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HAL Id: hal-00749606
https://hal.archives-ouvertes.fr/hal-00749606

Submitted on 8 Nov 2012

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Abstract

This paper presents an original method for evaluating reliability indices for Multi-State Systems (MSSs) in the presence of aleatory and epistemic uncertainties. In many real world MSSs an insufficiency of data makes it difficult to estimate precise values for component state probabilities. The proposed approach applies the Transferable Belief Model (TBM) interpretation of the Dempster-Shafer theory to represent component state beliefs and to evaluate the MSS reliability indices. We use the example of an oil transmission system to demonstrate the proposed approach and we compare it with the Universal Generating Function method. The value of the Dempster-Shafer theory lies in its ability to use several combination rules in order to evaluate reliability indices for MSSs that depend on the reliability of the experts' opinions as well as their independence.

Key Words

Multi-State System (MSS), reliability indices, Dempster-Shafer (D-S) theory, Transferable Belief Model (TBM), Experts' opinion.

I. Introduction

In traditional binary reliability both systems and components have only two possible states: perfect functioning and complete failure [9], [25]. However, a system and its components can have different states

characterized by different levels of performance. Such systems are referred to as Multi-State Systems (MSSs). For example, a power station may have states 0, 1, 2, 3 and 4 that correspond to generating electricity at 0, 25, 50, 75 and 100 percent of its full capacity [61]. Therefore, reliability analysis of MSSs is much more complex than for binary state systems.

MSS theory has been a subject of investigation since 1978 in [8], [21]. A comprehensive presentation of MSS reliability theory and its applications can be found in the first book devoted to the reliability analysis of MSSs [35]. A recent review of the literature can be found in [39]. Practical methods of MSS reliability assessment are based on four different approaches [17], [43]: the structure function [44], [58], the Monte Carlo simulation technique [45], [69], the Markov approach [34], [62] and the Universal Generating Function (UGF) method [17], [31], [32]. The structure function approach based on the extension of Boolean models to multi-valued models was the first method to be developed for MSS reliability assessment. Monte Carlo simulation can represent any real world problem for the purposes of reliability assessment. In order to use the Markov method we need to generate all the possible states of a system. However, the number of states can be extremely large, even for a relatively small number of system elements. The UGF method was introduced in [57] and proved to be very effective in evaluating the reliability of different types of MSSs [35]. The UGF function extends the moment-generating function and allows the entire set of MSS performances to be obtained, based on the performance of its components for several system configurations. This can be done by introducing different composition operators over the UGF functions. [56] presents an interesting comparison of the four approaches.

Conventional MSS reliability theory makes two fundamental assumptions [17]: (i) every state probability of an MSS element can be fully characterized by probability measures; and (ii) the performance rate of any MSS element can be precisely determined. However, for some MSSs, there are different types of uncertainties about the state probabilities and performance rates of elements [12]. In recent years a general Fuzzy Multi-State Systems (FMSS) approach has been proposed to handle uncertainties in reliability assessments of MSSs [19]. This approach assumes that the state probabilities and/or the state performances of components can be represented by fuzzy values [18], [19], whereas the D-S approach developed in this paper assumes that the state beliefs of components are represented by belief functions and that the state performances of components are discrete values. The literature includes several attempts to extend belief functions to fuzzy events. The first extension of D-S theory to the general framework of

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fuzzy set theory was proposed by Zadeh in the context of information granularity and possibility theory [67]. A number of different generalizations were then proposed, according to how a measure of inclusion among fuzzy sets is used to define the belief functions [26], [36], [41], [52], [63], [65], [66]. In order to take fuzzy numbers into account, fuzzy numbers can be assigned to each focal element of the belief structure. This may be achieved by considering the upper and lower bounds of α-cuts of fuzzy numbers. The structure obtained, introduced by Denœux [15], is called a Fuzzy-Valued Belief Structure (FVBS). This structure is defined as a fuzzy set of belief structures on the frame of discernment Ω, whose belief masses are restricted by fuzzy numbers. Fuzzy credibility and plausibility can then be evaluated.

In this paper we are only interested in uncertainties about the state probabilities of elements. Uncertainties are classified into two categories: aleatory uncertainty and epistemic uncertainty. Aleatory uncertainty is the inherent variation associated with the physical system or the environment, such as the inherent variability of component failure. It is referred to as variability, random uncertainty, and stochastic uncertainty [40]. Knowledge provided by experts cannot be expected to reduce aleatory uncertainty. This type of uncertainty is thus also known as irreducible uncertainty. Epistemic uncertainty is an uncertainty that is due to a lack of knowledge of quantities or processes within the system or the environment. It is also referred to as reducible uncertainty or subjective uncertainty [27], [37]. Epistemic uncertainty can be eliminated by obtaining knowledge that was originally lacking, and expert opinion may be useful here. Over the last few years the risk assessment community has generally held that distinguishing between these types of uncertainty is useful and important when evaluating the reliability of systems [5]. In case of large amount of reliability data, the classical probabilistic approach was widely used to manage uncertainties in risk and reliability assessments [4]–[6]. This approach was based on the definition given by Laplace of the probability of an event as the ratio of the number of cases favorable to it, to the number of all possible cases when all cases are equally possible [30]. The frequentist probabilistic approach introduced by Venn [59] which defined the event probability as the limit of its relative frequency in a large number of trials was also widely used [4]–[6]. However, in the case of components that fail only rarely (nuclear systems, chemical processes, railway systems, etc.) or components that have not been operated long enough to generate a sufficient quantity of reliability data, expert judgment is required and both classical and frequentist probabilistic approaches become not suitable in these cases [4]–[6], [46]. For this reason, several methods were proposed to manage uncertainties such as Bayesian approach, interval approach,

evidence theory, possibility theory, etc.

The Bayesian approach was based on the use of subjective probabilities to represent expert judgment [3]. The subjective probabilities of an event indicate the degree to which the expert believes it [23]. The probability distributions representing the aleatory uncertainties are proposed such as for example the representation of a lifetime component by an exponential distribution. The epistemic uncertainties about the parameter values of the distributions are then represented by prior subjective probability distributions. The equation of Bayes is used to compute the new epistemic uncertainties in terms of the posterior distributions in case of new reliability data. Finally, the predictive distributions of the quantities of interest such as the lifetime of new components are derived by using the total probability law [3], [22], [42], [68]. The predictive distributions are subjective but they also take into account the aleatory uncertainties represented by the prior probability models. However, There are some critics about the Bayesian approach exposed particularly by Walley [60] and Caselton and Luo [11]. In a situation of ignorance a Bayesian approach must equally allocate subjective probabilities over the frame of discernment. Thus there is no distinction between uncertainty and ignorance.

The D-S theory also known as the evidence theory or belief functions theory is a generalization of the Bayesian theory of subjective probability. Whereas the Bayesian theory requires probabilities for each question of interest, belief functions allow us to base degrees of belief for one question on probabilities for a related question [50]. To illustrate the idea of obtaining degrees of belief for one question from subjective probabilities for another, we propose an example in risk assessment inspired from the example of limb given by Shafer [50]. Suppose we have subjective probabilities for the reliability of a risk expert A. The probability that A is reliable is 0.75, and the probability that A is not reliable is 0.25. The risk expert A reports us that a component i is failed. This information which must be true if A is reliable, is not necessarily false if A is not reliable. The risk expert testimony justifies a 0.75 degree of belief that the component i is failed, but only a 0 degree of belief (not a 0.25 degree of belief) that the component i is not failed. This value does not mean that we are sure that the component i is failed, as a zero probability would. It means that the risk expert's testimony gives us no reason to believe that the component i is failed. The 0.75 and the 0 constitute a belief function. Thus there is no requirement that belief not committed to a given proposition should be committed to its negation. The second important issue in D-S theory is that belief measures of uncertainty may be assigned to overlapping sets and subsets of

hypotheses, events or propositions as well as to individual hypothesis. There are several interpretations of D-S theory (Dempster's model [14], the Theory of Hints [28], etc.). In this work the Transferable Belief Model (TBM) interpretation of the D-S theory is proposed.

The paper is organized as follows: Section 2 introduces basic concepts of D-S theory and TBM. Section 3 presents and discusses the proposed MSS model based on the TBM, the corresponding structure function and reliability indices. Section 4 presents an example comparing the UGF method and the proposed approach in the case of aleatory uncertainty. The same example with expert opinion is then studied using the TBM in presence of both aleatory and epistemic uncertainties. Finally section 5 presents some conclusions.

II. Background of D-S theory and TBM

D-S is a theory for uncertain reasoning under both aleatory and epistemic uncertainties. It was first developed by Dempster [14] and extended by Shafer [50]. The first work using D-S theory in reliability assessment was presented by Dempster and Kong [13]. In recent years D-S theory has been used by numerous researchers to quantify uncertainty in the reliability analysis of binary state systems [1], [7], [33], [48], [49], [51], [55]. Several different interpretations of D-S theory have been put forward: Dempster's model [14], the Theory of Hints [28], the probability of modal propositions model [47] and the Transferable Belief Model (TBM) [54]. Each model corresponds to a different understanding of the concept of uncertainty. The interpretation of D-S theory that we have adopted in this paper is based on the TBM, which is a model developed outside the scope of probability theory and can thus avoid the accusation that D-S theory is understood as a special form of upper and lower probability theory [5].

A. Basic Belief Assignment (BBA)

The definition domain of the variable of interest x is called the frame of discernment Ω, where all of the possible events are mutually exclusive elementary propositions. As an example, let us consider a frame of discernment $\Omega = \{x_1, x_2\}$, meaning that x_1 and x_2 are elementary propositions and mutually exclusive of each other. The power set 2^Ω is the set of all the subsets of Ω including itself, i.e.: $2^\Omega = \{\{\emptyset\}, \{x_1\}, \{x_2\}, \Omega\}$. A Basic Belief Assignment (BBA) on Ω is a function $m^\Omega : 2^\Omega \to [0, 1]$ which maps belief masses onto events or sets of events such that:

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$$\sum_{A \in 2^{\Omega}} m^{\Omega}(A) = 1$$

An agent holding a piece of evidence allocates unitary amounts of belief to the different subsets of Ω. The number $m^{\Omega}(A)$ represents the support to A given by the agent's belief [53]. There is a distinction between probabilities and BBAs: probability distribution functions are defined on Ω, whereas BBAs are defined on the power set 2^{Ω}. This means that there are $2^{card(\Omega)}$ possible hypotheses in D-S theory, while in probability theory there are only $card(\Omega)$ possible hypotheses. Furthermore, the sub-additivity hypothesis is not required in D-S theory like it is in probability theory.

The subsets $A \subset \Omega$ such that $m^{\Omega}(A) > 0$ are called focal sets of m^{Ω}. Full knowledge is represented by a BBA having a singleton $\{x\}$ ($x \in \Omega$) as a unique focal set. A Bayesian BBA is a special case where all of the focal sets are singletons and is equivalent to probabilities. Complete ignorance is represented by a BBA having only one focal element equal to Ω and which is termed *vacuous*.

According to Klir and Folger [24], the BBAs have some important properties which distinguishes it from being a probability function:

- It is not required that $m(\Omega) = 1$.

- It is not required that $m(A) \leq m(B)$ when $A \subseteq B$.

- No relationship between $m(A)$ and $m(\bar{A})$.

- Also $m(A) + m(\bar{A})$ does not always have to be 1.

For simplicity, let us consider a component i with two states. The frame of discernment L_i of the component i is then given by $L_i = \{0_i, 1_i\}$, where 0_i and 1_i denote respectively the failed and operational states of the component i. If an expert asserts a portion of belief x_i that the component i is working at time t and a portion of belief y_i that the component i is not working at time t, this will be expressed as follows: $m_{Expert}^{L_i}(\{1_i\}) = x_i$ and $m_{Expert}^{L_i}(\{0_i\}) = y_i$. The epistemic uncertainty is represented by $m_{Expert}^{L_i}(\{0_i, 1_i\}) = 1 - x_i - y_i$.
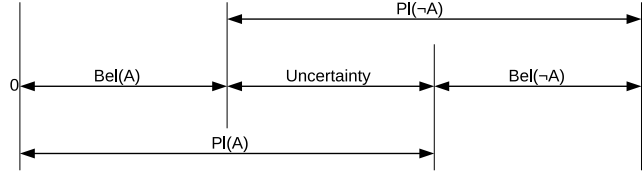
Fig. 1: The belief interval [29]

B. Belief and plausibility functions

The most important measures of uncertainty provided by D-S theory are known as belief and plausibility functions. The belief *Bel* and plausibility *Pl* functions for a subset A are defined as follows:

$$Bel(A) = \sum_{B \subseteq A} m^{\Omega}(B) \quad \forall A \subseteq \Omega, \ \forall B \subseteq \Omega \tag{1}$$

$$Pl(A) = \sum_{B \cap A \neq \emptyset} m^{\Omega}(B) \quad \forall A \subseteq \Omega, \ \forall B \subseteq \Omega \tag{2}$$

$Bel(A)$ is obtained by adding the BBAs of propositions that totally agree with A, whereas Pl is obtained by adding the BBAs of propositions that agree with A totally and partially (cf. Figure 1). $[Bel(A), Pl(A)]$ is the interval that describes the uncertainty of A. The functions Bel and Pl, although they are also functions mapping events A into $[0, 1]$, $\emptyset$ into 0 and Ω into 1, do not fulfill in the general case the sub-additivity properties given for probability. They are related to each other by the following equation:

$$Pl(A) + Bel(\overline{A}) = 1$$

where $\overline{A}$ represents the negation of the event A.

Let us consider the same component i. The belief measure concerning the functioning of component i at time t is given by: $Bel(\{1_i\}) = m^{L_i}_{Expert}(\{1_i\})$ and the plausibility measure is given by: $Pl(\{1_i\}) = m^{L_i}_{Expert}(\{1_i\}) + m^{L_i}_{Expert}(\{0_i, 1_i\})$. The availability of the component i at time t is then given by:

$$A_i = [\underline{A_i}, \overline{A_i}] = [Bel(\{1_i\}), Pl(\{1_i\})]$$

The quantity $\underline{A_i}$ represents the total amount of justified support given to the proposition "the component
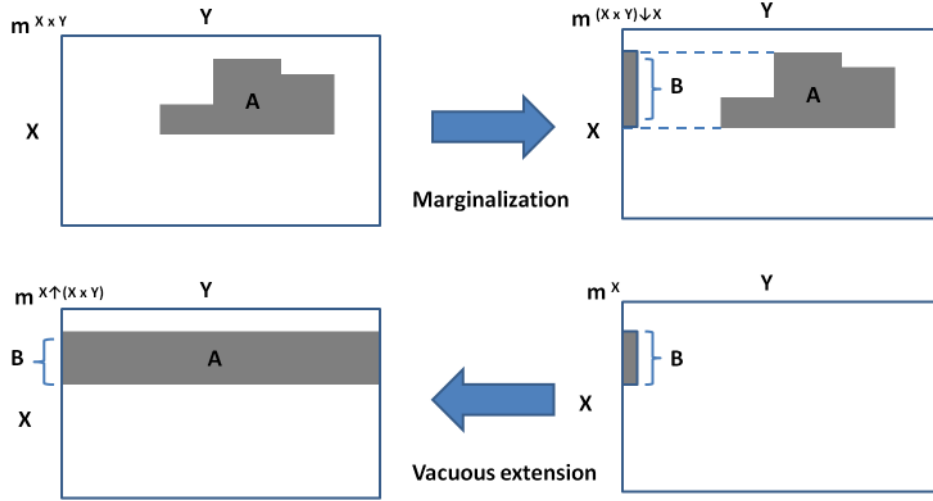
Fig. 2: Marginalization and vacuous extension operations

i is available at time t", while the quantity $\overline{A_i}$ represents the maximum amount of specific support that could be given to the proposition "the component i is available at time t" if justified by additional information. Finally, the quantity $\overline{A_i} - \underline{A_i}$ represents the epistemic uncertainty.

C. Marginalization and vacuous extension

The first step in TBM is to define the frame of discernment. As noticed by Shafer [50], the degree of granularity of the frame is always to some extent a matter of convention, since any element representing a state of nature can always be split into several possibilities. Hence, it is fundamental to examine how a belief function defined on a frame may be expressed in a finer or, conversely, in a coarser frame. Consider two finite sets X and Y. A mapping $\rho : 2^X \to 2^Y$ is called a refining if it satisfies:

- $\rho(B) = \bigcup_{y \in B} \rho(\{y\}) \qquad \forall B \subseteq Y$.
- The set $\{\rho(y), y \in Y\} \subseteq 2^X$ is a partition of X.

Y is called a coarsening of X, and X is called a refinement of Y. A BPA m^Y on Y can be transformed into a BBA m^X on a refinement X by transferring each mass $m^Y(B)$ for $B \subseteq Y$ to $m^X(A)$ for $A = \rho(B)$. This operation is called a vacuous extension of m^Y to X (cf. Figure 2). The inverse operation is the marginalization (cf. Figure 2). The formulas of vacuous extension and marginalization are given in the Appendix.

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	$\{0_i\}$	$\{1_i\}$	$\{0_i, 1_i\}$
Expert 1	0.1	0.3	0.6
Expert 2	0	0.4	0.6
Expert 1 $\oplus$ Expert 2	0.0625	0.5625	0.3750

TABLE I: Dempster combination of BBAs

D. Combination rules

The D-S evidence theory can aggregate multiple sources of information through the combination rules. The two most familiar rules of combination are the Conjunctive and Disjunctive rules [50] (the formulas for the different rules can be found in the Appendix). Dempster's rule is a widely-used rule that consists in applying a conjunctive combination followed by a normalization of the conflict factor k (the formula for Dempster's rule can be found in the Appendix). This conflict factor is equivalent to the BBA allocated to the empty set.

Let us suppose that Expert 1 asserts a 0.3 portion of belief that component i is working and a 0.1 portion of belief that it is not working at time t. Expert 2 asserts a 0.4 portion of belief that component i is working at time t and no belief at all that it is not working. Dempster's rule of combination applied to the BBAs from experts 1 and 2 gives the new BBAs shown in Table I. Also, since BBAs have been combined the level of conflict k can be gauged. In this case it is $k = 0.04$, indicating a minor conflict in the evidence from the two experts.

Several other combination rules have been defined in D-S theory that depend on the reliability of experts and the conflict between them (Yager rule [64], Dubois and Prade rule [20], Cautious rule [16], etc.). For more details, see [14], [16], [50], [53].

III. TBM OF MULTI-STATE SYSTEMS

A. General model

Let us consider a system S with n components. For $i = 1, ..., n$, X_i denotes the state of the i-th component. The set $L_i = \{0, 1, ..., m_i\}$ representing all states of the i-th component is linearly ordered $(L_i, \leq)$, i.e. there exists a linear order (total, antisymmetric, transitive binary relation) over each set L_i. Complete failure (the worst state) corresponds to state 0. Perfect functioning (the best state) corresponds to state m_i. A general model of MSS with a partial ordering over the set L_i was proposed by Montero

et al. [38]. In D-S theory, L_i can be considered as the frame of discernment of the i-th component. The state vector of the MSS is $X = (X_1, X_2, ..., X_n)$.

A central problem in reliability theory is how to determine the relationship between the system states and the states of its components. This relationship can be described by a structure function defined by the mapping $\varphi : L_1 \times L_2 \times ... \times L_n \to L_S$. The value $\varphi(X_1, X_2, ..., X_n)$ is the system state when each component i is at state X_i. The structure function φ is assumed to be non-decreasing, i.e.:

$$X \le Y \Rightarrow \varphi(X) \le \varphi(Y)$$

This means that improving the state of one or more components cannot lead to a lower system performance. We also assume that

$$\varphi(0, 0, ..., 0) = 0 \quad and \quad \varphi(m_1, m_2, ..., m_n) = m_S$$

The BBAs of the i-th component states are defined by the mapping $m^{L_i} : 2^{L_i} \to [0, 1]$ such that

$$\sum_{A \in 2^{L_i}} m^{L_i}(A) = 1 \tag{3}$$

Let $m^{L_i}_{\{j\}} = m^{L_i}(X_i = \{j\})$ denote the BBA that the i-th component is in state j. The BBA $m^{L_i}_{\{j,k\}} = m^{L_i}(X_i = \{j,k\})$ represents the fact that the i-th component is in state j or k.

For $j = 0, ..., m_i$, the vacuous extension is used to extend the BBAs m^{L_i} of each component i to the product space $L_1 \times L_2 ... \times L_n \times L_S$. The resulting BBAs of all components are combined using Dempster's rule. The resulting BBAs are then combined with the BBA representing the system configuration. The BBA $m^{L_1 \times L_2 ... \times L_n \times L_S}_{Conf}$ represents the relation between the state of system and the state of its components. To obtain the BBAs of system states the final results are marginalized on L_S:

$$
\begin{aligned}
m^{L_S} &= (m^{L_1 \uparrow L_1 \times L_2 \times ... \times L_n \times L_S} \oplus m^{L_2 \uparrow L_1 \times L_2 \times ... \times L_n \times L_S} \oplus ... \\
&\quad ... \oplus m^{L_n \uparrow L_1 \times L_2 ... \times L_n \times L_S} \oplus m^{L_1 \times L_2 ... \times L_n \times L_S}_{Config})^{\downarrow L_S}
\end{aligned}
\tag{4}
$$

For example, let us consider a system S with two components 1 and 2. For simplicity we consider that the system and its components have two states 0_i and 1_i. We use the vacuous extension to extend m^{L_1} and m^{L_2} to the product space $L_1 \times L_2 \times L_S$ and we combine the obtained BBAs using the Dempster

rule. The resulting BBAs are combined with $m_{Config}^{L_1 \times L_2 \times L_S}$.

For serial configuration (a failure of any component 1 or 2 results in failure of the entire system S) we have $m_{Series}^{L_1 \times L_2 \times L_S}(\{(1_1, 1_2, 1_S), (0_1, 0_2, 0_S), (0_1, 1_2, 0_S), (1_1, 0_2, 0_S)\}) = 1$.

For parallel configuration (at least one of the two components 1 or 2 must succeed for the system S to succeed) we have $m_{Parallel}^{L_1 \times L_2 \times L_S}(\{(1_1, 1_2, 1_S), (0_1, 0_2, 0_S), (0_1, 1_2, 1_S), (1_1, 0_2, 1_S)\}) = 1$.

To obtain the BBAs of the system S the final result is marginalized on L_S.

Belief and plausibility functions are then computed using Eq. 1 and Eq. 2. Finally, the lower and upper bounds (belief interval) for the system to be in state j are given by:

$$h_j = [\underline{h_j}, \overline{h_j}] = [Bel(\varphi(X) = j), Pl(\varphi(X) = j)] \qquad j = 0, ..., m_S \qquad (5)$$

System utility is an important performance measure introduced by Aven [2] for the study of multi-state systems. At each state j an MSS yields a particular gain. This gain is termed the system utility at state j and is denoted u_j. The overall system utility is the expected utility of the system, defined as follows:

$$U = [\underline{U}, \overline{U}] = [\sum_{j=0}^{m_S} u_j.Bel(\varphi(X) = j), \sum_{j=0}^{m_S} u_j.Pl(\varphi(X) = j)] \qquad j = 0, ..., m_S \qquad (6)$$

The overall system utility of the system S is defined as follows:

$$U = [u_0.Bel(\varphi(X) = 0) + u_1.Bel(\varphi(X) = 1), u_0.Pl(\varphi(X) = 0) + u_1.Pl(\varphi(X) = 1)]$$

where belief and plausibility functions are given by:

$$Bel(\varphi(X) = 0) = m^{L_S}(\{0_S\})$$

$$Bel(\varphi(X) = 1) = m^{L_S}(\{1_S\})$$

$$Pl(\varphi(X) = 0) = m^{L_S}(\{0_S\}) + m^{L_S}(\{0_S, 1_S\})$$

$$Pl(\varphi(X) = 1) = m^{L_S}(\{1_S\}) + m^{L_S}(\{0_S, 1_S\})$$

B. MSS reliability indices

MSS reliability measures were first introduced by Aven [2] and subsequently by Brunelle and Kapur [10]. In this work we consider the three measures most commonly used by reliability engineers:

1) MSS availability A.

2) MSS expected performance E_d.

3) MSS unsupplied demand E_u.

The aim of this section is to define the above reliability measures in the TBM framework. The set $g_i = \{g_{i0}, g_{i1}, ..., g_{im_i}\}$ represents the performance rates of component i at states j ($j = \{0, ..., m_i\}$). The set $G = \{G_0, G_1, ..., G_{m_S}\}$ represents the performance rates of the system. We define MSS availability A as the belief that the MSS will be in a state with a performance level greater than or equal to the demand w. MSS availability is given by:

$$A(w) = [Bel(\varphi(X) \geq j), Pl(\varphi(X) \geq j)] \tag{7}$$

where j corresponds to the first state which satisfies the demand w.

Another important measure of system performance is the MSS expected output performance E_d. This index determines the system's expected performance, and does not depend on the demand w. Therefore, E_d defines the average capacity (productivity) of the system. It can be obtained as follows:

$$E_d = [\sum_{j=0}^{m_S} Bel(\varphi(X) = j).G_j, \sum_{j=0}^{m_S} Pl(\varphi(X) = j).G_j] \tag{8}$$

In some cases the expected unsupplied demand E_u may be used as a measure of system output performance. This index can be formulated as:

$$E_u(w) = [\sum_{j=0}^{m_S} Bel(\varphi(X) = j).max(w - G_j, 0), \sum_{j=0}^{m_S} Pl(\varphi(X) = j).max(w - G_j, 0)] \tag{9}$$

It should be noted that the MSS expected output performance E_d and MSS expected unsupplied performance E_u are particular cases of the overall system utility defined in Eq.6 when the system utility functions u_j at state j are respectively equal to G_j and $max(w - G_j, 0)$.

C. Construction of BBAs

1) From observations: Let us consider n observations of events or sets of events in the frame of discernment $\Omega = \{A_1, ..., A_n\}$. Let c_i denote the number of occurrences of an event A_i. The BBAs related to the occurrence of A_i can then be obtained as follows:

$$m^\Omega(\{A_i\}) = \frac{c_i}{n}$$

If we lack any observations relating to the occurrence of the event A_i, then if $c_{i,j}$ denotes the number of occurrences of a set of events $\{A_i, A_j\}$, we obtain

$$m^\Omega(\{A_i, A_j\}) = \frac{c_{i,j}}{n}$$

The *Bel* and *Pl* functions are then computed from BBAs using Eq. 1 and Eq. 2. For example, let us consider a system with two components 1 and 2. The two components can have three states, i.e. $L_1 = \{0, 1, 2\}$ and $L_2 = \{0, 1, 2\}$. Out of 50 observations we have 30 observations indicating that component 1 is in state 0 or 1, 10 observations indicating that it is in state 2 and a further 10 indicating that it is in state 0, 1, or 2. At the same time 32 observations indicate that component 2 is in state 0, 8 observations indicate that it is in state 1 and 10 observations indicate that this component 2 is in states 0, 1 or 2. This is written as:

$$m^{L_1}_{\{0,1\}} = 30/50 = 0.6$$

$$m^{L_1}_{\{2\}} = 10/50 = 0.2$$

$$m^{L_1}_{\{0,1,2\}} = 10/50 = 0.2$$

$$m^{L_2}_{\{0\}} = 32/50 = 0.64$$

$$m^{L_2}_{\{1\}} = 8/50 = 0.16$$

$$m^{L_2}_{\{0,1,2\}} = 10/50 = 0.2$$

Using Eq.1 and Eq.2 the components' Bel and Pl functions (belief intervals) for being in each of these different states are computed (cf. Table II).

States	Component 1 [Bel, Pl]	Component 2 [Bel, Pl]
{0}	[0, 0.8]	[0.64, 0.84]
{1}	[0, 0.8]	[0.16, 0.36]
{2}	[0.2, 0.4]	[0, 0.2]
{0, 1}	[0.6, 0.8]	[0.8, 1]
{0, 2}	[0.2, 1]	[0.64, 0.84]
{1, 2}	[0.2, 1]	[0.16, 0.36]
{0, 1, 2}	1	1

TABLE II: Belief intervals of components 1 and 2

	Component 3		
States	Expert 1	Expert 2	Expert 3
{0}	0.3	0.4	0.45
{1}	0.2	0.2	0.15
{2}	0.1	0.2	0.25
{0, 1}	0	0	0
{0, 2}	0	0	0
{1, 2}	0	0	0
{0, 1, 2}	0.4	0.2	0.15

TABLE III: BBAs for component 3 according to experts' opinions

2) From expert opinion: Suppose that Expert 1 asserts a 0.3 portion of belief that component 3 is in state 0, a 0.2 portion of belief that it is in state 1 and a 0.1 portion of belief that it is in state 2 (the remaining 0.4 represents ignorance and will be allocated to the frame of discernment). The corresponding values asserted by Expert 2 are 0.4, 0.2 and 0.2, while for Expert 3 they are 0.45, 0.15 and 0.25. The BBAs provided by the experts are shown in Table III, and the BBAs obtained from the combination rules in Table IV. Finally, the components' Bel and Pl functions for each state are given in Table V.

The conflict factor k between BBAs is defined as follows:

$$k = \sum_{A \cap B = \emptyset, \forall A \subseteq L_i, B \subseteq L_j} m^{L_i}(A) m^{L_j}(B) \tag{10}$$

The factor k is equal to 0.3, 0.4 and 0.32 respectively between Expert 1 and Expert 2, Expert 2 and Expert 3, and Expert 1 and Expert 3. Dempster's rule therefore does not lead to obvious contradictions, because the factor k is not high. Dempster's rule assumes the three experts to be equally reliable and independent. The Disjunctive rule considers only one of the experts to be reliable, but we do not know

States	Dempster's rule	Disjunctive rule
{0}	0.6316	0.054
{1}	0.1579	0.006
{2}	0.1789	0.005
{0, 1}	0	0.12
{0, 2}	0	0.109
{1, 2}	0	0.037
{0, 1, 2}	0.0316	0.669

TABLE IV: The obtained BBAs for component 3

States	Dempster's rule [Bel, Pl]	Disjunctive rule [Bel, Pl]
{0}	[0.6316, 0.6632]	[0.0540, 0.9520]
{1}	[0.1579, 0.1895]	[0.0060, 0.8320]
{2}	[0.1789, 0.2105]	[0.0050, 0.8200]
{0, 1}	[0.7895, 0.8211]	[0.1800, 0.9580]
{0, 2}	[0.8105, 0.8421]	[0.1680, 0.9570]
{1, 2}	[0.3368, 0.3684]	[0.0480, 0.9460]
{0, 1, 2}	1	1

TABLE V: Interval beliefs for component 3

which one. It is very important to choose a combination rule in accordance with the hypothesis under consideration (independence, reliability, conflict, etc.).

IV. NUMERICAL EXAMPLES

A. Example 1

The TBM method proposed here has not been used before in MSS reliability assessment, and so we have chosen to compare it with the UGF method used by Ding and Lisianski for evaluating reliability indices in an oil transmission system [17]. In this example only aleatory uncertainty is considered (the components' state probabilities are precise values) [17].

The oil transmission system (cf. Figure 3) consists of three pipes. The oil flow is transmitted from point C to point E. The pipes' performance is measured by their transmission capacity (tons per minute). Elements 1 and 2 have three states. A state of total failure for both elements corresponds to a transmission capacity of zero, a state of partial failure corresponding to a capacity of 1 ton/min for element 1 and 1.5 tons/min for element 2, and the operational state corresponds to capacities of 1.5 tons/min for element 1 and 2
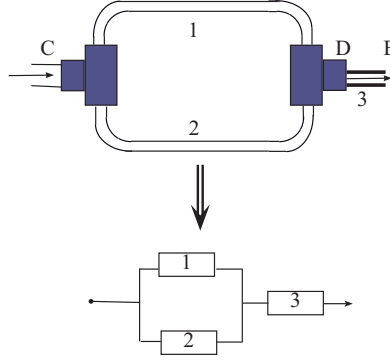
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Fig. 3: The oil transmission system and its Reliability Block Diagram

tons/min for element 2, so that $g_1 = \{0, 1, 1.5\}$ and $g_2 = \{0, 1.5, 2\}$. The corresponding probabilities for element 1 are $p_{10} = 0.1$, $p_{11} = 0.1$, and $p_{12} = 0.8$, and for element 2 are $p_{20} = 0.1$, $p_{21} = 0.2$, and $p_{22} = 0.7$. Element 3 is binary. It has a state of total failure corresponding to a capacity of zero and a fully operational state with a capacity of 4 tons/min so that $g_3 = \{0, 4\}$. The corresponding probabilities are $p_{30} = 0.04$ and $p_{31} = 0.96$. The system output performance rate is defined as the maximum flow that can be transmitted from C to E.

The total flow between points C and D through the parallel pipes 1 and 2 is equal to the sum of the flows in the two pipes. The flow from point D to point E is limited by the transmission capacity of element 3. This flow cannot however be greater than the flow between points C and D. Therefore, the flow between points C and E (the system performance) is:

$$G = \Phi(g_1(t), g_2(t), g_3(t))$$

where Φ is the function which maps component performance rates into system performance rates, as shown in Table VI. The set $G = \{0, 1, 1.5, 2, 2.5, 3, 3.5\}$ represents the system performance rates (the oil transmission system has 7 states). We shall now evaluate the reliability indices of the oil transmission system using both the UGF and the TBM methods.

1) UGF Method: The UGF Method (u-functions) was introduced by Ushakov in the mid-1980s [57], and it has proved to be very effective in evaluating the reliability of different types of MSSs [31], [34], [35]. It involves intuitively simple recursive procedures combined with simplification techniques. The UGF method includes the following steps:

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g_1	0	0	0	0	0	0	1	1	1	1	1	1	1.5	1.5	1.5	1.5	1.5	1.5
g_2	0	0	1.5	1.5	2	2	0	0	1.5	1.5	2	2	0	0	1.5	1.5	2	2
g_3	0	4	0	4	0	4	0	4	0	4	0	4	0	4	0	4	0	4
$G = \Phi(g_1, g_2, g_3)$	0	0	0	1.5	0	2	0	1	0	2.5	0	3	0	1.5	0	3	0	3.5

TABLE VI: Performance rates of the oil transmission system

- Evaluation of individual u-functions for each element.

- Evaluation of the resulting u-function for the whole MSS using composition operators.

- Evaluation of MSS reliability measures.

The u-function representing the probability mass function (pmf) of a random discrete variable X_i is given by

$$u_i(z) = \sum_{j=0}^{m_i} a_{ij} z^{x_{ij}}$$

where x_{ij} are the $m_i + 1$ possible values of X_i and a_{ij} is the probability that X_i is equal to x_{ij}. The u-function representing the pmf of a function Φ of n independent random variables $X_1, X_2, ..., X_n$ is defined using a composition operator Ω_Φ as follows:

$$
\begin{aligned}
U(z) &= \Omega_\Phi(u_1(z), u_2(z), ..., u_n(z)) \\
&= \Omega_\Phi(\sum_{j_1=0}^{m_1} a_{1j_1} z^{x_{1j_1}}, \sum_{j_2=0}^{m_2} a_{2j_2} z^{x_{2j_2}}, ..., \sum_{j_n=0}^{m_n} a_{nj_n} z^{x_{nj_n}}) \\
&= \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} ... \sum_{j_n=0}^{m_n} (\prod_{i=1}^{n} a_{ij_i} z^{\Phi(x_{1j_1}, x_{2j_2}, ..., x_{nj_n})})
\end{aligned}
$$

Using performance rates g_{ij} and the corresponding probabilities $p_{ij}(t)$ that component i ($i \in \{1, ..., n\}$) is in state j ($j \in \{0, ..., m_i\}$), the u-function for component i is defined as follows:

$$u_i(z) = \sum_{j=0}^{m_i} p_{ij} z^{g_{ij}}$$

The u-functions of the three elements of the oil transmission system are then given by:

$$
\begin{aligned}
u_1(z) &= p_{10} z^{g_{10}} + p_{11} z^{g_{11}} + p_{12} z^{g_{12}} \\
&= 0.1 z^0 + 0.1 z^1 + 0.8 z^{1.5}
\end{aligned}
$$

$$u_2(z) = p_{20}z^{g_{20}} + p_{21}z^{g_{21}} + p_{22}z^{g_{22}}$$

$$= 0.1z^0 + 0.2z^{1.5} + 0.7z^2$$

$$u_3(z) = p_{30}z^{g_{30}} + p_{31}z^{g_{31}}$$

$$= 0.04z^0 + 0.96z^4$$

The composition operators Ω_{Φ_S} for serial elements, Ω_{Φ_P} for parallel elements and Ω_{Φ_B} for elements connected in a bridge structure are defined in [35], where corresponding recursive procedures for their computation were introduced for different types of systems.

Applying the operator Ω_Φ with $\Phi(g_1(t), g_2(t), g_3(t))$ over the u-functions of the different elements comprising the oil transmission system, we obtain:

$$U(z) = \Omega_\Phi(u_1(z), u_2(z), u_3(z))$$

$$= \Omega_{\Phi_S}(\Omega_{\Phi_P}(u_1(z), u_2(z)), u_3(z))$$

The function Φ is defined by the type of connection between the elements. Here the function Φ_S is defined as the min function, and the function Φ_P is defined as the sum function:

$$\Phi(g_1, g_2, g_3) = \Phi_S(\Phi_P(g_1, g_2), g_3)$$

$$= min((g_1 + g_2), g_3))$$

The u-function $u_4(z)$ for elements 1 and 2 in parallel is determined as follows:

$$u_4(z) = \Omega_{\Phi_P}(u_1(z), u_2(z))$$

$$= \sum_{j_1=0}^{2} \sum_{j_2=0}^{2} p_{1j_1} p_{2j_2} . z^{g_{1j_1} + g_{2j_2}}$$

$$= 0.01z^0 + 0.01z^1 + 0.1z^{1.5} + 0.07z^2 + 0.02z^{2.5} + 0.23z^3 + 0.56z^{3.5}$$

Based on these procedures, the resulting UGF for the entire MSS can be obtained as follows:

$$
\begin{aligned}
U(z) &= \Omega_{\Phi_S}(u_4(z), u_3(z)) \\
&= \sum_{j_3=0}^{1} \sum_{j_4=0}^{6} p_{4j_4} p_{3j_3}.z^{min\{g_{4j_4}, g_{3j_3}\}} \\
&= 0.0496z^0 + 0.0096z^1 + 0.096z^{1.5} + 0.0672z^2 + 0.0192z^{2.5} + 0.2208z^3 + 0.5376z^{3.5}
\end{aligned}
$$

Applying the operators δ_A, δ_E and δ_C (introduced in [35]) over the resulting u-function of the whole MSS, the following MSS reliability indices are obtained:

- The MSS availability $A(w)$ for the arbitrary demand constant w can be obtained by:

$$
A(w) = \delta_A(U(z), w) = \delta_A(\sum_{j=0}^{m_S} p_j z^{G_j}, w) = \sum_{j=0}^{m_S} p_j 1(G_j - w \geq 0) \tag{11}
$$

where G_j is the performance rate of system at state j and p_j is the corresponding probability of the system being in state j.

- MSS Expected output performance E_d for the given $U(z)$ using the following δ_E:

$$
E_d = \delta_E(U(z)) = \delta_E(\sum_{j=0}^{m_S} p_j z^{G_j}) = \sum_{j=0}^{m_S} p_j G_j \tag{12}
$$

- MSS performance deficiency $E_u(w)$ for the given $U(z)$ and constant demand w using the following δ_D:

$$
E_u(w) = \delta_D(U(z), w) = \delta_D(\sum_{j=0}^{m_S} p_j z^{G_j}, w) = \sum_{j=0}^{m_S} p_j max(w - G_j, 0) \tag{13}
$$

Using Eq. 11, Eq. 12 and Eq. 13, the availability, expected output performance and performance deficiency of the oil transmission system obtained for each demand w are computed (cf. Table VII).

2) TBM method: The system consists of 3 components. Components 1 and 2 have 3 states $\{0, 1, 2\}$. Component 3 has 2 states $\{0, 1\}$. The state vector of the system is $X = (X_1, X_2, X_3)$. The frames of discernment of the components and of the system are:

$$
L_1 = \{0, 1, 2\}
$$

$$
L_2 = \{0, 1, 2\}
$$

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Demand w		UGF	
	A	E_d	E_u
1	0.9504	2.88	0.0496
1.5	0.9408	2.88	0.0792
2	0.8448	2.88	0.1568
2.5	0.7776	2.88	0.268
3	0.7584	2.88	0.3888
3.5	0.5376	2.88	0.6200

TABLE VII: Example 1: MSS reliability indices based on UGF method

X_1	0	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2
X_2	0	0	1	1	2	2	0	0	1	1	2	2	0	0	1	1	2	2
X_3	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$X_S = \varphi(X_1, X_2, X_3)$	0	0	0	2	0	3	0	1	0	4	0	5	0	2	0	5	0	6

TABLE VIII: Structure function of the oil transmission system

$$L_3 = \{0, 1\}$$

$$L_S = \{0, 1, 2, 3, 4, 5, 6\}$$

The corresponding BBAs for element 1, 2 and 3 are $m_{\{0\}}^{L_1} = 0.1$, $m_{\{1\}}^{L_1} = 0.1$, $m_{\{2\}}^{L_1} = 0.8$, $m_{\{0\}}^{L_2} = 0.1$, $m_{\{1\}}^{L_2} = 0.2$, $m_{\{2\}}^{L_2} = 0.7$, $m_{\{0\}}^{L_3} = 0.04$, and $m_{\{1\}}^{L_3} = 0.96$.

The vacuous extension defined in Eq.15 is used to extend the BBAs for the separate components to the product space $L_1 \times L_2 \times L_3 \times L_S$. The resulting BBAs are combined using Dempster's rule, and the new BBAs are then combined with the BBA m_{config}, which represents the structure function φ of the oil transmission system (cf. Table VIII). This BBA is given by:

$$m_{Config}(\{X_1 = 0, X_2 = 0, X_3 = 0, X_S = 0\}, \{X_1 = 0, X_2 = 0, X_3 = 1, X_S = 0\}, ...$$

$$..., \{X_1 = 2, X_2 = 2, X_3 = 0, X_S = 0\}, \{X_1 = 2, X_2 = 2, X_3 = 1, X_S = 6\}) = 1$$

The resulting BBAs are then marginalized to the frame of discernment L_S using Eq.14. Belief and plausibility functions are computed using Eq.1 and Eq.2, and the belief intervals of system states using Eq.4 and Eq.5 (cf. Table IX). Finally, using Eq.7, Eq.8 and Eq.9, we obtain the reliability indices summarized in Table X. In the absence of epistemic uncertainty we have $Bel(\varphi(X) = j) = Pl(\varphi(X) = j)$

System states j	Performance rates (ton/min)	$h_j = [\underline{h_j}, \overline{h_j}]$
0	0	0.0496
1	1	0.0096
2	1.5	0.096
3	2	0.0672
4	2.5	0.0192
5	3	0.1536
6	3.5	0.6048

TABLE IX: Example 1: Belief intervals of system states

Demand w		TBM	
	A	E_d	E_u
1	0.9504	2.88	0.0496
1.5	0.9408	2.88	0.0792
2	0.8448	2.88	0.1568
2.5	0.7776	2.88	0.268
3	0.7584	2.88	0.3888
3.5	0.5376	2.88	0.6200

TABLE X: Example 1: MSS reliability indices based on the TBM method

for each state j ($\underline{h_j} = \overline{h_j}$). Thus we obtain precise values instead of interval beliefs. As expected, the same reliability indices are obtained for the UGF and TBM methods (cf. Table VII and Table X).

B. Example 2

This example features the same oil transmission system used in Example 1, but epistemic uncertainty is also taken into account. The corresponding BBAs of component states are given by three experts (Expert 1, Expert 2 and Expert 3), as shown in Table XI. Three combination rules are used to aggregate the experts' opinions: Dempster's rule, the Disjunctive rule and the Cautious rule [16]. Table XII shows the BBAs of component states obtained using each of these combination rules.

Belief and plausibility functions are computed using Eq.1 and Eq.2. The belief intervals of system states for each rule are then computed using Eq.4 and Eq.5 (cf. Table XIII). As we can see, the Disjunctive rule gives a less precise belief interval than Dempster's rule and the Cautious rule.

The reliability indices of the system (cf. Table XIV) obtained for each demand w using the three combination rules are computed using Eq.7, Eq.8 and Eq.9.

Component i	Expert 1			Expert 2			Expert 3		
	$m_{\{0\}}$	$m_{\{1\}}$	$m_{\{2\}}$	$m_{\{0\}}$	$m_{\{1\}}$	$m_{\{2\}}$	$m_{\{0\}}$	$m_{\{1\}}$	$m_{\{2\}}$
1	0.19	0.2	0.58	0.16	0.3	0.52	0.14	0.32	0.52
2	0.15	0.24	0.56	0.18	0.29	0.53	0.10	0.26	0.6
3	0.29	0.58	0	0.26	0.52	0	0.34	0.52	0

TABLE XI: Example 2: BBAs of component states given by experts

Component i	Dempster's rule			Disjunctive rule			Cautious rule		
	$m_{\{0\}}$	$m_{\{1\}}$	$m_{\{2\}}$	$m_{\{0\}}$	$m_{\{1\}}$	$m_{\{2\}}$	$m_{\{0\}}$	$m_{\{1\}}$	$m_{\{2\}}$
1	0.0302	0.1196	0.8502	0.0043	0.0027	0.0256	0.1560	0.3144	0.5106
2	0.0212	0.1064	0.8724	0.0192	0.0181	0.1568	0.1802	0.2907	0.5116
3	0.2111	0.7798	0	0.1568	0.1781	0	0.3074	0.5662	0

TABLE XII: Example 2: The obtained BBAs after combination

Dempster's rule is based on the assumption that the BBAs to be combined come from reliable experts. The expected output performance $Ed = [2.572681, 2.603106]$ never exceeds 3 tons/min and the system unsupplied demand ever exceeds 1 ton/min. Suppose that the system safety standard requires that the system operation satisfies a required level of system availability greater than or equal to 0.6. This would imply that the oil transmission system cannot meet the system availability requirement if the demand w is greater than 3 ($A = [0.578374, 0.585170]$ for $w > 3$). The system unsupplied demand E_u is always lower than 1 ton/min.

The Disjunctive rule merely assumes that at least one source of information is reliable, and we do not know which one. In the case where a system availability of $A > 0.6$ is required we cannot reach a

System states j	Performance rates (ton/min)	Dempster's rule $[\underline{h_j}, \overline{h_j}]$	Disjunctive rule $[\underline{h_j}, \overline{h_j}]$	Cautious rule $[\underline{h_j}, \overline{h_j}]$
0	0	[0.211560,0.220665]	[0.025647,0.994379]	[0.325023,0.467572]
1	1	[0.001981,0.002005]	[0.000008,0.968740]	[0.032853,0.175403]
2	1.5	[0.016596,0.016795]	[0.000078,0.968810]	[0.076276,0.218825]
3	2	[0.020564,0.029713]	[0.000119,0.968850]	[0.040901,0.183450]
4	2.5	[0.009917,0.010038]	[0.000054,0.968786]	[0.052999,0.195549]
5	3	[0.151858,0.153676]	[0.000981,0.969713]	[0.178829,0.321379]
6	3.5	[0.578374,0.585170]	[0.004380,0.973112]	[0.150569,0.293119]

TABLE XIII: Example 2: Belief intervals of system states

Demand	A	Dempster's rule E_d	E_u
1	[0.779335,0.788440]	[2.572681,2.603106]	[0.211560,0.220665]
1.5	[0.777354,0.786435]	[2.572681,2.603106]	[0.318331,0.332001]
2	[0.760753,0.769645]	[2.572681,2.603106]	[0.433400,0.451734]
2.5	[0.740150,0.748840]	[2.572681,2.603106]	[0.558751,0.581888]
3	[0.730232,0.738807]	[2.572681,2.603106]	[0.689060,0.717062]
3.5	[0.578374,0.585170]	[2.572681,2.603106]	[0.895299,0.929075]

Demand	A	Disjunctive Rule E_d	E_u
1	[0.052958,0.974353]	[0.018774,11.158645]	[0.025647,0.947042]
1.5	[0.050538,0.974302]	[0.018774,11.158645]	[0.038475,1.749088]
2	[0.028428,0.971658]	[0.018774,11.158645]	[0.051342,2.942482]
2.5	[0.005416,0.967598]	[0.018774,11.158645]	[0.064268,4.528951]
3	[0.005361,0.967259]	[0.018774,11.158645]	[0.077222,6.450237]
3.5	[0.004380,0.931722]	[0.018774,11.158645]	[0.090666,8.841546]

Demand	A	Cautious Rule E_d	E_u
1	[0.532428,0.674977]	[1.425048,1.923970]	[0.325023,0.467572]
1.5	[0.499575,0.642124]	[1.425048,1.923970]	[0.503961,0.789059]
2	[0.423299,0.565848]	[1.425048,1.923970]	[0.721037,1.219959]
2.5	[0.382398,0.524947]	[1.425048,1.923970]	[0.958563,1.742584]
3	[0.329399,0.471948]	[1.425048,1.923970]	[1.222589,2.362983]
3.5	[0.150569,0.293119]	[1.425048,1.923970]	[1.576030,3.144072]

TABLE XIV: Example 2: MSS reliability indices

decision, because for each demand w the support of A is very large. Moreover, the supports of E_d and E_u are also very large and greater than the maximum capacity of the oil transmission system (3.5 tons/min). The Disjunctive rule does not generate any conflicts and does not reject any of the information asserted by the sources. As such, no normalization procedure is required. The drawback of this method is that it yields a less precise result.

However, both Dempster's rule and the Disjunctive rule assume that the experts are independent. The Cautious rule is commutative, associative and idempotent. The property of idempotence makes it suitable for combining non-distinct items of evidence (i.e evidence from dependent experts). In Table XIV we can see that using the Cautious rule the oil transmission system *may* (but we cannot be sure) meet the system availability requirement ($A > 0.6$) if the demand is lower than 2 ($A = [0.499575, 0.642124]$ for $w = 1.5$). On the other hand, the expected output performance $Ed = [1.425048, 1.923970]$ is always

lower than 2 tons/min.

V. Conclusion

The current work proves the applicability of D-S theory and TBM in determining the reliability indices of MSSs in the presence of both aleatory and epistemic uncertainties. New definitions are given for MSS reliability indices in the framework of TBM. In particular, in the case of aleatory uncertainty, the same reliability indices are obtained for the UGF and TBM methods. Our results show that the proposed model might be used in practical situations when there is a need to take into account epistemic uncertainties and experts' opinions. The use of several combination rules in the TBM framework is also discussed.

Appendix

A. Formula of marginalization

Consider a BBA $m^{\Omega_x \Omega_y}$ defined on the Cartesian product $\Omega_x \Omega_y$. The marginal BBA $m^{\Omega_x \Omega_y \downarrow \Omega_x}$ on Ω_x is defined by:

$$m^{\Omega_x \Omega_y \downarrow \Omega_x}(A) = \sum_{B \subseteq \Omega_x \Omega_y / Proj(B \downarrow \Omega_x) = A} m^{\Omega_x \Omega_y}(B)$$

(14)

$$\forall A \subseteq \Omega_x$$

Where $Proj(B \downarrow \Omega_x) = \{x \in \Omega_x / \exists y \in \Omega_y, (x, y) \in B\}$.

B. Formula of vacuous extension

Consider a BBA m^{Ω_x} defined on Ω_x. Its vacuous extension on $\Omega_x \Omega_y$ is defined by:

$$m^{\Omega_x \uparrow \Omega_x \Omega_y}(B) = \begin{cases} m^{\Omega_x}(A) & \text{if} \quad B = A \times \Omega_y \\ 0 & \text{otherwise.} \end{cases}$$

(15)

$$\forall A \subseteq \Omega_x$$

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C. Formulas of combination rules

The Conjunctive $\cap$ and Disjunctive $\cup$ rules are defined by:

$$m_{i\cap j}^{\Omega}(H) = \sum_{A \cap B = H, \forall A, B \subseteq \Omega} m_i^{\Omega}(A) m_j^{\Omega}(B), \forall H \subseteq \Omega \tag{16}$$

$$m_{i\cup j}^{\Omega}(H) = \sum_{A \cup B = H, \forall A, B \subseteq \Omega} m_i^{\Omega}(A) m_j^{\Omega}(B), \forall H \subseteq \Omega \tag{17}$$

Dempster's rule is given by:

$$m_{i\oplus j}^{X}(H) = \frac{\sum_{A \cap B = H, \forall A, B \subseteq X} m_i^{X}(A) m_j^{X}(B)}{1 - \sum_{A \cap B = \emptyset, \forall A, B \subseteq X} m_i^{X}(A) m_j^{X}(B)} \tag{18}$$

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