut level set of $\tilde{X}$ where $\alpha \in [0, 1]$. The interval of this set is written as $\tilde{X}_\alpha = [\tilde{X}_\alpha^L, \tilde{X}_\alpha^U]$, and $\tilde{X}_0$ is the closure of the set $\tilde{X}_0 = \{x | \mu_{\tilde{X}}(x) \ge 0\}$.

$\tilde{X}$ is called a fuzzy real number if: (1) it is a normal and convex fuzzy set; (2) its membership function is upper semi-continuous; (3) the 0-cut level set $\tilde{X}_0$ is bounded in $\Re$; (4) the 1-cut level set $\tilde{X}_1$ is a singleton set, and $\tilde{X}_1^L = \tilde{X}_1^U$; (5) the boundary functions $L(\alpha) = \tilde{X}_\alpha^L$ and $U(\alpha) = \tilde{X}_\alpha^U$ of membership functions are continuous with respect to $\alpha \in [0, 1]$.

The membership function of a typical triangle fuzzy number (TFN) $\tilde{X}$ parameterised by the triplet (a, b, c), is defined as:

$$\mu_{\tilde{X}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & x = b \\ \dfrac{x-c}{b-c}, & b < x \le c \\ 0, & \text{otherwise} \end{cases}, \tag{7}$$

and is plotted in Figure 1.

In this article, both the state transition intensity and performance rate of each element are treated as TFN, because it is straightforward to manipulate TFN in mathematical calculation, and it has been widely used in many practical situations and reliability engineering (Ding and Lisnianski 2008; Huang 1995; Verma, Srividya and Gaonkar 2004; Ding, Zuo, Lisnianski and Tian 2008; Pardo and Fuente 2008; Kleiner, Sadiq and Rajani 2006; Alex 2007; Chen 1994).

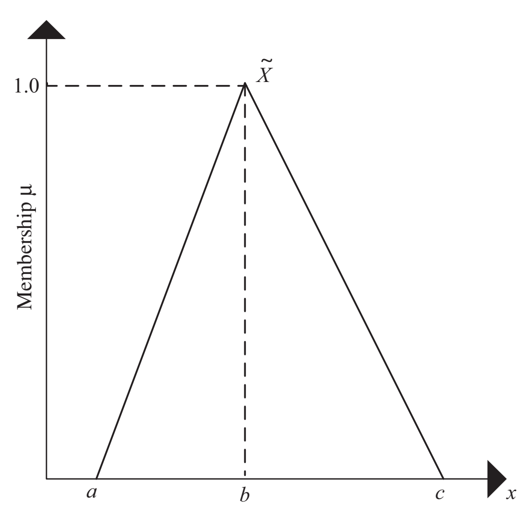

Figure 1. The membership function of a TFN.

3.2. *Extension principle and parametric programming technique*

Zadeh (1965, 1978) introduced the extension principle to obtain the membership function of a function with n fuzzy numbers as inputs:

$$\begin{aligned}\mu_{\tilde{p}(\tilde{\mathbf{x}})}(z) &= \sup_{\substack{\mathbf{x}\in\mathbf{R^n}\\ z=p(\mathbf{x})}} \min\{\mu_{\tilde{\mathbf{X}}}(\mathbf{x})\}\\ &= \sup_{\substack{x_1\in\Re_1,\ldots,x_n\in\Re_n\\ z=p(x_1,\ldots,x_n)}} \min\{\mu_{\tilde{X}_1}(x_1),\ldots,\mu_{\tilde{X}_n}(x_n)\},\end{aligned} \tag{8}$$

where $\tilde{\mathbf{X}}$ represents a set of input fuzzy numbers $\{\tilde{X}_1,\ldots,\tilde{X}_n\}$, $\mathbf{x}$ is a set of inputs variables $\{x_1,\ldots,x_n\}$. $\mathbf{R^n}$ is a set $\{\Re_1,\ldots,\Re_n\}$ representing the universal sets of real numbers, and $p(\cdot)$ is a function mapping inputs $\mathbf{x}$ to a output variable z. According to the extension principle, the interval of α-cut level set of fuzzy number $\tilde{p}(\tilde{\mathbf{x}})$ is given by:

$$\begin{aligned}\tilde{p}_\alpha(\tilde{\mathbf{x}}) &= [\min p(\mathbf{x};\, \mu_{\tilde{\mathbf{x}}}(\mathbf{x}) \geq \alpha),\, \max p(\mathbf{x};\, \mu_{\tilde{\mathbf{x}}}(\mathbf{x}) \geq \alpha)]\\ &= \left[\tilde{p}_\alpha^L, \tilde{p}_\alpha^U\right].\end{aligned} \tag{9}$$

Thus, the lower and upper bounds of $\tilde{p}(\mathbf{x})$ at α-cut level could be obtained by a pair of parametric programming as follows:

$$\tilde{p}_\alpha^L: \quad \begin{aligned}&\min\ p(x_1,\ldots,x_n)\\ &s.t. \quad \tilde{x}_{1\alpha}^L \leq x_1 \leq \tilde{x}_{1\alpha}^U\\ &\qquad\qquad \vdots \qquad ,\\ &\qquad \tilde{x}_{n\alpha}^L \leq x_n \leq \tilde{x}_{n\alpha}^U\end{aligned} \tag{10}$$

$$\tilde{p}_\alpha^U: \quad \begin{aligned}&\max\ p(x_1,\ldots,x_n)\\ &s.t. \quad \tilde{x}_{1\alpha}^L \leq x_1 \leq \tilde{x}_{1\alpha}^U\\ &\qquad\qquad \vdots \qquad .\\ &\qquad \tilde{x}_{n\alpha}^L \leq x_n \leq \tilde{x}_{n\alpha}^U\end{aligned} \tag{11}$$

This parametric programming problem can be realised by computer program, and it can easily find a couple of extreme values subjected to different intervals of input variables $\mathbf{x}$ determined by α-cut level.

4. FMSE, FMSS and reliability assessment

4.1. *Definitions of FMSE and FMSS*

FMSE is defined as the MSE in which the state performance rates, the corresponding state probabilities or transition intensities between each pair of states are presented as fuzzy values. In this case, any system element l has k_l different states characterised by fuzzy performance rates $\tilde{\mathbf{g}}_l = \{\tilde{g}_{(l,1)},\ldots,\tilde{g}_{(l,k_l)}\}$, and the instantaneous state probabilities are represented by fuzzy values $\tilde{\mathbf{p}}_l(t) = \{\tilde{p}_{(l,1)}(t),\ldots,\tilde{p}_{(l,k_l)}(t)\}$. The FUGF proposed by Ding and Lisnianski (2008) can be applied to describe the dynamic behaviour of each element at any time instant t, as:

$$\begin{aligned}\tilde{u}_l(z,t) &= \sum_{i_l=1}^{k_l} \tilde{p}_{(l,i_l)}(t)\cdot z^{\tilde{g}_{(l,i_l)}} = \tilde{p}_{(l,1)}(t)\cdot z^{\tilde{g}_{(l,1)}}\\ &\quad + \tilde{p}_{(l,2)}(t)\cdot z^{\tilde{g}_{(l,2)}} + \cdots + \tilde{p}_{(l,k_l)}(t)\cdot z^{\tilde{g}_{(l,k_l)}}.\end{aligned} \tag{12}$$

Since an MSS is consisting of more than one FMSE, this kind of MSS is defined as FMSS, for its state probability and performance rate inherit the fuzzy property from FMSE. Once the dynamic fuzzy state probability of each FMSE is available, the dynamic behaviour of the FMSS can be expressed through the combination rules based on the system structure function and the property of its performance rate under the fuzzy context.

4.2. *Fuzzy Markov Model for FMSE*

Based on the definition of FMSE, the state-space diagram of the l-th non-repairable FMSE takes the form presented in Figure 2, where state k_l is the best state, and state 1 is the worst state (total failure). The transition intensity between states i and j is presented as the fuzzy value $\tilde{\lambda}_{(i,j)}^l$.

With the fuzzy transition intensities, the state probability of elements l at time t is also a fuzzy value denoted as $\tilde{p}_{(l,i)}(t)$, where $1 \leq i \leq k_l$. The fuzzy Markov model is proposed to evaluate the dynamic fuzzy state probability $\tilde{p}_{(l,i)}(t)$ at any time instant.

The fuzzy transition intensity matrix for the l-th FMSE is given as:

$$|\tilde{\boldsymbol{\lambda}}^l| = \begin{array}{c} \\ 1 \\ \vdots \\ k_l \end{array}\!\!\begin{array}{c} \text{State } 1 \quad \ldots \quad k_i \\ \begin{pmatrix} \tilde{\lambda}_{(1,1)}^l & \cdots & \tilde{\lambda}_{(1,k_l)}^l \\ \vdots & \ddots & \vdots \\ \tilde{\lambda}_{(k_l,1)}^l & \cdots & \tilde{\lambda}_{(k_l,k_l)}^l \end{pmatrix}\end{array}. \tag{13}$$

where $\tilde{\lambda}_{(i,i)}^l = -\sum_{\substack{j=1\\ j\neq i}}^{k_l} \tilde{\lambda}_{(i,j)}^l$. For the l-th non-repairable FMSE, $\tilde{\lambda}_{(i,j)}^l = 0$ for $j > i$. Then, the Kolmogorov's equation with fuzzy transition intensities takes the

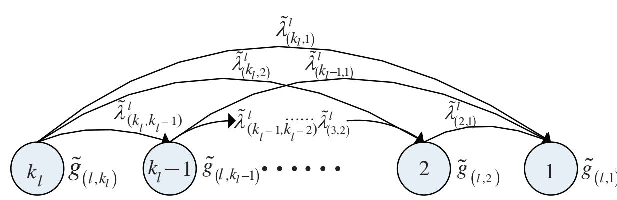

Figure 2. The state-space diagram of the l-th non-repairable FMSE.

form (Trivedi 2002):

$$\begin{cases} \dfrac{d\tilde{p}_{(l,k_l)}(t)}{dt} = -\tilde{p}_{(l,k_l)}(t)\sum_{j=1}^{k_l-1}\tilde{\lambda}^l_{(k_l,j)} \\ \dfrac{d\tilde{p}_{(l,i)}(t)}{dt} = \sum_{j=i+1}^{k_l}\tilde{\lambda}^l_{(j,i)}\tilde{p}_{(l,j)}(t) - \tilde{p}_{(l,i)}(t)\sum_{j=1}^{i-1}\tilde{\lambda}^l_{(i,j)}, \\ \qquad 1 < i < k_l, t \geq 0 \\ \dfrac{d\tilde{p}_{(l,1)}(t)}{dt} = \sum_{j=2}^{k_l}\tilde{\lambda}^l_{(j,1)}\tilde{p}_{(l,j)}(t) \end{cases} \tag{14}$$

with initial conditions: $\tilde{p}_{(l,k_l)}(0) = 1$, $\tilde{p}_{(l,i)}(0) = 0$ for $(i \neq k_l)$. Sometimes, it is too difficult to solve the first-order fuzzy differential equations. For the sake of reducing computational complexity, Laplace–Stieltjes transform is adopted to transform Equation (14) into linear equations:

$$\begin{cases} s\tilde{p}_{(l,k_l)}(s) - 1 = -\tilde{p}_{(l,k_l)}(s)\sum_{j=1}^{k_l-1}\tilde{\lambda}^l_{(k_l,j)} \\ s\tilde{p}_{(l,i)}(s) = \sum_{j=i+1}^{k_l}\tilde{\lambda}^l_{(j,i)}\tilde{p}_{(l,j)}(s) - \tilde{p}_{(l,i)}(s)\sum_{j=1}^{i-1}\tilde{\lambda}^l_{(i,j)}, \quad 1 < i < k_l \\ s\tilde{p}_{(l,1)}(s) = \sum_{j=2}^{k_l}\tilde{\lambda}^l_{(j,1)}\tilde{p}_{(l,j)}(s) \end{cases} \tag{15}$$

where $s\tilde{p}_{(l,i)}(s) - \tilde{p}_{(l,i)}(0) = L\left[(d\tilde{p}_{(l,i)}(t))/dt\right]$ and $L[\bullet]$ denotes Laplace–Stieltjes operator.

Solving (15) gives $\tilde{p}_{l,i}(s)$ as function of $\tilde{\boldsymbol{\lambda}}^l_{(i,j)}$ and s, and then the inverse Laplace–Stieltjes transform is executed to get the $\tilde{p}_{(l,i)}(t)$ in time domain:

$$\tilde{p}_{(l,i)}(t) = L^{-1}\left[\tilde{p}_{(l,i)}(s)\right] = f_{(l,i)}(\tilde{\boldsymbol{\lambda}}^l, t), \tag{16}$$

where $L^{-1}[\bullet]$ is reverse Laplace–Stieltjes operator, and $\tilde{p}_{(l,i)}(t)$ is a function in terms of fuzzy variables $\tilde{\boldsymbol{\lambda}}^l = \{\tilde{\boldsymbol{\lambda}}^l_{(k_l,k_l-1)}, \ldots, \tilde{\boldsymbol{\lambda}}^l_{(i,j)}, \ldots, \tilde{\boldsymbol{\lambda}}^l_{(2,1)}\}$ at any time t. The α-cut level interval of $\tilde{p}_{(l,i)}(t)$ can be obtained as:

$$\begin{aligned} \tilde{p}_{(l,i)\alpha}(t) &= \left[\min f_{(l,i)}(\boldsymbol{\lambda}^l, t; \mu_{\tilde{\lambda}^l}(\boldsymbol{\lambda}^l) \geq \alpha),\right. \\ &\qquad \left.\max f_{(l,i)}\left(\boldsymbol{\lambda}^l, t; \mu_{\tilde{\lambda}^l}(\boldsymbol{\lambda}^l) \geq \alpha\right)\right] \\ &= \left[\tilde{p}^L_{(l,i)\alpha}(t), \tilde{p}^U_{(l,i)\alpha}(t)\right], \quad (t \geq 0, 0 \leq \alpha \leq 1) \end{aligned} \tag{17}$$

The lower bound $\tilde{p}^L_{(l,i)\alpha}(t)$ and the upper bound $\tilde{p}^U_{(l,i)\alpha}(t)$ of any α-cut level set can be calculated by the parametric programming introduced in Section 3.2.

Lower bound:

$$\begin{aligned} \tilde{p}^L_{(l,i)\alpha}(t): \quad & \min \quad f_{(l,i)}(\boldsymbol{\lambda}^l, t), \quad (t \geq 0, 0 \leq \alpha \leq 1) \\ & s.t. \quad \tilde{\lambda}^{l,L}_{(k_l,k_l-1)\alpha} \leq \lambda^l_{(k_l,k_l-1)} \leq \tilde{\lambda}^{l,U}_{(k_l,k_l-1)\alpha} \\ & \qquad \vdots \\ & \qquad \tilde{\lambda}^{l,L}_{(2,1)\alpha} \leq \lambda^l_{(2,1)} \leq \tilde{\lambda}^{l,U}_{(2,1)\alpha} \end{aligned}, \tag{18}$$

Upper bound:

$$\begin{aligned} \tilde{p}^U_{(l,i)\alpha}(t): \quad & \max \quad f_{(l,i)}(\boldsymbol{\lambda}^l, t), \quad (t \geq 0, 0 \leq \alpha \leq 1) \\ & s.t. \quad \tilde{\lambda}^{l,L}_{(k_l,k_l-1)\alpha} \leq \lambda^l_{(k_l,k_l-1)} \leq \tilde{\lambda}^{l,U}_{(k_l,k_l-1)\alpha} \\ & \qquad \vdots \\ & \qquad \tilde{\lambda}^{l,L}_{(2,1)\alpha} \leq \lambda^l_{(2,1)} \leq \tilde{\lambda}^{l,U}_{(2,1)\alpha} \end{aligned}. \tag{19}$$

4.3. *Fuzzy universal generating function*

Following the basic concept of the UGF (Ushakov 1986; Levitin 2005), that the performance rate and probability of any system state can be evaluated through recursive composition operation once the behaviour of each element at time instant t is known, the FUGF of the FMSS is written as (Ding and Lisnianski 2008):

$$\begin{aligned} \tilde{U}_s(z,t) &= \tilde{\Omega}_\phi\left(\sum_{i_1=1}^{k_1}\tilde{p}_{(1,i_1)}(t)\cdot z^{\tilde{g}_{(1,i_1)}}, \ldots, \sum_{i_N=1}^{k_N}\tilde{p}_{(N,i_N)}(t)\cdot z^{\tilde{g}_{(N,i_N)}}\right) \\ &= \sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2}\cdots\sum_{i_N=1}^{k_N}\left(\prod_{l=1}^{N}\tilde{p}_{(l,i_l)}(t)\cdot z^{\phi(\tilde{g}_{(1,i_1)},\ldots,\tilde{g}_{(N,i_N)})}\right) \end{aligned} \tag{20}$$

The composition function $\phi(\cdot)$ that maps the fuzzy performance rates of all the elements into the fuzzy performance rate of the system is determined by the system structure and the property of performance rate (e.g. flow capacity, processing speed, etc.). In general, if the fuzzy performance rate $\tilde{g}_{si}$ of the FMSS is due to a certain combination of element states $\tilde{g}_{(1,i_1)}, \ldots, \tilde{g}_{(N,i_N)}$ and denoted as

$$\tilde{g}_{si} = \phi(\tilde{g}_{(1,i_1)}, \ldots, \tilde{g}_{(N,i_N)}), \tag{21}$$

the associated fuzzy state probability at time instant t is equal to:

$$\tilde{p}_{si}(t) = \prod_{l=1}^{N}\tilde{p}_{(l,i_l)}(t), \tag{22}$$

where $\tilde{p}_{(l,i_l)}(t)$ is a function of $\tilde{\boldsymbol{\lambda}}^l$ and t as shown in Equation (16).

The interval of α-cut level set of the system fuzzy performance rate $\tilde{g}_{si}$ is given by:

$$\begin{aligned} \tilde{g}_{si\alpha} &= \left[\min \phi(g_{(1,i_1)}, \ldots, g_{(N,i_N)}; \mu_{\tilde{g}_{(l,i_l)}}(g_{(l,i_l)}) \geq \alpha),\right. \\ &\qquad \left.\max \phi(g_{(1,i_1)}, \ldots, g_{(N,i_N)}; \mu_{\tilde{g}_{(l,i_l)}}(g_{(l,i_l)}) \geq \alpha)\right] \\ &= \left[\tilde{g}^L_{si\alpha}, \tilde{g}^U_{si\alpha}\right], \quad (1 \leq l \leq N, 0 \leq \alpha \leq 1). \end{aligned} \tag{23}$$

The lower bound $\tilde{g}^L_{si\alpha}$ and the upper bound $\tilde{g}^U_{si\alpha}$ of the α-cut level set can be calculated by the parametric programming as follows.

Lower bound:

$$\tilde{g}^{L}_{si\alpha}: \quad \min \phi(g_{(1,i_1)},\ldots,g_{(N,i_N)};\mu_{\tilde{g}_{(l,i_l)}}(g_{(l,i_l)}) \geq \alpha), \\ (0 \leq \alpha \leq 1, 1 \leq l \leq N) \\ s.t. \quad \tilde{g}^{L}_{(1,i_1)\alpha} \leq g_{(1,i_1)} \leq \tilde{g}^{U}_{(1,i_1)\alpha} \\ \vdots \quad , \\ \tilde{g}^{L}_{(N,i_N)\alpha} \leq g_{(N,i_N)} \leq \tilde{g}^{U}_{(N,i_N)\alpha} \tag{24}$$

Upper bound:

$$\tilde{g}^{U}_{si\alpha}: \quad \max \phi(g_{(1,i_1)},\ldots,g_{(N,i_N)};\mu_{\tilde{g}_{(l,i_l)}}(g_{(l,i_l)}) \geq \alpha), \\ (0 \leq \alpha \leq 1, 1 \leq l \leq N) \\ s.t. \quad \tilde{g}^{L}_{(1,i_1)\alpha} \leq g_{(1,i_1)} \leq \tilde{g}^{U}_{(1,i_1)\alpha} \\ \vdots \quad . \\ \tilde{g}^{L}_{(N,i_N)\alpha} \leq g_{(N,i_N)} \leq \tilde{g}^{U}_{(N,i_N)\alpha} \tag{25}$$

Assume the fuzzy performance rate of any element is presented by the TFN $\tilde{g}_{(l,i_l)}(a_{(l,i_l)}, b_{(l,i_l)}, c_{(l,i_l)})$, $(1 \leq l \leq N, 1 \leq i_l \leq k_l)$. Some typical composition functions usually applied in the flow transmission and task processing MSS are given in the fuzzy context as follows:

(1) Flow transmission type of FMSE connected in parallel

If any two FMSE are connected in parallel, the composition function for the arbitrary element state performance rate is given as:

$$\phi(\tilde{g}_{(1,i_2)}, \tilde{g}_{(2,i_2)}) = \tilde{g}_{(1,i_2)} + \tilde{g}_{(2,i_2)}, \quad 1 \leq i_1 \leq k_1, 1 \leq i_2 \leq k_2. \tag{26}$$

where $\tilde{g}_{(1,i_1)}$ and $\tilde{g}_{(2,i_2)}$ denote the fuzzy transmission capacities when elements 1 and 2 are in the state i_1 and i_2, respectively. According to the mathematical calculation rule of fuzzy numbers (Chen 1994) or the parametric programming at any α-cut level as formulated in Equations (24) and (25), the output of composition function is exactly equal to a TFN denoted as $((a_{(1,i_1)} + a_{(2,i_2)}), (b_{(1,i_1)} + b_{(2,i_2)}), (c_{(1,i_1)} + c_{(2,i_2)}))$, which represents the total fuzzy performance rate if the elements are connected in parallel with the fuzzy performance rates of their current states.

(2) Flow transmission type of FMSE connected in series

If the two FMSE are connected in series, the composition function for the arbitrary element state performance rate is given as:

$$\phi(\tilde{g}_{(1,i_2)}, \tilde{g}_{(2,i_2)}) = \min\{\tilde{g}_{(1,i_2)}, \tilde{g}_{(2,i_2)}\}, \quad 1 \leq i_1 \leq k_1, \\ 1 \leq i_2 \leq k_2. \tag{27}$$

This function is no longer a linear operation on fuzzy numbers as Equation (26), and calculating the membership function of the composition function output can only resort to the parametric programming as Equations (24) and (25). It is a tedious and time-consuming work (Ding and Lisnianski 2008). To overcome the deficiency, an approximating approach is considered to be applied to facilitate the computation in the non-linear fuzzy number operations (Chen 1994; Ding and Lisnianski 2008; Lee 2005; Verma, Srividya and Gaonkar 2004). Thus, the output of Equation (27) is approximated by a TFN expressed as $(\min\{a_{(1,i_1)}, a_{(2,i_2)}\}, \min\{b_{(1,i_1)}, b_{(2,i_2)}\}, \min\{c_{(1,i_1)}, c_{(2,i_2)}\})$, to void the computational complexity from using Equations (24) and (25).

(3) Task processing type of FMSE connected in parallel

If $\tilde{g}_{(1,i_1)}$ and $\tilde{g}_{(2,i_2)}$ represent the fuzzy task processing speeds of the two parallel-connected elements at their states i_1 and i_2, respectively, the composition function is written as (Levitin 2004):

$$\phi(\tilde{g}_{(1,i_1)}, \tilde{g}_{(2,i_2)}) = \tilde{g}_{(1,i_1)} + \tilde{g}_{(2,i_2)}, \quad 1 \leq i_1 \leq k_1, 1 \leq i_2 \leq k_2, \tag{28}$$

where the function output is a fuzzy processing speed, which is regarded as the total processing time needed when the two elements process a unit task in a simultaneous mode. Thus, one has $\phi(\tilde{g}_{(1,i_1)}, \tilde{g}_{(2,i_2)}) = ((a_{(1,i_1)} + a_{(2,i_2)}), (b_{(1,i_1)} + b_{(2,i_2)}), (c_{(1,i_1)} + c_{(2,i_2)}))$, which is solved in the same manner as Equation (26).

(4) Task processing type of FMSE connected in series

If the two task processing elements are connected in series, the total processing time is equal to the summation of the operation times on these two elements (Levitin 2004). The total fuzzy task processing speed in this scenario is formulated as:

$$\phi(\tilde{g}_{(1,i_1)}, \tilde{g}_{(2,i_2)}) = (\tilde{g}_{(1,i_1)} \cdot \tilde{g}_{(2,i_2)}) / (\tilde{g}_{(1,i_1)} + \tilde{g}_{(2,i_2)}), \\ 1 \leq i_1 \leq k_1, 1 \leq i_2 \leq k_2. \tag{29}$$

With the assistance of the approximating method for non-linear fuzzy number operations (Chen 1994), the total fuzzy processing speed can be also presented as a TFN written as $((a_{(1,i_1)} \cdot a_{(2,i_2)})/(c_{(1,i_1)} + c_{(2,i_2)}), (b_{(1,i_1)} \cdot b_{(2,i_2)})/b_{(1,i_1)} + b_{(2,i_2)}, (c_{(1,i_1)} \cdot c_{(2,i_2)})/(a_{(1,i_1)} + a_{(2,i_2)}))$.

The performance rate of FMSS can be computed through the above composition functions recursively, while the interval of α-cut level set of the corresponding fuzzy state probability $\tilde{p}_{si}(t)$ is given by:

$$\begin{aligned}\tilde{p}_{si\alpha}(t) &= \left[\min\left(\prod_{l=1}^{N} p_{(l,i_l)}(t);\ \mu_{\tilde{\lambda}^l}(\boldsymbol{\lambda}^l)\geq\alpha\right),\right.\\ &\qquad \left.\max\left(\prod_{l=1}^{N} p_{(l,i_l)}(t);\ \mu_{\tilde{\lambda}^l}(\boldsymbol{\lambda}^l)\geq\alpha\right)\right]\\ &= \left[\tilde{p}^L_{si\alpha}(t), \tilde{p}^U_{si\alpha}(t)\right], \quad (0\leq\alpha\leq 1).\end{aligned} \tag{30}$$

In the same manner, a couple of parametric programming formulas are established to find out the lower bound $\tilde{p}^L_{si\alpha}(t)$ and the upper bound $\tilde{p}^U_{si\alpha}(t)$ at any α-cut level:

Lower bound:

$$\begin{aligned}\tilde{p}^L_{si\alpha}(t):\quad &\min \prod_{l=1}^{N} p_{(l,i_l)}(t), \quad (t\geq 0, 0\leq\alpha\leq 1)\\ &s.t.\quad \tilde{\lambda}^{l,L}_{(k_l,k_l-1)\alpha}\leq\lambda^l_{(k_l,k_l-1)}\leq\tilde{\lambda}^{l,U}_{(k_l,k_l-1)\alpha}\\ &\qquad\vdots\\ &\qquad \tilde{\lambda}^{l,L}_{(2,1)\alpha}\leq\lambda^l_{(2,1)}\leq\tilde{\lambda}^{l,U}_{(2,1)\alpha}\end{aligned}, \tag{31}$$

Upper bound:

$$\begin{aligned}\tilde{p}^L_{si\alpha}(t):\quad &\max \prod_{l=1}^{N} p_{(l,i_l)}(t), \quad (t\geq 0, 0\leq\alpha\leq 1)\\ &s.t.\quad \tilde{\lambda}^{l,L}_{(k_l,k_l-1)\alpha}\leq\lambda^l_{(k_l,k_l-1)}\leq\tilde{\lambda}^{l,U}_{(k_l,k_l-1)\alpha}\\ &\qquad\vdots\\ &\qquad \tilde{\lambda}^{l,L}_{(2,1)\alpha}\leq\lambda^l_{(2,1)}\leq\tilde{\lambda}^{l,U}_{(2,1)\alpha}\end{aligned}. \tag{32}$$

The instantaneous expected performance rate inherits the fuzzy property from fuzzy state probability and performance rate at any time instant, and it is written as:

$$\tilde{E}(t) = \sum_{i=1}^{K}\tilde{p}_{si}(t)\cdot\tilde{g}_{si}. \tag{33}$$

Thus, the α-cut level set of the fuzzy instantaneous expected performance rate $\tilde{E}(t)$ is given by:

$$\begin{aligned}\tilde{E}_\alpha(t) &= \left[\min\left(\sum_{i=1}^{K}p_{si}(t)\cdot g_{si};\ \mu_{\tilde{p}_{si}(t)}(p_{si}(t))\geq\alpha, \mu_{\tilde{g}_{si}}(g_{si})\geq\alpha\right),\right.\\ &\qquad\left.\max\left(\sum_{i=1}^{K}p_{si}(t)\cdot g_{si};\ \mu_{\tilde{p}_{si}}(p_{si}(t))\geq\alpha, \mu_{\tilde{g}_{si}}(g_{si})\geq\alpha\right)\right]\\ &= \left[\tilde{E}^L_\alpha(t), \tilde{E}^U_\alpha(t)\right], \quad (0\leq\alpha\leq 1)\end{aligned} \tag{34}$$

The parametric programming technique can also be resorted to solve the upper and lower bound of any α-cut level set.

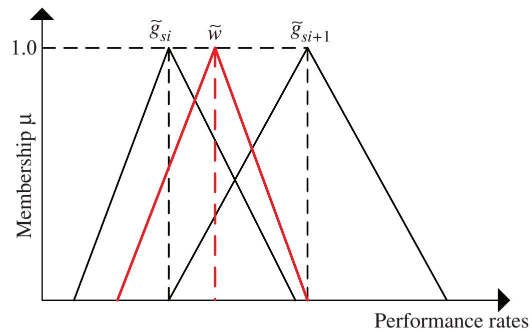

Figure 3. Fuzzy performance rates and demand with overlapping membership functions.

4.4. *Fuzzy availability assessment for FMSS*

For an MSS with crisp performance rate and user demand w where $g_{si} < w \leq g_{si+1}$ $(i = 1,\ldots,K-1)$, the acceptable states are the states $i+1,\ldots,K$, and the instantaneous availability of MSS is equal to:

$$A(t,w) = \sum_{j=i+1}^{K} p_{sj}(t). \tag{35}$$

Considering an FMSS where the user demand is presented as fuzzy value $\tilde{w}$, and availability of the FMSS is defined as probability that the performance of the FMSS satisfies the fuzzy demand $\tilde{w}$. However, the membership functions of the fuzzy performances rates and user demand may overlap as shown in Figure 3. As we consider the value of α as the level (or degree) of 'fuzzy risk' (or fuzzy interval) one is willing to tolerate, the availability of the FMSS will vary at different α-cut levels. In other words, if α is increased, the fuzzy divergence of indices of interest will shrink and finally the availability will become a crisp value when $\alpha = 1$.

The relative cardinality $|ar_i|_{rel}$ originally proposed by Ding and Lisnianski (2008), is defined as:

$$|ar_i|_{rel} = \frac{|ar_i|}{|r_i|}, \tag{36}$$

where $|ar_i|$ represents the cardinality of the $\tilde{ar_i}$, which is a fuzzy set representing that the system performance rate in state i is not less than the user demand w, and $|r_i|$ is the cardinality of the fuzzy number $\tilde{r}_i = \tilde{g}_{si} - \tilde{w}$. However, the definition of relative cardinality neglects the α-cut level of fuzzy demand and fuzzy performance under different 'fuzzy risk', and the membership function of fuzzy availability is defined as $\tilde{A}(\tilde{w}) = \sum_{i=1}^{K}\tilde{p}_i\cdot|ar_i|_{rel}$. Ignoring the effect of 'fuzzy risk' may result in some problematic issues. Suppose at time t, the fuzzy state probabilities and fuzzy performance rates of a two-state FMSS are $\tilde{p}_{s1}(t) = (0.15, 0.2, 0.25)$, $\tilde{p}_{s2}(t) =$ (0.75, 0.8, 0.85) and $\tilde{g}_{s1} = (0, 5, 10)$,

$\tilde{g}_{s2} = (10, 15, 20)$, respectively. If one examines the availability under the user demand $\tilde{w} = 15$, the availability is just equal to 0.8 if $\alpha = 1$, because it can be considered as a crisp value case. However, according to the relative cardinality approach in Ding and Lisnianski (2008), one has $\tilde{r}_2 = \tilde{g}_2 - \tilde{w} = (-5, 0, 5)$, and the availability is equal to 0.4 $(\tilde{A}_{\alpha=1}(t, \tilde{w}) = \tilde{p}_{s2\alpha=1}(t) \times \frac{|ar_2|}{|r_2|} = 0.8 \times 0.5 = 0.4)$. Apparently, the outcome from the relative cardinality approach with $\alpha = 1$ is not consistent with the one in the crisp context. Therefore, we propose a modified approach to overcome this drawback.

Define the α-cut level cardinality of fuzzy set $\tilde{r}_i = \tilde{g}_i - \tilde{w}$ by:

$$|r_i|_\alpha = \sum_{\substack{r_i \in R_i \\ \mu_{\tilde{r}_i}(r_i) \geq \alpha}} \mu_{\tilde{r}_i}(r_i), \quad 0 \leq \alpha \leq 1, \tag{37}$$

where R_i is range of r_i, and define the α-cut level cardinality of fuzzy set $a\tilde{r}_i$ as:

$$|ar_i|_\alpha = \sum_{\substack{r_i \in R_i \\ r_i \geq 0 \\ \mu_{\tilde{r}_i}(r_i) \geq \alpha}} \mu_{\tilde{r}_i}(r_i), \quad 0 \leq \alpha \leq 1. \tag{38}$$

Thus, the α-cut level relative cardinality of adequate demand fuzzy set is given by:

$$|ar_i|_\alpha^{rel} = \frac{|ar_i|_\alpha}{|r_i|_\alpha}. \tag{39}$$

Hence, the α-cut level of the instantaneous availability $\tilde{A}(t, \tilde{w})$ is given by $\tilde{A}_\alpha(t, \tilde{w}) = [\tilde{A}_\alpha^L(t, \tilde{w}), \tilde{A}_\alpha^U(t, \tilde{w})]$ with lower bound:

$$\tilde{A}_\alpha^L(t, \tilde{w}): \quad \min \quad \sum_{i=1}^{K} p_{si}(t) \cdot |ar_i|_\alpha^{rel} \quad s.t. \quad \tilde{p}_{si\alpha}^L(t) \leq p_{si}(t) \leq \tilde{p}_{si\alpha}^U(t), \quad \sum_{i=1}^{K} p_{si}(t) = 1 \tag{40}$$

and upper bound:

$$\tilde{A}_\alpha^L(t, \tilde{w}): \quad \max \quad \sum_{i=1}^{K} p_{si}(t) \cdot |ar_i|_\alpha^{rel} \quad s.t. \quad \tilde{p}_{si\alpha}^L(t) \leq p_{si}(t) \leq \tilde{p}_{si\alpha}^U(t), \quad \sum_{i=1}^{K} p_{si}(t) = 1. \tag{41}$$

It should be noted that the reason for adding a pair of equality constraints into the parametric programming is to ensure that at any time instant, the summation of the state probabilities of all possible states is equal to one. Recall the two-state FMSS as an example: if $\tilde{w} = 0$, both the fuzzy availability with and without considering the equality constraint are plotted in

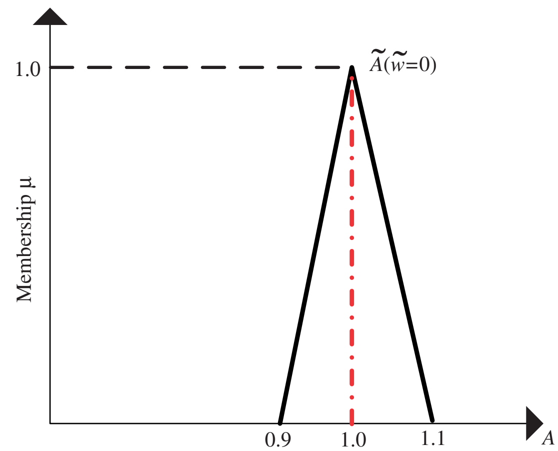

Figure 4. The membership of system availability when $\tilde{w} = 0$.

Figure 4 by solid and dot-dash lines, respectively. The availability is exactly equal to one when the equality constraints are included, so the result is more acceptable and reasonable. The explanation is that although the performance rates are fuzzy values, they are still absolutely greater than the user demand. The system availability in this scenario would be equal to one at anytime; however, ignoring the equality constraint may result in an irrational outcome in which the availability is not equal to one. Because the state transition intensities are presented as fuzzy numbers in our studied case, the summation of state probabilities determined by any possible transition intensity is definitely equal to one, even without these constraints. However, if the steady/instantaneous state probabilities are presented as fuzzy numbers like the cases in Ding and Lisnianski (2008), the couple of equality constraints are very important and necessary.

To solve the non-linear parametric programming in Equations (18, 19, 24, 25, 31, 32, 40, 41), optimization routines, such as the steepest descent method, the Newton–Raphson method, etc. can be realised either in Matlab or other commercial optimisation softwares. The function 'fmincon' in the Matlab optimisation toolbox is adopted to solve the constrained non-linear problems (more details can be found in the Matlab manual handbook).

5. Illustrative example

Consider a power generation system with three non-repairable generators as shown in Figure 5. The generators 1 and 2 are assumed to be binary capacity elements with nominal performance rate in state 2, and zero performance rate in failure state 1. The performance rates of generator 3 are divided into three levels

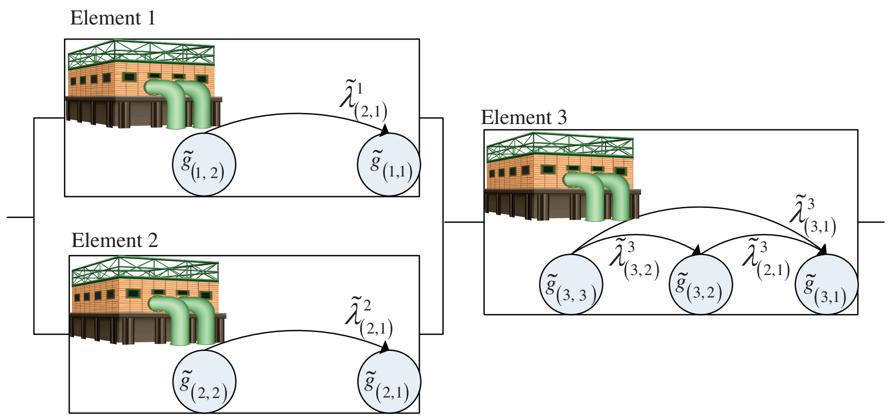

Figure 5. The structure of the three-element FMSS.

Table 1. TFN for fuzzy transition intensities (per year).

Element 1	$\tilde{\lambda}^1_{(2,1)}$	–	–
Fuzzy intensity	(0.4, 0.5, 0.6)	–	–
Element 2	$\tilde{\lambda}^2_{(2,1)}$	–	–
Fuzzy intensity	(0.7, 0.8, 0.9)	–	–
Element 3	$\tilde{\lambda}^3_{(3,2)}$	$\tilde{\lambda}^3_{(3,1)}$	$\tilde{\lambda}^3_{(2,1)}$
Fuzzy intensity	(0.4, 0.5, 0.7)	(0.2, 0.3, 0.4)	(0.4, 0.5, 0.6)

named perfect (state 3), partially failed (state 2) and completely failed (state 1) by domain engineers. The statistical estimation method is problematic because of the lack of sufficient data, especially for a new system. Thus, crisp values are not suitable to assess the transition intensities. On the other hand, the performance rate may fluctuate around its expected value in each state. For example the capacity of a solar generator is affected by the weather conditions and the power of a water-turbine generator fluctuates with water flow. It is reasonable to use fuzzy numbers to evaluate the transition intensities and performance rates according to expert's knowledge (Verma et al. 2004), thus each power generator can be considered as an FMSE, and the entire system is regarded as an FMSS.

The fuzzy transition intensities and fuzzy performance rates of each FMSE are treated as TFN, and tabulated in Tables 1 and 2.

For element 1:

The FUGF representing the dynamic fuzzy performance rate distribution is written as:

$$\begin{aligned}\tilde{u}_1(z,t) &= \tilde{p}_{(1,2)}(t)\cdot z^{\tilde{g}_{(12)}} + \tilde{p}_{(1,1)}(t)\cdot z^{\tilde{g}_{(1,1)}}\\ &= \tilde{p}_{(1,2)}(t)\cdot z^{(120,150,180)} + \tilde{p}_{(1,1)}(t)\cdot z^0\end{aligned}$$

where the dynamic fuzzy state probability can be obtained throughout the exponential distribution, and one has $\tilde{p}_{(1,2)}(t) = e^{-\tilde{\lambda}^1_{(2,1)}t}$ and $\tilde{p}_{(1,1)}(t) = 1 - e^{-\tilde{\lambda}^1_{(2,1)}t}$.

Table 2. TFN for fuzzy performance rates ($\times 10^3$ kW).

State	1	2	3
Element 1	0	(120, 150, 180)	–
Element 2	0	(240, 260, 300)	–
Element 3	0	(280, 320, 350)	(400, 450, 500)

For element 2:

In the same manner, the FUGF of the element 2 is given as:

$$\begin{aligned}\tilde{u}_2(z,t) &= \tilde{p}_{(2,2)}(t)\cdot z^{\tilde{g}_{(22)}} + \tilde{p}_{(2,1)}(t)\cdot z^{\tilde{g}_{(2,1)}}\\ &= \tilde{p}_{(2,2)}(t)\cdot z^{(240,260,300)} + \tilde{p}_{(2,1)}(t)\cdot z^0\end{aligned}$$

where $\tilde{p}_{(2,2)}(t) = e^{-\tilde{\lambda}^2_{(2,1)}t}$ and $\tilde{p}_{(2,1)}(t) = 1 - e^{-\tilde{\lambda}^2_{(2,1)}t}$.

For element 3:

The FUGF for the element is given as:

$$\begin{aligned}\tilde{u}_3(z,t) &= \tilde{p}_{(3,3)}(t)\cdot z^{\tilde{g}_{(3,3)}} + \tilde{p}_{(3,2)}(t)\cdot z^{\tilde{g}_{(3,2)}} + \tilde{p}_{(3,1)}(t)\cdot z^{\tilde{g}_{(3,1)}}\\ &= \tilde{p}_{(3,3)}(t)\cdot z^{(400,450,500)} + \tilde{p}_{(3,2)}(t)\cdot z^{(280,320,350)}\\ &\quad + \tilde{p}_{(3,1)}(t)\cdot z^0\end{aligned}$$

The corresponding Kolmogorov's equation for solving the fuzzy state probability of element 3 takes the form:

$$\begin{cases}\dfrac{d\tilde{p}_{(3,1)}(t)}{dt} = \tilde{\lambda}^3_{(3,1)}\tilde{p}_{(3,3)}(t) + \tilde{\lambda}^3_{(2,1)}\tilde{p}_{(3,2)}(t),\\[2mm] \dfrac{d\tilde{p}_{(3,2)}(t)}{dt} = \tilde{\lambda}^3_{(3,2)}\tilde{p}_{(3,3)}(t) - \tilde{\lambda}^3_{(2,1)}\tilde{p}_{(3,2)}(t), \quad t \ge 0,\\[2mm] \dfrac{d\tilde{p}_{(3,3)}(t)}{dt} = -\left(\tilde{\lambda}^3_{(3,1)} + \tilde{\lambda}^3_{(3,2)}\right)\tilde{p}_{(3,3)}(t),\end{cases}$$

with initial condition $\tilde{p}_{(3,3)}(0) = 1$ and $\tilde{p}_{(3,2)}(0) = \tilde{p}_{(3,1)}(0) = 0$. After using the Laplace–Stieltjes transform, the following linear equations are obtained:

$$\begin{cases} s\tilde{p}_{(3,1)}(s) = \tilde{\lambda}^3_{(3,1)}\tilde{p}_{(3,3)}(s) + \tilde{\lambda}^3_{(2,1)}\tilde{p}_{(3,2)}(s) \\ s\tilde{p}_{(3,2)}(s) = \tilde{\lambda}^3_{(3,2)}\tilde{p}_{(3,3)}(s) - \tilde{\lambda}^3_{(2,1)}\tilde{p}_{(3,2)}(s) \\ s\tilde{p}_{(3,3)}(s) - 1 = -\left(\tilde{\lambda}^3_{(3,1)} + \tilde{\lambda}^3_{(3,2)}\right)\tilde{p}_{(3,3)}(s) \end{cases},$$

and one has:

$$\tilde{p}_{(3,3)}(s) = \frac{1}{s + \tilde{\lambda}^3_{(3,1)} + \tilde{\lambda}^3_{(3,2)}},$$

$$\tilde{p}_{(3,2)}(s) = \frac{\tilde{\lambda}^3_{(3,2)}}{\left(s + \tilde{\lambda}^3_{(2,1)}\right)\left(s + \tilde{\lambda}^3_{(3,1)} + \tilde{\lambda}^3_{(3,2)}\right)},$$

$$\tilde{p}_{(3,1)}(s) = \frac{s\tilde{\lambda}^3_{(3,1)} + \tilde{\lambda}^3_{(2,1)}\tilde{\lambda}^3_{(3,1)} + \tilde{\lambda}^3_{(2,1)}\tilde{\lambda}^3_{(3,2)}}{s\left(s + \tilde{\lambda}^3_{(2,1)}\right)\left(s + \tilde{\lambda}^3_{(3,1)} + \tilde{\lambda}^3_{(3,2)}\right)}.$$

Using the inverse Laplace–Stieltjes transform, the dynamic fuzzy state probabilities can be obtained, as functions of time t in the form of:

$$\tilde{p}_{(3,3)}(t) = e^{-(\tilde{\lambda}^3_{(3,1)} + \tilde{\lambda}^3_{(3,2)})t},$$

$$\tilde{p}_{(3,2)}(t) = \frac{\tilde{\lambda}^3_{(3,2)}\left(e^{-(\tilde{\lambda}^3_{(3,2)} + \tilde{\lambda}^3_{(3,1)})t} - e^{-\tilde{\lambda}^3_{(2,1)}t}\right)}{\tilde{\lambda}^3_{(2,1)} - \tilde{\lambda}^3_{(3,2)} - \tilde{\lambda}^3_{(3,1)}},$$

$$\tilde{p}_{(3,1)}(t) = 1 - \tilde{p}_{(3,2)}(t) - \tilde{p}_{(3,3)}(t).$$

Thus, the FUGF of the FMSS is written as:

$$\tilde{U}_s(z,t) = \tilde{\Omega}_{\phi_{ser}}\left(\tilde{\Omega}_{\phi_{par}}(\tilde{u}_1(z,t), \tilde{u}_2(z,t)),\ \tilde{u}_3(z,t)\right),$$

where $\tilde{\Omega}_{\phi_{ser}}$ and $\tilde{\Omega}_{\phi_{par}}$ represent the fuzzy composition operators for series and parallel connected elements, respectively. Following the composition algorithm introduced in Section 4.3, the FUGF can be written as:

$$\begin{aligned} \tilde{U}_s(z,t) = {} & \tilde{p}_{(1,1)}(t)\tilde{p}_{(2,1)}(t)\tilde{p}_{(3,1)}(t)\cdot z^{(0)} \\ & + \tilde{p}_{(1,1)}(t)\tilde{p}_{(2,1)}(t)\tilde{p}_{(3,2)}(t)\cdot z^{(0)} \\ & + \tilde{p}_{(1,1)}(t)\tilde{p}_{(2,1)}(t)\tilde{p}_{(3,3)}(t)\cdot z^{(0)} \\ & + \tilde{p}_{(1,1)}(t)\tilde{p}_{(2,2)}(t)\tilde{p}_{(3,1)}(t)\cdot z^{(0)} \\ & + \tilde{p}_{(1,1)}(t)\tilde{p}_{(2,2)}(t)\tilde{p}_{(3,2)}(t)\cdot z^{(240,260,300)} \\ & + \tilde{p}_{(1,1)}(t)\tilde{p}_{(2,2)}(t)\tilde{p}_{(3,3)}(t)\cdot z^{(240,260,300)} \\ & + \tilde{p}_{(1,2)}(t)\tilde{p}_{(2,1)}(t)\tilde{p}_{(3,1)}(t)\cdot z^{(0)} \\ & + \tilde{p}_{(1,2)}(t)\tilde{p}_{(2,1)}(t)\tilde{p}_{(3,2)}(t)\cdot z^{(120,150,180)} \\ & + \tilde{p}_{(1,2)}(t)\tilde{p}_{(2,1)}(t)\tilde{p}_{(3,3)}(t)\cdot z^{(120,150,180)} \\ & + \tilde{p}_{(1,2)}(t)\tilde{p}_{(2,2)}(t)\tilde{p}_{(3,1)}(t)\cdot z^{(0)} \\ & + \tilde{p}_{(1,2)}(t)\tilde{p}_{(2,2)}(t)\tilde{p}_{(3,2)}(t)\cdot z^{(280,320,350)} \\ & + \tilde{p}_{(1,2)}(t)\tilde{p}_{(2,2)}(t)\tilde{p}_{(3,3)}(t)\cdot z^{(360,410,480)}. \end{aligned}$$

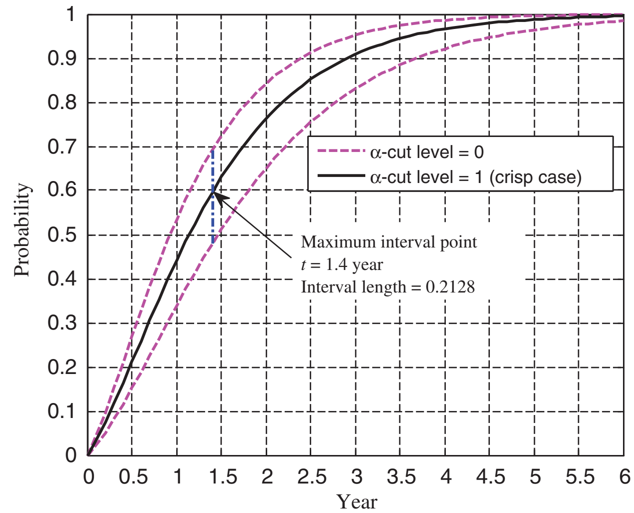

Figure 6. The probability $\tilde{p}_{s1}(t)$ with $\alpha = 1$ and $\alpha = 0$.

Simplify the FUGF via combining the terms with identical performance rate. Thus, the FMSS only has five different system states corresponding to its performance rate, and they are:

State 1: $\tilde{g}_{s1} = 0$ and

$$\begin{aligned} \tilde{p}_{s1} = {} & \tilde{p}_{(1,1)}(t)\tilde{p}_{(2,1)}(t)\tilde{p}_{(3,1)}(t)\cdot z^{(0)} \\ & + \tilde{p}_{(1,1)}(t)\tilde{p}_{(2,1)}(t)\tilde{p}_{(3,2)}(t)\cdot z^{(0)} \\ & + \tilde{p}_{(1,1)}(t)\tilde{p}_{(2,1)}(t)\tilde{p}_{(3,3)}(t)\cdot z^{(0)} \\ & + \tilde{p}_{(1,1)}(t)\tilde{p}_{(2,2)}(t)\tilde{p}_{(3,1)}(t)\cdot z^{(0)} \\ & + \tilde{p}_{(1,2)}(t)\tilde{p}_{(2,1)}(t)\tilde{p}_{(3,1)}(t)\cdot z^{(0)} \\ & + \tilde{p}_{(1,2)}(t)\tilde{p}_{(2,2)}(t)\tilde{p}_{(3,1)}(t)\cdot z^{(0)}; \end{aligned}$$

State 2: $\tilde{g}_{s2} = (120, 150, 180)$ and $\tilde{p}_{s2} = \tilde{p}_{(1,2)}(t)\tilde{p}_{(2,1)}(t)\times \tilde{p}_{(3,2)}(t) + \tilde{p}_{(1,2)}(t)\tilde{p}_{(2,1)}(t)\tilde{p}_{(3,3)}(t)$;

State 3: $\tilde{g}_{s3} = (240, 260, 300)$ and $\tilde{p}_{s3} = \tilde{p}_{(1,1)}(t)\tilde{p}_{(2,2)}(t)\times \tilde{p}_{(3,2)}(t) + \tilde{p}_{(1,1)}(t)\tilde{p}_{(2,2)}(t)\tilde{p}_{(3,3)}(t)$;

State 4: $\tilde{g}_{s3} = (280, 320, 350)$ and $\tilde{p}_{s4} = \tilde{p}_{(1,2)}(t)\tilde{p}_{(2,2)}(t)\times \tilde{p}_{(3,2)}(t)$;

State 5: $\tilde{g}_{s3} = (360, 410, 480)$ and $\tilde{p}_{s5} = \tilde{p}_{(1,2)}(t)\tilde{p}_{(2,2)}(t)\times \tilde{p}_{(3,3)}(t)$.

The parametric programming as Equations (31) and (32) is executed to calculate the intervals $\tilde{p}_{si\alpha}(t) = \left[\tilde{p}^L_{si\alpha}(t), \tilde{p}^U_{si\alpha}(t)\right]$ at any α-cut level of the $\tilde{p}_{si}(t)$. The fuzzy system state probabilities $\tilde{p}_{si}(t)$ at cut levels $\alpha = 0$ and $\alpha = 1$ are plotted in Figures 6 to 10, respectively, and the fuzzy instantaneous expect performance rate is depicted in Figure 11. In these figures, the possible value of the indices of interest falls into the interval bounded by the outer lines obtained

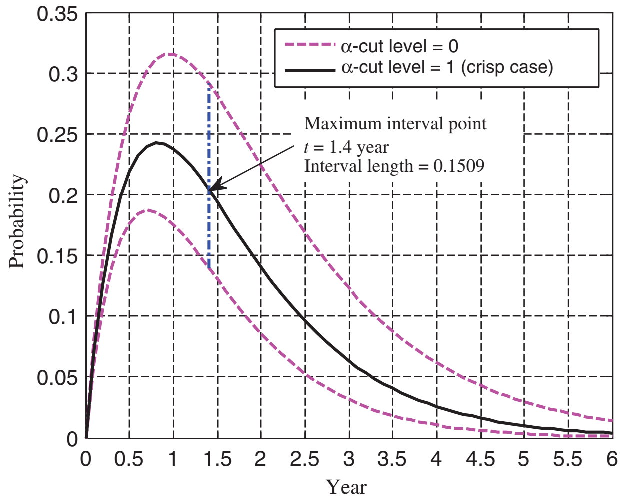

Figure 7. The probability $\tilde{p}_{s2}(t)$ with $\alpha = 1$ and $\alpha = 0$.

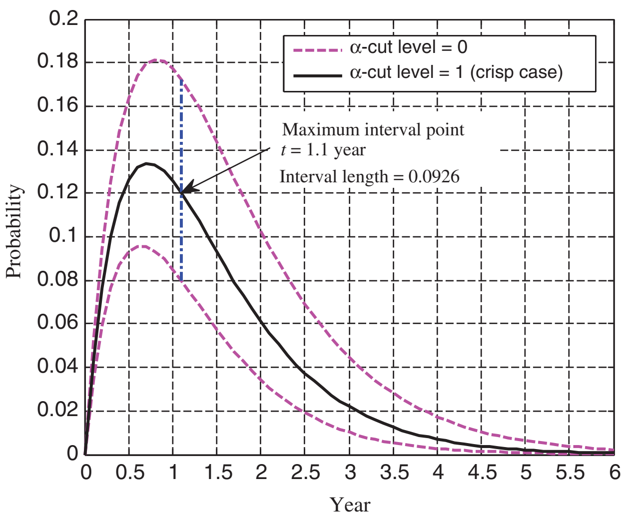

Figure 8. The probability $\tilde{p}_{s3}(t)$ with $\alpha = 1$ and $\alpha = 0$.

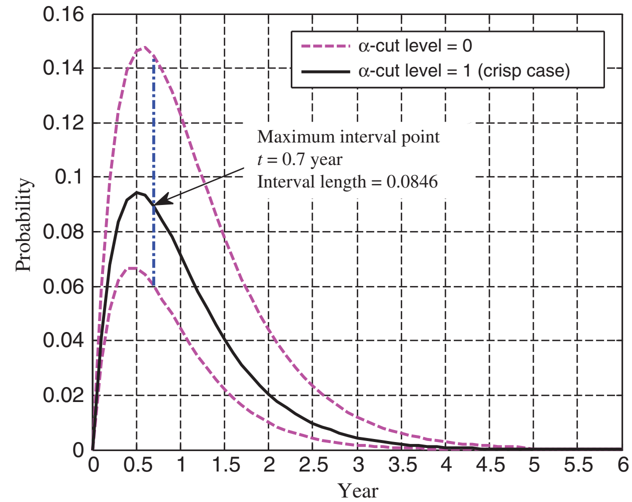

Figure 9. The probability $\tilde{p}_{s4}(t)$ with $\alpha = 1$ and $\alpha = 0$.

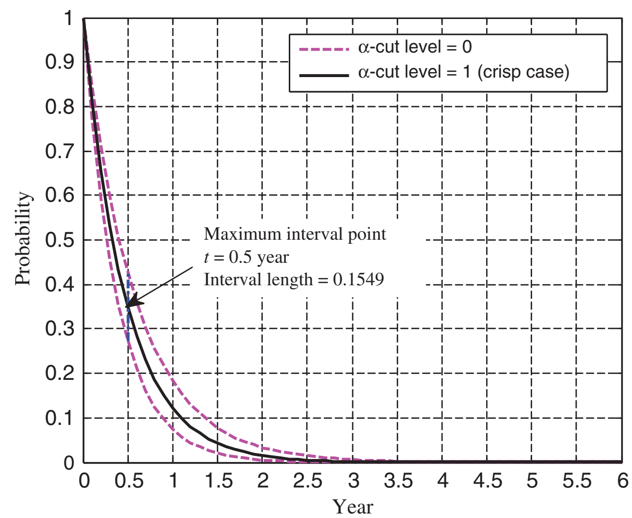

Figure 10. The probability $\tilde{p}_{s5}(t)$ with $\alpha = 1$ and $\alpha = 0$.

with $\alpha = 0$. When $\alpha = 1$, the value of indices is equal to the results regarding transition intensities and performance rates as crisp values. Furthermore, one can find out the time point with the maximum interval between the upper and lower bound when $\alpha = 0$. It means when one considers the fuzzy uncertainty, the largest uncertainty divergence (fuzzy interval) is at this time point during the whole system working period. However, the time points with maximum fuzzy interval for different indices of interest are not identical. Examining the divergence incurred by fuzzy uncertainty is significant when the robustness is greatly concerned.

Figure 12 plots the membership function of fuzzy state probability $\tilde{p}_{si}(t)$ at different α-cut level when $t = 0.8$ years, and the membership function of $\tilde{E}(t = 0.8)$ is illustrated in Figure 13.

Assume that the fuzzy demand for the FMSS is $\tilde{w} = (160, 180, 210) \times 10^3$ kW, and fuzzy performance rate of state 2 overlaps with user demand. According to the approach proposed in Section 4.4, the membership function of availability at $t = 0.8$ years is presented in Figure 14 compared with the one using the relative cardinality, as well as the one using the approach proposed in Ding and Lisnianski (2008) where the equality constraints are not considered.

When $\alpha = 1$, all the fuzzy values, including element state transition intensity and performance rates, as well as user demand become crisp numbers. The availability is also a crisp value by solving the traditional MSS formulation, and it is equal to 0.4037 (note that since the crisp performance rates of the system states are 0 kW, 150×10^3 kW, 260×10^3 kW, 320×10^3 kW and

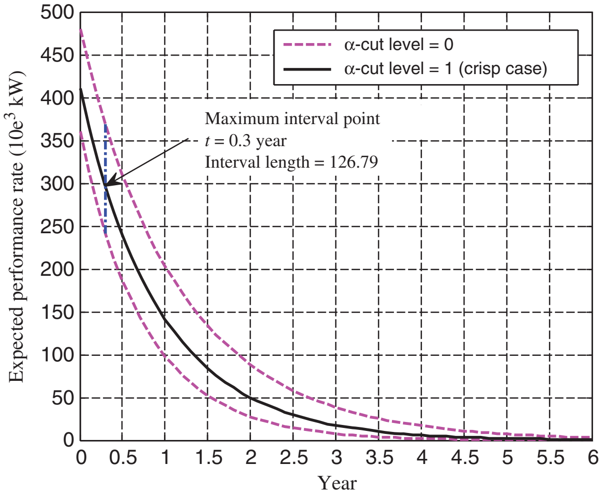

Figure 11. The fuzzy instantaneous expected performance rate $\tilde{E}(t)$ with $\alpha = 1$ and $\alpha = 0$.

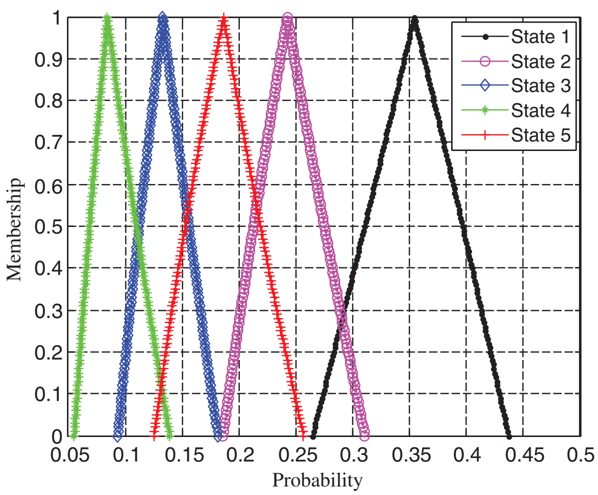

Figure 12. The membership function of $\tilde{p}_{si}(t)$ at $t = 0.8$ year.

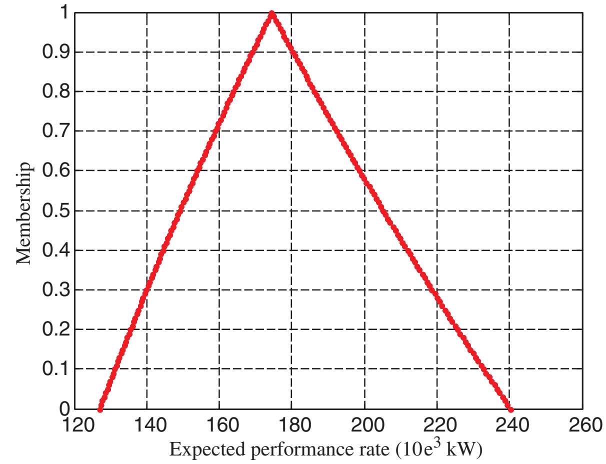

Figure 13. The membership function of the expected performance rate $\tilde{E}(t)$ at $t = 0.8$ year.

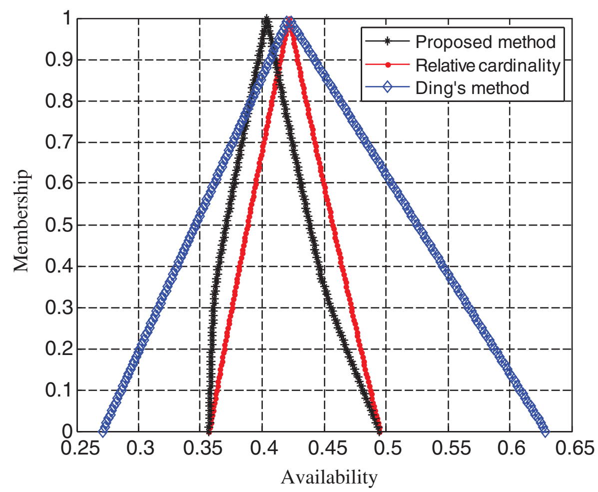

Figure 14. The membership function of $\tilde{A}(t, \tilde{w})$ at $t = 0.8$ year when $\tilde{w} = (160, 180, 210) \times 10^3$ kW.

410×10^3 kW, respectively, the performance rates of the three states (states 5, 4, 3) are greater than the crisp user demand of 180×10^3 kW, and the availability is the summation of the three crisp state probabilities). The result is identical with our proposed method when one sets $\alpha = 1$ as shown in Figure 14, but the relative cardinality methods provide a larger value (0.4214). It is because the relative cardinality is constant even if the α-cut level increases, and when $\alpha = 1$, the relative cardinality for state 2 still regards the performance rates as a fuzzy value and the product of relative cardinality and state probability $\tilde{p}_{s2}(t)$ provides a partial contribution to the availability index. Furthermore, the membership function according to relative cardinality gives a proper triangle while our proposed approach does not, especially at α between 0.3 and 0.4. This is caused by the variation of $|ar_i|_\alpha^{rel}$ which is not a constant in our proposed method. The fuzzy availability can be regarded as a fixed-weight sum of state fuzzy probabilities in the relative cardinality approach, and it still looks like a TFN when each probability is TFN-like. However, the proposed method is like a variable-weight sum which will not form a TFN, preventing the probability of state 2 to contribute to the availability when $\alpha \geq 0.4 (|ar_i|_\alpha^{rel} = 0)$. To demonstrate the effect of the equality constraints in Equstions (40) and (41), we directly regard the element state probabilities as TFN calculated from fuzzy Markov model, and using the approach in Ding and Lisnianski (2008) to evaluate the system availability without considering the equality constraints. The result is plotted in Figure 14 by the line with diamond marks. Apparently, the scenario without equality constraints involved provides a larger fuzzy uncertainty interval, and for some irrational cases where the summation of the possible system state probability is less or greater

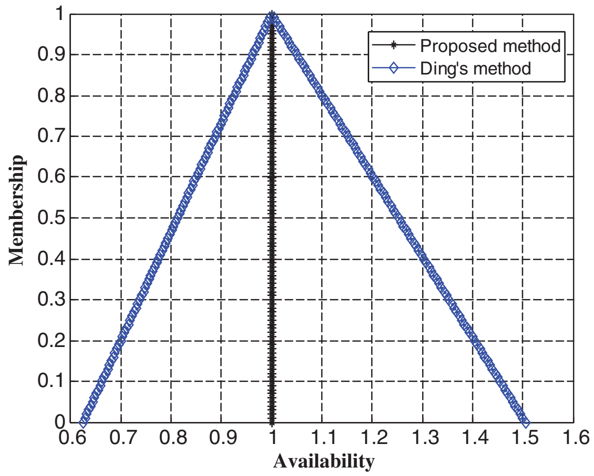

Figure 15. The membership function of $\tilde{A}(t, \tilde{w})$ at $t = 0.8$ year when $\tilde{w} = 0$.

than one are included, which enlarges the set size of possible availability value in fuzzy context. Furthermore, an extreme case that user demand $\tilde{w} = 0$ is examined to illustrate the advantage of the proposed method. Because all the state performance rates are greater than user demand and system availability should be exactly equal to one at any α-cut level. As shown in Figure 15, the result provided by our method matches above judgment that the availability is equal to one. On the other hand, the availability computed by Ding–Lisnianski (2008) method which does not consider the equality constraints forms a TFN, produces a fuzzy interval where availability is less or greater than one, which makes no sense.

6. Conclusions

This article introduces the FMSS that extends the MSS model to cases when the transition rates and performance rates of MSE are uncertain and/or imprecise. These uncertain parameters are presented as fuzzy values. A fuzzy Markov model is proposed to establish the dynamic state probabilities of the FMSE. The MSS state probability, which inherits the fuzzy property from FMSE, is evaluated through the FUGF, and the composition rules for the fuzzy performance rate in both flow transmission type and task processing type systems are discussed. The parametric programming formulas are presented to obtain membership functions of the instantaneous state probability and expected performance rate at any time instant. A modified availability evaluation method is developed when system performance rate and user demand are presented as fuzzy values. A power generation system with three FMSE is studied, and it shows how the proposed method provides a more effective and reasonable outcome. This technique is suitable for the reliability and performance evaluation of MSS where the accurate data are not available and need to be approximated by fuzzy values, and it provides engineers with more useful information about possible system behaviour and, thereby enabling better decisions to be made about safety issues. Further research areas include developing an effective methodology to reduce the computational complexity, and incorporating the maintenance decision into the FMSS.

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